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A Discourse on Galilean Invariance, SUPG Stabilization, and the Variational Multiscale Framework

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A Discourse on Galilean Invariance, SUPG Stabilization, and the Variational Multiscale Framework

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Abstract

Galilean invariance is one of the key requirements of many physical models adopted in theoretical and computational mechanics. Spurred by recent research developments in shock hydrodynamics computations [16], a detailed analysis on the principle of Galilean invariance in the context of SUPG operators is presented. It was observed in [16] that lack of Galilean invariance can yield catastrophic instabilities in Lagrangian computations. Here, the analysis develops at a more general level, and an arbitrary Lagrangian-Eulerian (ALE) formulation is used to explain how to consistently derive Galilean invariant SUPG operators. Stabilization operators for Lagrangian and Eulerian mesh computations are obtained as limits of the stabilization operator for the underlying ALE formulation. In the case of Eulerian meshes, it is shown that most of the SUPG operators designed for compressible flow computations to date *are not consistent* with Galilean invariance. It is stressed that Galilean invariant SUPG formulations can provide consistent advantages in the context of complex engineering applications, due to the simple modifications needed for their implementation.

Acknowledgments

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Appendix

A Discourse on Galilean Invariance, SUPG Stabilization, and the Variational Multiscale Framework

1 Introduction

The Galilean invariance principle states that the form of the equations of motion of an isolated system should be invariant when a change of observer, consisting of a translation with constant velocity $\boldsymbol{V}^G$, is applied.

In the case of numerical computations, it is advisable for the discretized equations of motion to maintain the same invariance properties of the continuum. Bubnov- and Petrov-Galerkin finite element methods are obtained by enforcing the so-called *Galerkin orthogonality property*, which states that the equations of motion (i.e., the *residual*) must be orthogonal to the test function space. In this case, the Galilean principle readily translates into the requirement that the residual must remain orthogonal to the Bubnov- and Petrov-Galerkin test spaces, after a Galilean transformation is performed. It is straightforward to prove that if the equations of the continuum are invariant, so are the discrete equations generated by a Bubnov- or Petrov-Galerkin method. This is due to the fact that the constant velocity $\boldsymbol{V}^G$ factors out of all the integrals in the variational statement, as will be clear from the discussion in section 7.

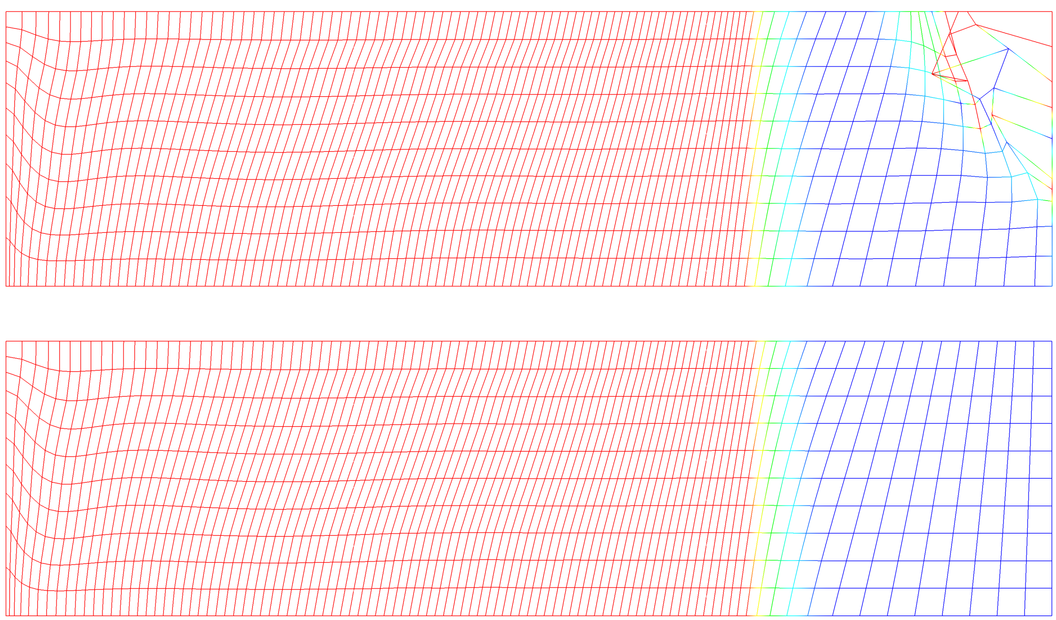

Figure 1. Results from the computations from [16]. Mesh distortion plot: The color scheme represents the pressure. Above: SUPG formulation violating Galilean invariance. Below: SUPG abiding the Galilean invariance principle. A classical quadrilateral Saltzmann mesh is used in an implosion computation. The initial velocity is of unit magnitude and directed horizontally from right to left, except the left boundary which is held fixed. The initial density is unity and the initial specific internal energy is 10^{-1}. A shock forms at the left boundary and advances to the right. Note the *mesh coasting* phenomenon on the top right corner of the upper domain, absent in the SUPG formulation satisfying Galilean invariance, below.

SUPG and variational multiscale stabilized methods [8, 10, 11, 12, 13, 14] can be interpreted as Petrov-Galerkin methods in which the test space depends on the *local* structure of the partial differential equations simulated. A typical stabilized method is derived from the corresponding Bubnov-Galerkin method by *perturbing* the test function space on element interiors. The stability properties of SUPG-stabilized methods depend on the structure of the test function perturbation. In this case, invariance of the test function perturbation has to be ensured to avoid the paradox of having the stability properties of the method depending on the *observer*. As was shown in [16], "standard" stabilization procedures – which usually lack Galilean invariance – were found to generate catastrophic instabilities when applied in compressible Lagrangian hydrodynamics computations (see, e.g., Fig. 1).

More recent work of the author has been focusing on exploring the development of SUPG-stabilized methods for shock hydrodynamics applications on arbitrary Lagrangian-Eulerian meshes, and the question of invariance was posed again. An important aspect of the ongoing investigation is related to what happens when the Eulerian rather than the Lagrangian limit of the ALE equations is taken. To the best of the author's knowledge, the *large majority* of the stabilization operators developed to date in the context of Eulerian (fixed) meshes for compressible flow computations are *not* consistent with Galilean invariance.

A new approach that obviates this issue will be presented and compared to the old approach. As a point of note, the instabilities documented in [16] manifested themselves whenever the inconsistent terms became predominant in the stabilization operator, while were absent in all other conditions. Therefore, it is quite possible that a milder form of such instabilities might have been experienced by other researchers, and erroneously attributed to "weakenesses" in the design of the stabilization tensor $\boldsymbol{\tau}$, whose definition has a substantial degree of arbitrariness. This statement cannot be made more precise, and may be considered as the author's "reasonable doubt".

It will be shown that, for certain definitions of the set of solution variables, conformity with Galilean invariance can be achieved by a number of straightforward *simplifications*, which imply a conspicuous reduction in the computational cost of the stabilization operator. As will become clear from the forthcoming discussion, it is *easier* and computationally *more efficient* to develop Galilean invariant stabilized operators, which, in addition, have the potential for improved reliability in complex geometry, multi-physics applications.

Since SUPG methods have proved to be a well-established and reliable tool for compressible flow simulations, questions about the importance of Galilean invariance may arise. An answer to these concerns is provided in [16], where the pitfalls caused by neglecting the main principles of physics are carefully presented and analyzed. Consequently, one should always wonder whether to take risks on invariance issues, considering the ever-increasing level of complexity of modern numerical computations.

The rest of the material is organized as follows: A very general discussion of the issue of Galilean invariance in the context of ALE equations and its Eulerian and Lagrangian limits is presented in section 2. The ALE description of the kinematics of motion is developed in section 3. Section 4 presents an example of the invariance issue in the context of a one dimensional scalar advection equation, and a brief survey of Galilean invariant SUPG methods for incompressible flows. A stabilized space-time variational formulation of the ALE compressible Euler equations is developed in section 6. Section 7 presents an analysis of the invariance properties of the residuals and their effect on the approximation to the subgrid-scale solution. In section 8, a Galilean consistency analysis shows that standard SUPG formulations for compressible flows yield a non-invariant test function space. A new, invariant approach is also developed, and its advantages are analyzed in detail. Conclusions are summarized in section 9.

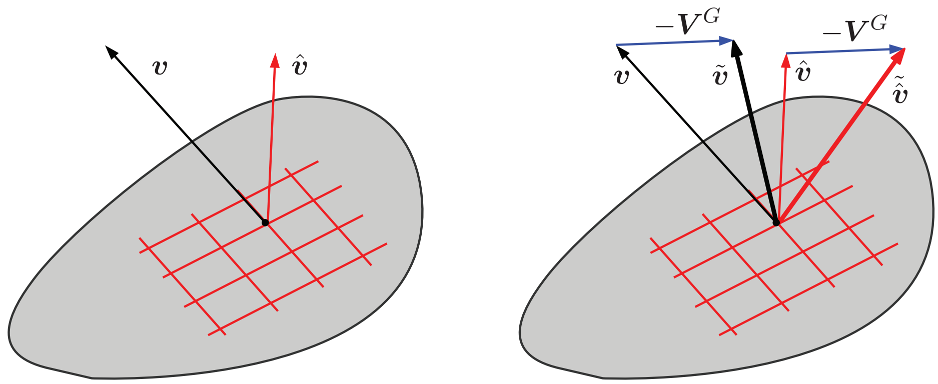

Figure 2. Sketch of a Galilean transformation for a generic ALE mesh. Left: A material domain, and the corresponding mesh (the red grid), are moving with velocity $\boldsymbol{v}$ (black arrow) and $\hat{\boldsymbol{v}}$ (red arrow), respectively. Left: After a Galilean transformation is applied, the material and the mesh are moving with velocities $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}^G$ and $\tilde{\hat{\boldsymbol{v}}} = \hat{\boldsymbol{v}} - \boldsymbol{V}^G$, respectively. The relative velocity of the material with respect to the mesh is an invariant: $\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{v}} - \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G - \hat{\boldsymbol{v}} + \boldsymbol{V}^G = \boldsymbol{v} - \hat{\boldsymbol{v}} = \boldsymbol{c}$.

2 Galilean transformations in the ALE context

This section presents an overview on how Galilean invariance applies to SUPG formulations in various reference frames. In order to explore the computational implications of the Galilean principle, we need to think about the mesh as a (possibly moving) *laboratory* which is used to sample the numerical data. In this sense, we cannot completely separate the numerical aspects from the physics of the problem, since SUPG forces act to stabilize advection and pressure perturbations *across* the mesh.

With respect to the Eulerian (current configuration) reference frame, a Galilean transformation can be expressed by the *affine* mapping

$$G : \mathbb{R}^+ \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \longrightarrow \mathbb{R}^+ \times \mathbb{R}^{n_d} \times \mathbb{R}^{n_d} \tag{1}$$

$$\left[t\ \boldsymbol{x}^T\ \boldsymbol{v}^T\right]^T \mapsto \left[\tilde{t}\ \tilde{\boldsymbol{x}}^T\ \tilde{\boldsymbol{v}}^T\right]^T \tag{2}$$

or, in matrix form,

$$\begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ -\boldsymbol{V}_G & \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix} \begin{bmatrix} t \\ \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{0}_{3\times1} \\ \boldsymbol{V}_G \end{bmatrix} \tag{3}$$

which can be easily inverted as

$$\begin{bmatrix} t \\ \boldsymbol{x} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \boldsymbol{V}_G & \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times1} & \mathbf{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix} \begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{x}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{0}_{3\times1} \\ \boldsymbol{V}^G \end{bmatrix} \tag{4}$$

and consists of a spatial coordinate *shift* by $\boldsymbol{V}^G t$. Therefore, if $\boldsymbol{x}$ represents the coordinate in the original reference frame, and $\tilde{\boldsymbol{x}}$ the coordinate in the transformed reference frame, $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{V}^G t$ indicates that the

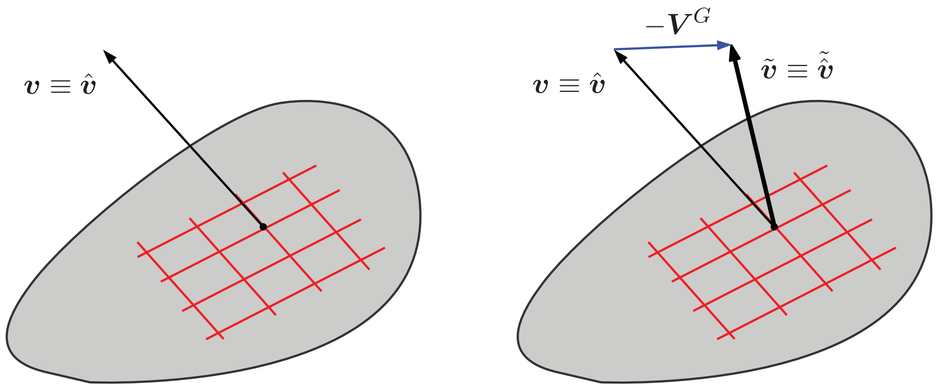

Figure 3. Sketch of a Galilean transformation for a Lagrangian mesh. Left: A material domain, and the corresponding mesh (the red grid), are moving with the same velocity $\boldsymbol{v} \equiv \hat{\boldsymbol{v}}$ (black arrow). Left: After a Galilean transformation is applied, the material and the mesh are moving with velocity $\tilde{\boldsymbol{v}} \equiv \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G = \hat{\boldsymbol{v}} - \boldsymbol{V}^G$. Therefore a Lagrangian mesh is transformed into a Lagrangian mesh by a Galilean transformation.

coordinate $\tilde{\boldsymbol{x}}$ is shifted by $\boldsymbol{V}^G t$ with respect to the coordinate $\boldsymbol{x}$. Analogously, $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}^G$, while the time coordinate is unchanged. Galilean transformations are routinely used to check the consistency of models in physics and computational sciences. In particular, a well-designed model or numerical scheme must be invariant to Galilean transformations. This can be symbolically expressed by saying that any generalized functional form $\mathcal{M}$ representing a model, either physical or numerical, should transform as

$$\mathcal{M}(\boldsymbol{v}, \boldsymbol{x}, t, \dots) \xrightarrow{G} \mathcal{M}(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{x}}, \tilde{t}, \dots) \tag{5}$$

If a mesh is used to discretize a spatial domain by a computational model, Galilean transformations apply to both the mesh and the domain, and the relative velocities between the mesh and the domain are preserved. This simple observation has clear implications when considering the Galilean invariance properties of SUPG formulations. For example, in the generic ALE context (see Fig. 2), an invariant SUPG perturbation to the Bubnov-Galerkin test function *can only* depend on thermodynamic variables and their gradients, the difference $\boldsymbol{c}$ between the material velocity $\boldsymbol{v}$ and the mesh velocity $\hat{\boldsymbol{v}}$, and derivatives of the velocities and position vectors, the only invariant quantities.

In the Lagrangian limit the mesh is tied to the material ($\boldsymbol{v} = \hat{\boldsymbol{v}}$, i.e., $\boldsymbol{c} = \boldsymbol{0}$, see Fig. 3), and the *absolute* velocity of the material $\boldsymbol{v}$ *cannot* enter the expression for the SUPG perturbation to the test space.

In the Eulerian limit, the mesh, seen from the transformed coordinate system, is moving with constant velocity $-\boldsymbol{V}^G$ (see Fig. 4). Therefore, an Eulerian mesh transforms into a uniformly moving mesh after a Galilean change of coordinates is performed. This situation is not paradoxical, but a simple consequence of the general principle of invariance applied to the ALE framework.

Remark 1 *Developing SUPG operators for Eulerian meshes is somewhat problematic, since it is not possible to discern from the equations whether the meaning of "$\boldsymbol{v}$" is $\boldsymbol{v} - \hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{0} = \boldsymbol{c}$, a relative velocity, or simply $\boldsymbol{v}$, the absolute material velocity. In this sense the best way to develop SUPG operators for Eulerian computations is to start from the ALE formulation and then take the limit for a fixed (Eulerian) mesh.*

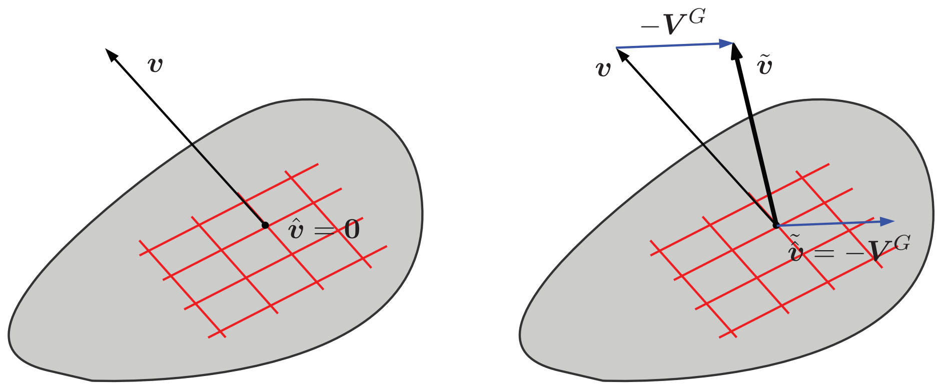

Figure 4. Sketch of a Galilean transformation for an Eulerian mesh. Left: The material is moving with velocity $\boldsymbol{v}$, while the mesh is fixed ($\hat{\boldsymbol{v}} = 0$). Left: After a Galilean transformation is applied, the material is moving with velocity $\tilde{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}^G$ and the mesh is undergoing a rigid body translation with velocity $\tilde{\hat{\boldsymbol{v}}} = -\boldsymbol{V}^G$. Therefore a motionless mesh, after transformation, assumes uniform translational motion. However, $\tilde{\boldsymbol{c}} = \tilde{\boldsymbol{v}} - \tilde{\hat{\boldsymbol{v}}} = \boldsymbol{v} - \boldsymbol{V}^G + \boldsymbol{V}^G = \boldsymbol{v} = \boldsymbol{c}$, as for a generic ALE mesh.

3 Kinematics of motion in the ALE context

The purpose of the present section is to fix the notation for arbitrary Lagrangian Eulerian equations and recall a number of very important results. The notation used in [1] is adopted in what follows, with minor differences. The reader can also refer to [3] or [4] for further details. A point of departure in the discussion of the arbitrary Lagrangian-Eulerian approach is to define the *material* (or *Lagrangian*), *referential*, and *Eulerian* reference frames. Let Ω_0, $\hat{\Omega}$, and Ω be open sets in $\mathbb{R}^{n_d}$ (see, e. g., Fig. 5). The *deformation* $\boldsymbol{\varphi}$ is the transformation from the material to the Eulerian reference frame

$$\begin{aligned} \boldsymbol{\varphi} : \Omega_0 \quad &\rightarrow \quad \Omega = \boldsymbol{\varphi}(\Omega_0), && (6)\\ \boldsymbol{X} \quad &\mapsto \quad \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t), \quad \forall \boldsymbol{X} \in \Omega_0,\ t \geq 0, && (7) \end{aligned}$$

Here $\boldsymbol{X}$ is the material coordinate (which usually corresponds to the point vector in the initial configuration of the body), and $\boldsymbol{x}$ is the point vector in the Eulerian frame. Ω_0 is the domain occupied by the body in the material reference frame. $\boldsymbol{\varphi}$ maps Ω_0 to Ω, the domain occupied by the body in the current configuration (Eulerian frame). It is also useful to define the *deformation gradient*, and the *Jacobian determinant*:

$$\begin{aligned} \boldsymbol{F} \quad &= \quad \nabla_{\boldsymbol{X}}\, \boldsymbol{\varphi} = \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j} && (8)\\ J \quad &= \quad \det(\boldsymbol{F}) && (9) \end{aligned}$$

The *referential map* $\hat{\boldsymbol{\varphi}}$, from the referential frame to the Eulerian frame, is defined as

$$\begin{aligned} \hat{\boldsymbol{\varphi}} : \hat{\Omega} \quad &\rightarrow \quad \Omega = \hat{\boldsymbol{\varphi}}(\hat{\Omega}), && (10)\\ \boldsymbol{\chi} \quad &\mapsto \quad \boldsymbol{x} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t), \quad \forall \boldsymbol{\chi} \in \hat{\Omega},\ t \geq 0, && (11) \end{aligned}$$

where $\boldsymbol{\chi}$ is the point vector in the referential frame. $\hat{\Omega}$, the domain occupied by the body in the referential frame, is mapped to Ω by $\hat{\boldsymbol{\varphi}}$. In addition, the *mesh deformation gradient* and the *mesh Jacobian*

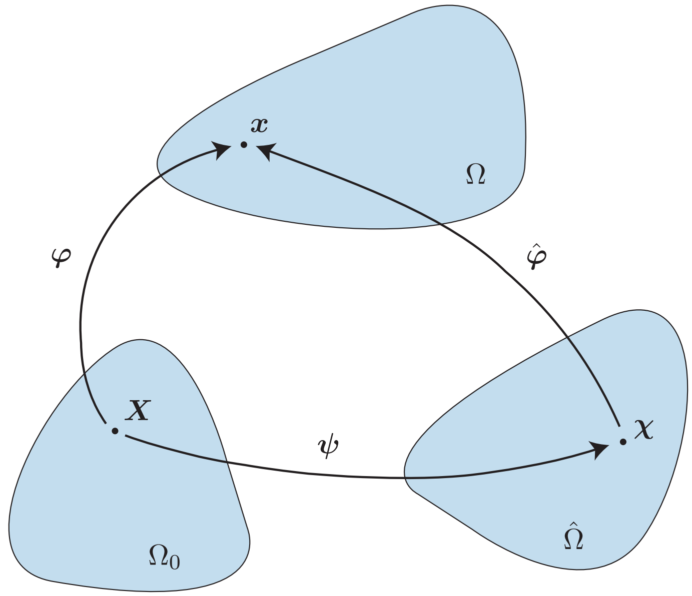

Figure 5. Sketch of the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\boldsymbol{\psi}$ for the generalized ALE framework.

determinant are defined as:

$$\hat{\boldsymbol{F}} = \nabla_{\chi} \hat{\boldsymbol{\varphi}} = \frac{\partial \hat{\varphi}_i}{\partial \chi_j} = \frac{\partial x_i}{\partial \chi_j} \tag{12}$$

$$\hat{J} = \det(\hat{\boldsymbol{F}}) \tag{13}$$

The referential frame of reference lies on a mesh which is not fixed in space (Eulerian) nor attached to the material (Lagrangian), but moves in time with an arbitrary motion. The transformation from the material to the referential frame will also be needed, namely

$$\boldsymbol{\psi} : \Omega_0 \rightarrow \hat{\Omega} = \boldsymbol{\psi}(\Omega_0), \tag{14}$$

$$\boldsymbol{X} \mapsto \boldsymbol{\chi} = \boldsymbol{\psi}(\boldsymbol{X}, t), \quad \forall \boldsymbol{X} \in \Omega_0,\ t \geq 0, \tag{15}$$

The definition of the *referential deformation gradient* reads

$$\nabla_{\mathbf{X}} \boldsymbol{\psi} = \frac{\partial \psi_i}{\partial X_j} = \frac{\partial \chi_i}{\partial X_j} \tag{16}$$

Displacements can then be defined as

$$\boldsymbol{u} = \boldsymbol{\varphi}(\boldsymbol{X}, t) - \boldsymbol{\varphi}(\boldsymbol{X}, 0) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} \tag{17}$$

$$\hat{\boldsymbol{u}} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, 0) = \boldsymbol{x}(\boldsymbol{\chi}, t) - \boldsymbol{X} \tag{18}$$

with the practical assumption, $\boldsymbol{\chi}(\boldsymbol{X}, t = 0) = \boldsymbol{X}$. The referential displacement $\hat{\boldsymbol{u}}$ is the displacement undergone by the mesh. Analogously, material and mesh velocities can be defined:

$$\boldsymbol{v} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{X}} = \left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{X}} = \dot{\boldsymbol{u}}, \tag{19}$$

$$\hat{\boldsymbol{v}} = \left.\frac{\partial \boldsymbol{x}}{\partial t}\right|_{\boldsymbol{\chi}} = \left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{\chi}} \tag{20}$$

Using the chain rule, it is possible to derive two very important expressions for the Lagrangian time derivative of a scalar-valued function f:

$$\dot{f}(\boldsymbol{\chi}, t) = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{w} \cdot \nabla_{\boldsymbol{\chi}} f = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} f \tag{21}$$

where $\nabla_{\boldsymbol{\chi}}$ and $\nabla_{\boldsymbol{x}}$ are the gradients in the referential and Eulerian frames, respectively. $\boldsymbol{w} = \partial_t \boldsymbol{\chi}|_{\boldsymbol{X}} = \dot{\boldsymbol{\psi}}(\boldsymbol{X}, t) = \dot{\boldsymbol{\chi}}$ is the *particle referential velocity*, that is the velocity of a material point seen from the referential frame. The *convective velocity* $\boldsymbol{c}$ is the velocity of the material relative to the mesh, and is related to $\boldsymbol{w}$ through

$$\boldsymbol{c} = \boldsymbol{v} - \hat{\boldsymbol{v}} = \hat{\boldsymbol{F}} \boldsymbol{w} \tag{22}$$

or, in index notation, $c_i = v_i - \hat{v}_i = \hat{F}_{ij} w_j$, with $\hat{F}_{ij} = \partial_{\chi_j} \hat{\varphi}_i(\boldsymbol{\chi}, t) = \partial_{\chi_j} x_i$.

3.1 Limit behavior

In the Lagrangian limit, $\boldsymbol{\chi} \equiv \boldsymbol{X}$, $\hat{\boldsymbol{v}} = \boldsymbol{v}$, and $\hat{\boldsymbol{F}} \equiv \boldsymbol{F}$, $\forall t$, so that $\boldsymbol{w} = \dot{\boldsymbol{\chi}} = \dot{\boldsymbol{X}} = \boldsymbol{0}$, and, in addition, $\boldsymbol{c} = \hat{\boldsymbol{F}} \boldsymbol{w} = \boldsymbol{0}$.

In the Eulerian limit, $\boldsymbol{\chi} \equiv \boldsymbol{x}$, $\hat{\boldsymbol{v}} = \boldsymbol{0}$, and $\hat{\boldsymbol{F}} \equiv \boldsymbol{I}$, $\forall t$, so that $\boldsymbol{w} = \dot{\boldsymbol{\chi}} = \dot{\boldsymbol{x}} = \boldsymbol{v}$, and, in addition, $\boldsymbol{c} = \hat{\boldsymbol{F}} \boldsymbol{w} = \boldsymbol{I} \boldsymbol{w} = \boldsymbol{w} = \boldsymbol{v}$.

4 Preamble: Linear scalar advection equation in one dimension

The discussion in the case of compressible flow equations will involve a large number of algebraic manipulations. However, the issues object of the discussion can be easily explained in the case of a scalar advection equation of the type

$$\left.\begin{array}{c} \dot{\phi} \\[1ex] \left.\dfrac{\partial \phi}{\partial t}\right|_{\chi} + w \dfrac{\partial \phi}{\partial \chi} \\[2ex] \left.\dfrac{\partial \phi}{\partial t}\right|_{x} + v \dfrac{\partial \phi}{\partial x} \end{array}\right\} = f \tag{23}$$

where the Lagrangian, ALE, and Eulerian descriptions of motion have been adopted. To avoid including boundary conditions in the discussion, the domains are infinite, namely, $\Omega_0 = \hat{\Omega} = \Omega = (-\infty, \infty)$. Assuming w and v constant (i.e., the material and the mesh velocities are constants, possibly different from one another) the Galerkin formulations corresponding to problem (23) in the three reference frames are given by:

$$0 = \int_{\Omega_0} \Psi^h \left(\dot{\phi}^h - f \right) dX \tag{24}$$

$$0 = \int_{\hat{\Omega}} \left(\hat{\psi}^h \left. \frac{\partial \phi^h}{\partial t} \right|_{\chi} - \frac{\partial \hat{\psi}^h}{\partial \chi} w \phi^h - \hat{\psi}^h f \right) \ d\chi \tag{25}$$

$$0 = \int_{\Omega} \left(\psi^h \left. \frac{\partial \phi^h}{\partial t} \right|_{x} - \frac{\partial \psi^h}{\partial x} v \phi^h - \psi^h f \right) \ dx \tag{26}$$

where ϕ^h is the numerical approximation to ϕ and Ψ^h, $\hat{\psi}^h$, and ψ^h are the test functions, collocated at the nodes of the Lagrangian, ALE, and Eulerian meshes, respectively. To fix the ideas, we can think of solving the Galerkin formulations above in the space of continuous functions which are piecewise linear over each element of the discretization. Let us then ask ourselves the question: "What is the correct way of stabilizing the Galerkin discretizations (24)–(26)?" The answer is easy in this simple case.

For the Lagrangian form (24), there is no advection across the mesh, and a simple ordinary differential equation does not need stabilization. Anticipating a later discussion, the case of compressible Euler equations is more complicated, since there is still no advection of material across the computational grid, but acoustic waves do propagate through the mesh and need stabilization.

For the ALE equation (25), the advection is given by the particle referential velocity w, and, applying the SUPG method originally developed by Brooks and Hughes in [2], the stabilization term reads:

$$\mathcal{SUPG}(\hat{\psi}^h, \phi^h) = \sum_{e=1}^{n_{el}} \int_{\hat{\Omega}^e} \left(w \frac{\partial \hat{\psi}^h}{\partial \chi} \right) \tau_e \left(\left. \frac{\partial \phi^h}{\partial t} \right|_{\chi} + w \frac{\partial \phi^h}{\partial \chi} - f \right) \ d\chi \tag{27}$$

where a typical choice for τ is given by (see also the recent, concurrent work of Masud [15])

$$\tau_e = \left(\left(\frac{2}{\Delta t} \right)^{\beta} + \left| \frac{2w}{\Delta \chi_e} \right|^{\beta} \right)^{-1/\beta} \tag{28}$$

with $\beta \geq 1$. $\Delta \chi_e$ is the element length of the referential mesh.

Remark 2 *The perturbation to the test function, $w \partial_{\chi} \hat{\psi}^h \tau_e$, is clearly invariant. Therefore, for this simple example, the Petrov-Galerkin method generated by the SUPG approach is invariant at the discrete level. Notice that the residual $\left. \frac{\partial \phi^h}{\partial t} \right|_{\chi} + w \frac{\partial \phi^h}{\partial \chi} - f$ is also invariant. For more complicated sets of nonlinear equations, this last condition may not always be verified, as explained in section 7.*

In the Eulerian case, the advection is due to the material velocity v, and the derivations are analogous to the ALE case:

$$\mathcal{SUPG}(\psi^h,\phi^h) = \sum_{e=1}^{n_{el}} \int_{\Omega^e} \left(v\frac{\partial \psi^h}{\partial x} \right) \tau_e \left(\frac{\partial \phi^h}{\partial t}\bigg|_x + v\frac{\partial \phi^h}{\partial x} - f \right) \, dx \tag{29}$$

$$\tau_e = \left(\left(\frac{2}{\Delta t} \right)^\beta + \left| \frac{2v}{\Delta x_e} \right|^\beta \right)^{-1/\beta} \tag{30}$$

Remark 3 *The Eulerian and Lagrangian cases are limits of the ALE case. In fact, (27) vanishes for $w = 0$, and transforms to (29), for $w = v$.*

The previous derivations are clearly consistent with the principle of Galilean invariance: If a Galilean transformation is applied to the Eulerian mesh, we will recover an ALE formulation as in (27)–(28), with a transformed mesh velocity $\tilde{\hat{v}} = -V^G$, so that $\tilde{w} = \tilde{v} - \tilde{\hat{v}} = v - V^G + V^G = v$. With these substitutions, the transformed SUPG operator is *exactly identical* to the original Eulerian SUPG operator, and for a very important reason: *it is the advection relative to the mesh that needs stabilization, and not the absolute advection.*

Remark 4 *It would have been utterly incorrect to say that for the transformed Eulerian case, the form of (29)–(30) would hold unchanged, with $\tilde{v}$ in place of v. In this case, the SUPG operator would change even if the advection relative to the mesh were unchanged: This is the key point of the entire discussion.*

Remark 5 *It is clear that if a general approach needs to be developed, it is crucial to start from the ALE equations and take Lagrangian and Eulerian limits, rather than trying to generalize a concept developed for the Eulerian equations. For additional considerations on stabilization issues in the ALE context, see [15].*

5 A brief survey on stabilized methods for incompressible flow

Although the present work is mainly focused on compressible flows, it is worthwhile to briefly discuss the incompressible case, for which *all* the most commonly used stabilization techniques *are* Galilean invariant.

5.1 SUPG stabilization of the incompressible Navier-Stokes equations, Brooks and Hughes [2]

Considering the incompressible Navier-Stokes equations in ALE advective form, it is straightforward to see that Galilean invariance is preserved:

$$\mathcal{SUPG}(\hat{\boldsymbol{\psi}}^h_{\boldsymbol{v}}; \rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) = \sum_{e=1}^{n_{el}} \int_{\hat{\Omega}^e_n} (\tau \boldsymbol{w} \cdot \nabla_{\boldsymbol{\chi}} \hat{\boldsymbol{\psi}}^h_{\boldsymbol{v}}) \cdot \hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) \, d\hat{\Omega} \tag{31}$$

Here $\hat{\boldsymbol{\psi}}_{\boldsymbol{v}}^h$ is the test function vector, and

$$\begin{aligned}\hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) &= \hat{J}\rho \left.\frac{\partial \boldsymbol{v}^h}{\partial t}\right|_{\chi} + \hat{J}\rho \boldsymbol{w} \cdot \nabla_{\boldsymbol{\chi}} \boldsymbol{v} + \mathbf{cof}\hat{\boldsymbol{F}} \nabla_{\boldsymbol{\chi}} p - \hat{J}\rho \boldsymbol{g} \\ &= \hat{J}\left(\rho \left.\frac{\partial \boldsymbol{v}^h}{\partial t}\right|_{\chi} + \rho \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} + \nabla_{\boldsymbol{x}} p - \rho \boldsymbol{g}\right)\end{aligned} \tag{32}$$

where $\mathbf{cof}\hat{\boldsymbol{F}} = \hat{J}\hat{\boldsymbol{F}}^{-T}$. In the Eulerian limit, $\boldsymbol{w} = \boldsymbol{v}$, $\boldsymbol{\chi} = \boldsymbol{x}$, and the familiar expression for the stabilization is recovered. Notice that Galilean invariance holds as long as

$$\tau = \tau(\boldsymbol{w}, \nu, \Delta t, \Delta\chi_e) \tag{33}$$

where ν is the physical viscosity, and $\Delta\chi_e$ is a mesh length scale.

5.2 PSPG stabilization, Tezduyar [19]

PSPG-type terms (see, e.g., [18, 20, 19]) are also Galilean consistent. In the ALE context, their form is:

$$\mathcal{PSPG}(\hat{\psi}_\rho^h; \rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h) = \sum_{e=1}^{n_{el}} \int_{\hat{\Omega}_n^e} \frac{\tau_{PSPG}}{\rho} \nabla_{\boldsymbol{\chi}} \hat{\psi}_\rho^h \cdot \hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\rho, \boldsymbol{v}^h, \boldsymbol{w}, p^h)\, d\hat{\Omega} \tag{34}$$

where $\hat{\psi}_\rho^h$ is the test function for the mass conservation (divergence-free velocity field constraint), and

$$\tau_{PSPG} = \tau_{PSPG}(\boldsymbol{w}, \nu, \Delta t, \Delta\chi_e) \tag{35}$$

5.3 Advanced multiscale concepts and turbulence [9]

Recent developments in the application of multiscale methods to stabilization and turbulence subgrid modeling hinge upon substituting the subgrid-scale approximation

$$\boldsymbol{v}' \approx -\tau \hat{\boldsymbol{Res}}^{\boldsymbol{v}}(\boldsymbol{v}^h, \boldsymbol{w}, p^h) \tag{36}$$

in the Galerkin mesh-scale equations. As long as the τ parameter is in the form (33), the overall approach is Galilean invariant.

6 ALE equations of compressible flows

The present section contains the derivation and discretization of the ALE equations, using the space-time formulation developed in [17].

6.1 Generalized Reynolds transport theorem

In order to derive useful integral forms of conservation laws, a generalized version of the classical Reynolds transport theorem is needed. The transport theorem is simply an integral equation between the material reference frame and an arbitrary reference frame, which may or may not correspond to the Eulerian frame. Hence, if referential coordinates are used,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}} f\hat{J}\; d\hat{\Omega} = \int_{\hat{\Omega}} \left.\frac{\partial(f\hat{J})}{\partial t}\right|_{\boldsymbol{\chi}} d\hat{\Omega} + \int_{\hat{\Gamma}=\partial\hat{\Omega}} f\boldsymbol{w}\cdot\hat{\boldsymbol{n}}\;\hat{J}\; d\hat{\Gamma} \tag{37}$$

Equation (37) can be derived noticing that it corresponds to the standard form in Eulerian coordinates with $\boldsymbol{\chi}$ in place of $\boldsymbol{x}$ and $\boldsymbol{w}$ in place of $\boldsymbol{v}$.

6.2 Integral form of the ALE equations in the referential coordinate frame

Applying (37) to the mass, momentum, and total energy, it is easily derived (see [1], pp. 443–447):

$$0 = \int_{\hat{\Omega}} \left.\frac{\partial\hat{\rho}}{\partial t}\right|_{\boldsymbol{\chi}} d\hat{\Omega} + \int_{\hat{\Gamma}} \hat{\rho}\boldsymbol{w}\cdot\hat{\boldsymbol{n}}\; d\hat{\Gamma} \tag{38}$$

$$0 = \int_{\hat{\Omega}} \left.\frac{\partial(\hat{\rho}\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} d\hat{\Omega} + \int_{\hat{\Gamma}} (\hat{\rho}\boldsymbol{v}\otimes\boldsymbol{w} - \hat{\boldsymbol{P}})\hat{\boldsymbol{n}}\; d\hat{\Gamma} - \int_{\hat{\Omega}} \hat{\rho}\boldsymbol{g}\; d\hat{\Omega} \tag{39}$$

$$\begin{aligned} 0 = {} & \int_{\hat{\Omega}} \left.\frac{\partial(\hat{\rho}E)}{\partial t}\right|_{\boldsymbol{\chi}} d\hat{\Omega} + \int_{\hat{\Gamma}} (\hat{\rho}E\boldsymbol{w} - \hat{\boldsymbol{P}}^T\boldsymbol{v} + \hat{\boldsymbol{Q}})\cdot\hat{\boldsymbol{n}}\; d\hat{\Gamma} \\ & - \int_{\hat{\Omega}} \hat{\rho}(\boldsymbol{v}\cdot\boldsymbol{g} + s)\; d\hat{\Omega} \end{aligned} \tag{40}$$

where $\hat{\rho} = \rho\hat{J}$, $\boldsymbol{g}$ is the body force term per unit mass (e.g., the gravitational acceleration), $\hat{\boldsymbol{P}} = \hat{J}\boldsymbol{\sigma}\hat{\boldsymbol{F}}^{-T} = \boldsymbol{\sigma}\;\mathbf{cof}\hat{\boldsymbol{F}}$, $\boldsymbol{\sigma}$ is the Cauchy stress tensor in Eulerian coordinates, $E = e + \boldsymbol{v}\cdot\boldsymbol{v}/2$ is the total energy per unit mass, e is the internal energy per unit mass, $\hat{\boldsymbol{Q}} = (\boldsymbol{q}^T\mathbf{cof}\hat{\boldsymbol{F}})^T = (\mathbf{cof}\hat{\boldsymbol{F}})^T\boldsymbol{q} = \hat{J}\hat{\boldsymbol{F}}^{-1}\boldsymbol{q}$, $\boldsymbol{q}$ is the heat flux in the Eulerian frame, and s is a heat source ($s > 0$) or sink ($s < 0$) per unit mass. Let us introduce the following definitions:

$$\hat{\boldsymbol{U}} = \hat{J}\boldsymbol{U}, \qquad \boldsymbol{U} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E \end{bmatrix}, \qquad \hat{\boldsymbol{Z}} = \begin{bmatrix} 0 \\ -\hat{J}\rho g_1 \\ -\hat{J}\rho g_2 \\ -\hat{J}\rho g_3 \\ -\hat{J}\rho v_i g_i - \hat{J}\rho s \end{bmatrix} \tag{41}$$

$$\hat{\boldsymbol{G}}_i = \begin{bmatrix} \hat{J}\rho w_i \\ \hat{J}\rho v_1 w_i - \hat{P}_{1i} \\ \hat{J}\rho v_2 w_i - \hat{P}_{2i} \\ \hat{J}\rho v_3 w_i - \hat{P}_{3i} \\ \hat{J}\rho E w_i - v_k\hat{P}_{ik} + \hat{Q}_i \end{bmatrix} = \begin{bmatrix} \hat{J}\rho w_i \\ \hat{J}\rho v_1 w_i - \sigma_{1k}\;\mathrm{cof}\hat{F}_{ki} \\ \hat{J}\rho v_2 w_i - \sigma_{2k}\;\mathrm{cof}\hat{F}_{ki} \\ \hat{J}\rho v_3 w_i - \sigma_{3k}\;\mathrm{cof}\hat{F}_{ki} \\ \hat{J}\rho E w_i + (q_k - v_j\sigma_{jk})\;\mathrm{cof}\hat{F}_{ki} \end{bmatrix} \tag{42}$$

with $i = 1, 2, 3$. $\boldsymbol{U}$ is the vector of conserved variables, $\hat{\boldsymbol{U}}$ is the vector of generalized ALE conserved variables, $\hat{\boldsymbol{G}}_i$ is the Euler flux in the i-th direction, and $\hat{\boldsymbol{Z}}$ is a vector-valued source term.

Remark 6 *Notice that*

$$\hat{\boldsymbol{G}}_i \;=\; w_i \hat{\boldsymbol{U}} + \hat{\boldsymbol{G}}_i^L, \qquad \hat{\boldsymbol{G}}_i^L = \begin{bmatrix} 0 \\ -\sigma_{1k}\ \mathrm{cof}\hat{F}_{ki} \\ -\sigma_{2k}\ \mathrm{cof}\hat{F}_{ki} \\ -\sigma_{3k}\ \mathrm{cof}\hat{F}_{ki} \\ (q_k - v_j\sigma_{jk})\ \mathrm{cof}\hat{F}_{ki} \end{bmatrix} \tag{43}$$

where $\hat{\boldsymbol{G}}_i^L$ *is the Lagrangian limit of the Euler flux Jacobians, as* $\boldsymbol{w} \rightarrow \mathbf{0}$.

Equations (38)–(40) can be expressed more succinctly in vector form:

$$\partial_t|_{\boldsymbol{\chi}} \hat{\boldsymbol{U}}(\boldsymbol{Y}) + \partial_{\chi_i} \hat{\boldsymbol{G}}_i(\boldsymbol{Y}) + \hat{\boldsymbol{Z}} \;=\; \mathbf{0} \tag{44}$$

where the Gauss divergence theorem has been applied, as well as the fact that (38)–(40) hold on an arbitrary domain. $\boldsymbol{Y}$ is the vector of solution variables to be specified subsequently.

6.3 Mie-Grüneisen constitutive laws

It is assumed that the materials under consideration do not possess deformation strength, so that the Cauchy stress tensor $\boldsymbol{\sigma}$ reduces to an isotropic tensor, dependent only on the thermodynamic pressure:

$$\sigma_{ij} \;=\; -p\ \delta_{ij} \tag{45}$$

with δ_{ij}, the Kronecker tensor. Mie-Grüneisen materials satisfy an equation of state of the form $p = f_1(\rho; \rho_r, e_r) + f_2(\rho; \rho_r, e_r)e$, where ρ_r and e_r are fixed reference thermodynamic states. More succinctly,

$$p \;=\; f_1(\rho) \;+\; f_2(\rho)\ e \tag{46}$$

If $f_1 = 0$ and $f_2 = (\gamma - 1)\ \rho$, the equation of state for an ideal gas, $p \;=\; (\gamma - 1)\ \rho\ e$, is recovered. Thanks to the Mie-Grüneisen constitutive equations, a quasi-linear form of (44) can be developed, namely,

$$\hat{\boldsymbol{A}}_0\ \partial_t|_{\boldsymbol{\chi}} \boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y})\ \partial_{\chi_i} \boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y})\ \boldsymbol{Y} \;=\; \mathbf{0} \tag{47}$$

The definitions of $\hat{\boldsymbol{A}}_0$, $\hat{\boldsymbol{A}}_i$, and $\hat{\boldsymbol{C}}$ will be given in section 8, and depend on the choice of the solution vector $\boldsymbol{Y}$.

6.4 A space-time variational formulation in referential coordinates

In order to lay the foundations for the subsequent discussion, a space-time variational formulation in the referential frame is presented. The analysis of Galilean invariance is not strictly dependent on the variational formulation adopted, and, for example, similar conclusions hold for alternative space-time or semi-discrete formulations. In this paper, the approach developed in [17] for the purely Lagrangian case is extended to the ALE equations.

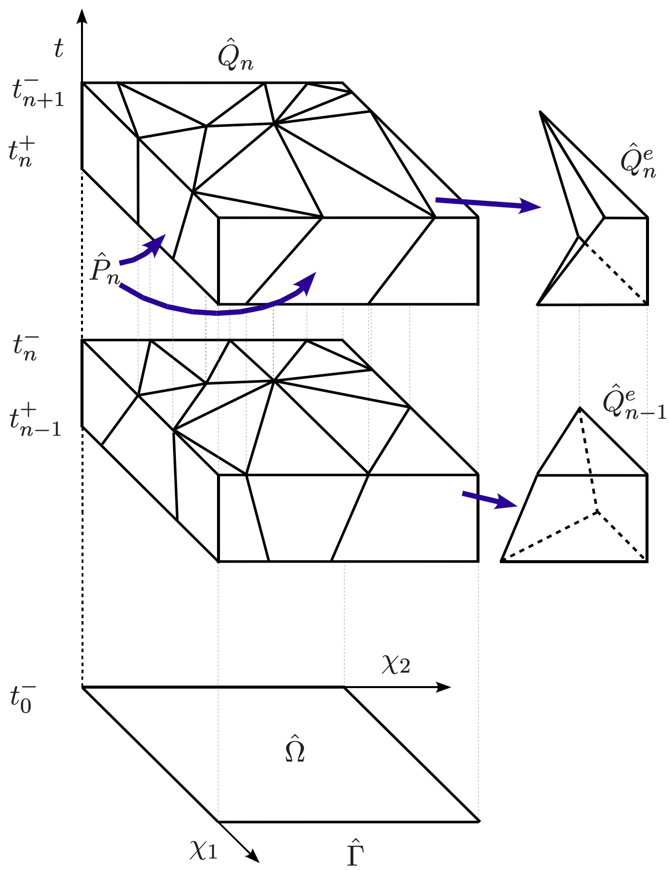

Figure 6. General finite element discretization in space-time.

Given a partition $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ of the time interval $I =]0, T]$, let $I_n =]t_n, t_{n+1}]$, so that $]0, T] = \bigcup_{n=0}^{N-1} I_n$. The space-time domain $\hat{Q} = \hat{\Omega} \times I$ can be divided into time slabs

$$\hat{Q}_n = \hat{\Omega} \times I_n \tag{48}$$

with "lateral" boundary $\hat{P}_n = \hat{\Gamma} \times I_n$ ($\hat{\Gamma} = \partial\hat{\Omega}$ is the boundary of $\hat{\Omega}$). A sketch of the general discretization in space-time is presented in Figure 6. We will only make use of discretizations *prismatic* in time. The material domain $\hat{\Omega}$ is further divided into material-subdomains $\hat{\Omega}^e$ (elements in space, a partition of the initial configuration). Thus $\hat{\Omega} = \overline{\bigcup_{e=1}^{n_{el}} \hat{\Omega}^e}$, and, consequently, a typical space-time element is given by the prism (i.e., tensor product domain)

$$\hat{Q}^e_n = \hat{\Omega}^e \times I_n \tag{49}$$

It is also assumed that the space-time boundary is partitioned as $\hat{P}_n = \hat{P}^g_n \cup \hat{P}^h_n$, $\hat{P}^g_n \cap \hat{P}^h_n = \emptyset$ (i.e., $\hat{P}_n$ is divided into a Dirichlet boundary $\hat{P}^g_n$ and a Neumann boundary $\hat{P}^h_n$). Let us define the test and trial function spaces as follows:

$$
\begin{aligned}
\hat{\mathcal{S}}^h &= \Big\{ \hat{\boldsymbol{V}}^h : \hat{\boldsymbol{V}}^h \in (C^0(\hat{Q}))^m, \\
&\qquad \hat{\boldsymbol{V}}^h\Big|_{\hat{Q}^e_n} \in (\mathcal{P}_1(\hat{\Omega}^e) \times \mathcal{P}_1(I_n))^m, \hat{\boldsymbol{V}}^h = \boldsymbol{G}_{bc}(t) \text{ on } \hat{P}^g_n \Big\} \qquad (50)\\
\hat{\mathcal{V}}^h &= \Big\{ \hat{\boldsymbol{W}}^h : \hat{\boldsymbol{W}}^h\Big|_{\hat{\Omega}} \in (C^0(\hat{\Omega}))^m, \\
&\qquad \hat{\boldsymbol{W}}^h\Big|_{\hat{Q}^e_n} \in (\mathcal{P}_1(\hat{\Omega}^e) \times \mathcal{P}_0(I_n))^m, \hat{\boldsymbol{W}}^h = \boldsymbol{0} \text{ on } \hat{P}^g_n \Big\} \qquad (51)
\end{aligned}
$$

where $\boldsymbol{G}_{bc}(t)$ is the vector of Dirichlet boundary conditions, $\mathcal{P}_k$ is the set of polynomials up to degree k, and $m = n_d + 2$, $n_d \in \{1, 2, 3\}$. The trial function space $\hat{\mathcal{S}}^h$ is given by the piecewise-linear, continuous functions on $\hat{Q} = \hat{\Omega} \times]0, T[$, while the test function space $\hat{\mathcal{V}}^h$ is given by functions that are continuous piecewise-linear in space and discontinuous, piecewise-constant in time. The variational statement reads:

Find $\boldsymbol{Y}^h \in \hat{\mathcal{S}}^h$, such that $\forall \hat{\boldsymbol{W}}^h \in \hat{\mathcal{V}}^h$

$$
\mathcal{B}(\hat{\boldsymbol{W}}, \boldsymbol{Y}^h) \; + \; \mathcal{SUPG}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) + \mathcal{DC}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) = \mathcal{F}(\hat{\boldsymbol{W}}) \qquad (52)
$$

with

$$
\begin{aligned}
\mathcal{B}(\hat{\boldsymbol{W}}, \boldsymbol{Y}^h) &= \int_{\hat{\Omega}} \hat{\boldsymbol{W}}^h(\boldsymbol{\chi}) \cdot \hat{\boldsymbol{U}}(\boldsymbol{Y}^h(\boldsymbol{\chi}, t_{n+1})) - \hat{\boldsymbol{W}}^h(\boldsymbol{\chi}) \cdot \hat{\boldsymbol{U}}(\boldsymbol{Y}^h(\boldsymbol{\chi}, t_n)) \;\; d\hat{\Omega} \\
&+ \int_{\hat{Q}_n} \left(-\hat{\boldsymbol{W}}^h_{,i} \cdot \hat{\boldsymbol{G}}_i(\boldsymbol{Y}^h) + \hat{\boldsymbol{W}}^h \cdot \hat{\boldsymbol{Z}}(\boldsymbol{Y}^h) \right) \;\; d\hat{Q} \\
&+ \int_{\hat{P}^g_n} \hat{\boldsymbol{W}}^h \cdot \hat{\boldsymbol{G}}_i(\boldsymbol{Y}^h) \hat{n}_i d\hat{P} \qquad (53)\\
\mathcal{F}(\hat{\boldsymbol{W}}) &= -\int_{\hat{P}^h_n} \hat{\boldsymbol{W}}^h \cdot \hat{\boldsymbol{H}}_i \hat{n}_i \; d\hat{P} \qquad (54)
\end{aligned}
$$

$\hat{\boldsymbol{W}}$ is the vector-valued test function, $\hat{n}_i$ is the i-th component of the normal to the space-time boundary, and $\hat{\boldsymbol{H}}_i$ is the Neumann flux across the boundary in the i-th direction. The SUPG operator $\mathcal{SUPG}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h)$ will be defined subsequently. The discontinuity capturing operator $\mathcal{DC}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h)$, will be omitted in the following discussion, which applies to regions of smooth flows, away from discontinuities.

Remark 7 *The proposed formulation is second-order-in-time and, following derivations analogous to [17], it can be easily proven to embed global conservation of mass, momentum and total energy.*

6.5 SUPG Stabilization

The SUPG stabilization operator can be abstractly defined as

$$
\mathcal{SUPG}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) \;=\; -\sum_{e=1}^{(n_{el})_n} \int_{\hat{Q}^e_n} (\hat{\mathcal{L}}^*_{SH} \hat{\boldsymbol{W}}_h) \cdot \hat{\boldsymbol{\tau}} \; \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h) d\hat{Q} \qquad (55)
$$

where

$$
\begin{aligned}
\hat{\boldsymbol{Res}} &= \hat{\mathcal{L}} = \hat{\boldsymbol{A}}_0\, \partial_t|_{\boldsymbol{\chi}} + \hat{\boldsymbol{A}}_i \partial_{\chi_i} + \hat{\boldsymbol{C}} && (56)\\
\hat{\mathcal{L}}_{SH} &= \hat{\boldsymbol{A}}_0\, \partial_t|_{\boldsymbol{\chi}} + \hat{\boldsymbol{A}}_i \partial_{\chi_i} && (57)\\
\hat{\mathcal{L}}^*_{SH} &= -\hat{\boldsymbol{A}}_0^T\, \partial_t|_{\boldsymbol{\chi}} - \hat{\boldsymbol{A}}_i^T \partial_{\chi_i} && (58)\\
\hat{\boldsymbol{\tau}} &= \hat{\boldsymbol{\tau}}(\Delta t, h_e, \hat{\boldsymbol{A}}_0, \hat{\boldsymbol{A}}_i, \hat{\boldsymbol{C}}, \dots) && (59)
\end{aligned}
$$

Δt is the time increment, and h_e is the e-th element mesh scale. For the discussion that follows, a precise definition of $\hat{\boldsymbol{\tau}}$ is not needed. Instead, its functional dependence on the parameters and various terms in the formulation is sufficient to fully understand the issues under investigation.

Remark 8 *The rest of the discussion will be focused on assessing whether or not the perturbation to the Bubnov-Galerkin test function,* $-(\hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}_h) \cdot \hat{\boldsymbol{\tau}}$*, is Galilean invariant.*

6.5.1 A multiscale view on Galilean invariance

The SUPG stabilization is obtained from a linearized multiscale decomposition of the Galerkin discretization, according to the following equations:

$$
\mathcal{B}(\hat{\boldsymbol{W}}^h, \boldsymbol{Y}^h) \quad + \quad \int_{\hat{Q}_n} \hat{\mathcal{L}}^*_{SH} \hat{\boldsymbol{W}}^h \cdot \boldsymbol{Y}'\, d\hat{Q} = 0 \qquad (60)
$$

$$
\boldsymbol{Y}' = \hat{\mathcal{L}}^{-1}_{SH}(-\hat{\boldsymbol{Res}}(\boldsymbol{Y}^h)) \quad = \quad -\int_{\hat{Q}_n} \hat{\boldsymbol{G}}'_{SH}\; \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h)\, d\hat{Q}, \quad \text{in } \mathcal{V}'(\hat{Q}_n) \qquad (61)
$$

where $\hat{\boldsymbol{G}}'_{SH}$ is the subgrid-scale element Green's function for the ALE equations of shock hydrodynamics, and $\hat{\mathcal{L}}^*_{SH}$ and $\hat{\mathcal{L}}_{SH}$ are obtained using the full Fréchet derivative of the Galerkin residual. Ideally, $\boldsymbol{Y}'$ is also an invariant of the Galilean transformation, since it is defined as the difference of $\boldsymbol{Y}$ and $\boldsymbol{Y}^h$ (see [16] for the trivial proof). However, due to the linearization, both (60)–(61) may yield second-order Galilean inconsistencies. Consequently, when the *exact* solution $\boldsymbol{Y}'$ to the linearized subgrid problem (61) is inserted into the mesh-scale equation (60), second-order Galilean inconsistencies are to be expected.

Furthermore, the approximation to the Green's function operator adopted in SUPG methods, namely

$$
\hat{\boldsymbol{G}}'_{SH} \approx \hat{\boldsymbol{\tau}}(\boldsymbol{\chi}; t)\delta(\tilde{\boldsymbol{\chi}} - \boldsymbol{\chi}; \tilde{t} - t) \qquad (62)
$$

may result too coarse in certain instances, preventing the expression for

$$
\boldsymbol{Y}' \approx -\hat{\boldsymbol{\tau}}(\boldsymbol{\chi}; t)\; \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h(\boldsymbol{\chi}; t)) \qquad (63)
$$

to retain first- or zeroth-order invariance properties. Since $\boldsymbol{Y}'$ is tied to the expression of the test function perturbation through $\hat{\boldsymbol{\tau}}$, stability can be at risk, as documented in [16]. The proposed extension to the ALE equations of the approach developed in [17, 16] is designed to remove Galilean inconsistencies from the SUPG operator, with specific emphasis on the construction of the Petrov-Galerkin test space.

Remark 9 *In addition, it should be understood that an invariant approximation of* $\boldsymbol{Y}'$ *may be advisable, at least from the theoretical point of view, as will be further discussed in later sections.*

7 Galilean invariance and the role of the subgrid-scale solution

Before undertaking an exhaustive discussion on the construction of the SUPG operator, it is important to understand how the numerical Galerkin residuals transform. It will be shown that for some choices of the solution variables, it is possible to maintain invariance properties if the residuals are in advective form, independently of the numerical quadrature adopted. This result *does not* hold for *any* solution vector, as will be clear in the case of conservation variables. Therefore, a key point to be made is the following: *Not all forms of the numerical, non-vanishing residuals transform correctly.*

Let us review how the Euler equations of gas dynamics transform. Namely,

$$0 = \left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{\partial \hat{J}\rho w_j}{\partial \chi_j} \tag{64}$$

$$0 = \left.\frac{\partial(\hat{J}\rho v_i)}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{\partial}{\partial \chi_j}(\hat{J}\rho v_i w_j - \hat{P}_{ij}) - \rho\hat{J}g_i \tag{65}$$

$$0 = \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{\partial}{\partial \chi_j}(\hat{J}\rho E w_j - v_i\hat{P}_{ij}) - \rho\hat{J}v_i g_i - \rho\hat{J}s \tag{66}$$

or, more simply,

$$0 = \hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \tag{67}$$

$$\begin{aligned} 0 &= \hat{Res}_i^{\rho\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \\ &= v_i\,\hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) + \hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \end{aligned} \tag{68}$$

$$\begin{aligned} 0 &= \hat{Res}^{E}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \\ &= \left(e + \frac{v_k v_k}{2}\right)\hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \\ &\quad + v_i\,\hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) + \hat{Res}^{e}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) \end{aligned} \tag{69}$$

where the mass conservation residual $\hat{Res}^{\rho}$, the momentum equation advective residual $\hat{Res}^{\boldsymbol{v}}$, and the internal energy equation residual $\hat{Res}^{e}$ are defined as

$$\hat{Res}^{\rho}(\rho;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) = \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial\rho}{\partial\chi_j} + \rho\,\mathrm{cof}\hat{F}_{ij}\frac{\partial v_i}{\partial\chi_j} \tag{70}$$

$$\hat{Res}_i^{\boldsymbol{v}}(\rho,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) = \hat{J}\rho\left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j\frac{\partial v_i}{\partial\chi_j} + \frac{\partial p}{\partial\chi_j}\,\mathrm{cof}\hat{F}_{ij} - \hat{J}\rho g_i \tag{71}$$

$$\hat{Res}^{e}(\rho,e,p;\boldsymbol{\chi},\boldsymbol{w},\boldsymbol{v},t) = \hat{J}\rho\left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j\frac{\partial e}{\partial\chi_j} + \frac{\partial v_i}{\partial\chi_j}p\,\mathrm{cof}\hat{F}_{ij} - \hat{J}\rho s \tag{72}$$

Here, the identity

$$\dot{\hat{J}} = \frac{\partial v_i}{\partial\chi_j}\mathrm{cof}\hat{F}_{ij} - \hat{J}\,\frac{\partial w_j}{\partial\chi_j} \tag{73}$$

has been used to rearrange the mass conservation equation (64) as follows:

$$
\begin{aligned}
0 &= \hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \hat{J}\rho\frac{\partial w_j}{\partial \chi_j} + \rho\left(\left.\frac{\partial \hat{J}}{\partial t}\right|_{\boldsymbol{\chi}} + w_j\frac{\partial \hat{J}}{\partial \chi_j}\right) \\
&= \hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \hat{J}\rho\frac{\partial w_j}{\partial \chi_j} + \rho\dot{\hat{J}} \qquad (74) \\
&= \hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \rho\frac{\partial v_i}{\partial \chi_j}\mathrm{cof}\hat{F}_{ij} \qquad (75)
\end{aligned}
$$

Recalling that $\boldsymbol{x}(\boldsymbol{X}, t=0) = \boldsymbol{\chi}(\boldsymbol{X}, t=0) = \boldsymbol{X}$, it easy to verify that a Galilean transformation applied to the referential coordinate system reads

$$
\begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{\chi}} \\ \tilde{\boldsymbol{w}} \\ \tilde{\boldsymbol{v}} \end{bmatrix} = \begin{bmatrix} t \\ \boldsymbol{\chi} \\ \boldsymbol{w} \\ \boldsymbol{v} - \boldsymbol{V}^G \end{bmatrix}, \quad \text{or,} \quad \begin{bmatrix} t \\ \boldsymbol{\chi} \\ \boldsymbol{w} \\ \boldsymbol{v} \end{bmatrix} = \begin{bmatrix} \tilde{t} \\ \tilde{\boldsymbol{\chi}} \\ \tilde{\boldsymbol{w}} \\ \tilde{\boldsymbol{v}} + \boldsymbol{V}^G \end{bmatrix} \qquad (76)
$$

Hence,

$$
\begin{aligned}
\hat{Res}^{\rho}(\rho; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) &\xrightarrow{G} \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (77) \\
\hat{Res}_i^{\boldsymbol{v}}(\rho, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) &\xrightarrow{G} \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (78) \\
\hat{Res}^{e}(\rho, e, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) &\xrightarrow{G} \hat{Res}^{e}(\rho, e, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (79)
\end{aligned}
$$

$$
\begin{aligned}
\hat{Res}_i^{\rho\boldsymbol{v}}(\rho, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \xrightarrow{G} \; & \tilde{v}_i \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + V_i^G \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
\hat{Res}^{E}(\rho, e, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \xrightarrow{G} \; & \left(e + \frac{\tilde{v}_k \tilde{v}_k}{2}\right) \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \tilde{v}_i \; \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + \hat{Res}^{e}(\rho, e, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& + V_i^G \Big(\tilde{v}_i \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
& \qquad + \hat{Res}_i^{\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t})\Big) \\
& + \frac{V_k^G V_k^G}{2} \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (81)
\end{aligned}
$$

The last two equations can be written more compactly as:

$$\begin{aligned}
\hat{Res}_i^{\rho\boldsymbol{v}}(\rho, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \quad &\xrightarrow{G} \quad \hat{Res}_i^{\rho\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
&\qquad + V_i^G \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (82) \\
\hat{Res}^{E}(\rho, e, p; \boldsymbol{\chi}, \boldsymbol{v}, \boldsymbol{w}, t) \quad &\xrightarrow{G} \quad \hat{Res}^{E}(\rho, e, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
&\qquad + V_i^G \hat{Res}_i^{\rho\tilde{\boldsymbol{v}}}(\rho, p; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \\
&\qquad + \frac{V_k^G V_k^G}{2} \hat{Res}^{\rho}(\rho; \tilde{\boldsymbol{\chi}}, \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{w}}, \tilde{t}) \qquad (83)
\end{aligned}$$

As expected, the equations transform appropriately, since the terms multiplied by the transformation velocity $\boldsymbol{V}^G$ annihilate exactly. In other words, if an exact multiscale decomposition of the solution is applied, the resulting equations would satisfy the invariance principle.

If, however, as already mentioned in [16], the subgrid-scale problem is solved only approximately, the situation is different and the numerical residuals are not necessarily invariant.

On the one hand, no matter the numerical quadrature used, the numerical approximations to the "advective" residuals $\hat{Res}^{h;\rho}$, $\hat{\boldsymbol{Res}}^{h;\boldsymbol{v}}$, and $\hat{Res}^{h;e}$ would transform correctly, if, for example, the set of solution variables is given by $[\rho,\ \boldsymbol{v}^T,\ p]^T$, $[\rho,\ \boldsymbol{v}^T,\ e]^T$, or $[e,\ \boldsymbol{v}^T,\ p]^T$. More generally, if $\boldsymbol{v}$ is a variable in the solution vector, and the remaining two entries are given by functions of the thermodynamic quantities, the resulting advective form of the residuals would transform correctly. It is important to notice that, in the advective form of the residual, the velocity $\boldsymbol{v}$ appears only in differentiated form.

Instead, $\hat{Res}^{h;\rho\boldsymbol{v}}$ and $\hat{Res}^{h;E}$ would not transform correctly, because $\boldsymbol{V}^G$ multiplies some of the non-vanishing residual terms. Hence, the approximation to the subgrid-scale solution $\boldsymbol{Y}' \approx -\hat{\boldsymbol{\tau}}\, \hat{\boldsymbol{Res}}(\boldsymbol{Y}^h)$ would not be invariant if these residuals were used in its construction. In addition, not all solution vectors yield invariant advective forms, as will become clear in the case of conservation variables.

It is also important to realize, however, that residuals are usually higher-order corrections: In the computations performed in [17, 16], virtually no difference in the results was observed between SUPG operators with and without invariant residuals in the approximation to $\boldsymbol{Y}'$. The fact that instabilities were experienced only for a non-invariant SUPG test function perturbation indicates that the latter is far more stringent then Galilean consistency of the residual terms. Nonetheless, it should be advisable to preserve invariance also for the approximations to $\boldsymbol{Y}'$.

8 Quasi-linear forms and invariance

Quasi-linear differential forms of the ALE equations have a central role in the design of SUPG operators, which make use of a fairly arbitrary combination of the Euler flux Jacobians to define a perturbation to the Bubnov-Galerkin test function space. Hence, a key requirement to be respected is that *every* Euler flux Jacobian must be invariant, or one cannot expect the perturbed test space to be independent of the observer. In what follows, two examples will be presented.

First, the case of density-pressure variables ($\boldsymbol{Y} = [\rho\ \boldsymbol{v}^T\ p]^T$) will be analyzed, and it will be shown how to successfully address the issue of lack of Galilean invariance affecting classical SUPG formulations. Specifically, invariance in the SUPG operator can be recovered by using the advective form of the residuals when constructing the Euler flux Jacobians. For the sake of completeness, the derivations in the case of the pressure primitive variables ($\boldsymbol{Y} = [e\ \boldsymbol{v}^T\ p]^T$), and density-internal energy variables ($\boldsymbol{Y} = [\rho\ \boldsymbol{v}^T\ e]^T$), are presented in appendix A.1. Pressure primitive variables are of greater interest in the aerospace community, since they allow to span the compressible and incompressible limit of the Euler equations. Instead, density-internal energy variables are traditionally used in the community developing hydrocodes.

Second, the discussion will proceed with the conservation variables ($\boldsymbol{U} = [\rho\ \rho\boldsymbol{v}^T\ \rho E]^T$). In this case, the advective form of the residuals does not lead to invariance properties for the Euler flux Jacobians. This fact does not imply that there is no hope to recover invariance when conservation variables are used, rather, that a successful approach is still to be developed.

8.1 Density-pressure variables

A quasi-linear form of the Euler equations using pressure variables ($\boldsymbol{Y} = [\rho\ \boldsymbol{v}^T\ p]^T$) will be derived using the traditional Fréchet differentiation approach and the new minimal approach of [16]. The structure and invariance properties of the resulting SUPG perturbations of the Galerkin test function will be analyzed.

For simplicity, heat fluxes are assumed absent. Applying the Piola identity

$$\frac{\partial(\mathrm{cof}\hat{F}_{ij})}{\partial \chi_j} \equiv \frac{\partial}{\partial \chi_j}\left(\hat{J}\hat{F}^{-1}_{ji}\right) = 0 \tag{84}$$

to the stress terms in (65)–(66) yields

$$\frac{\partial \hat{P}_{ij}}{\partial \chi_j} = \frac{\partial(\hat{J}\sigma_{ik}\hat{F}^{-1}_{jk})}{\partial \chi_j} = \frac{\partial \sigma_{ik}}{\partial \chi_j}\hat{J}\hat{F}^{-1}_{jk} = -\frac{\partial p}{\partial \chi_j}\hat{J}\hat{F}^{-1}_{ji} = -\frac{\partial p}{\partial \chi_j}\mathrm{cof}\hat{F}_{ij} \tag{85}$$

$$\frac{\partial v_i\hat{P}_{ij}}{\partial \chi_j} = \frac{\partial(v_i\hat{J}\sigma_{ik}\hat{F}^{-1}_{jk})}{\partial \chi_j} = \frac{\partial(v_i\sigma_{ik})}{\partial \chi_j}\hat{J}\hat{F}^{-1}_{jk} = -\frac{\partial (v_i p)}{\partial \chi_j}\mathrm{cof}\hat{F}_{ij} \tag{86}$$

Using (73),

$$0 = \hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial \rho}{\partial \chi_j} + \rho\ \mathrm{cof}\hat{F}_{ij}\frac{\partial v_i}{\partial \chi_j} \tag{87}$$

$$\begin{aligned} 0 = {} & v_i\left(\hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j\frac{\partial \rho}{\partial \chi_j} + \rho\ \mathrm{cof}\hat{F}_{kj}\frac{\partial v_k}{\partial \chi_j}\right) \\ & + \hat{J}\rho\left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j\frac{\partial v_i}{\partial \chi_j} + \frac{\partial p}{\partial \chi_j}\ \mathrm{cof}\hat{F}_{ij} - \hat{J}\rho g_i \end{aligned} \tag{88}$$

$$
\begin{aligned}
0 &= \left(e + \frac{v_k v_k}{2}\right) \left(\hat{J} \frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial \rho}{\partial \chi_j} + \rho \, \mathrm{cof} \hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j} \right) \\
& +v_i \left(\hat{J}\rho \, \frac{\partial v_i}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial v_i}{\partial \chi_j} + \frac{\partial p}{\partial \chi_j} \, \mathrm{cof} \hat{F}_{ij} - \hat{J}\rho g_i \right) \\
& +\hat{J}\rho \, \frac{\partial e}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial e}{\partial \chi_j} + \frac{\partial v_i}{\partial \chi_j} p \, \mathrm{cof} \hat{F}_{ij} - \hat{J}\rho s
\end{aligned} \tag{89}
$$

Remark 10 *The momentum and energy equations contain the mass conservation equation multiplied by the velocity component v_i, and internal energy e, respectively. In addition, the energy equation contains the kinetic-energy equation, the scalar product of the velocity times the momentum equation.*

In the present case, the following identity will become very useful:

$$
\begin{aligned}
\hat{J} \, \frac{\partial e}{\partial t}\bigg|_{\boldsymbol{\chi}} + \hat{J} w_j \frac{\partial e}{\partial \chi_j} &= \hat{J} \, \frac{\partial e}{\partial p}\bigg|_{\rho} \left(\frac{\partial p}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial p}{\partial \chi_j} \right) + \hat{J} \, \frac{\partial e}{\partial \rho}\bigg|_{p} \left(\frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial \rho}{\partial \chi_j} \right) \\
&= \hat{J} e_{,p} \left(\frac{\partial p}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial p}{\partial \chi_j} \right) + \hat{J} e_{,\rho} \left(\frac{\partial \rho}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial \rho}{\partial \chi_j} \right) \\
&= \hat{J} e_{,p} \left(\frac{\partial p}{\partial t}\bigg|_{\boldsymbol{\chi}} + w_j \frac{\partial p}{\partial \chi_j} \right) - e_{,\rho}\rho \, \mathrm{cof} \hat{F}_{ij} \frac{\partial v_i}{\partial \chi_j}
\end{aligned} \tag{90}
$$

where (64) has been used in the last step. Here, $e_{,p} = \frac{\partial e}{\partial p}\big|_{\rho}$, and $e_{,\rho} = \frac{\partial e}{\partial \rho}\big|_{p}$.

8.1.1 The "standard", non-invariant approach

The quasi-linear vector form reads

$$
\hat{\boldsymbol{A}}_0(\boldsymbol{Y}) \, \partial_t|_{\boldsymbol{\chi}} \boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y}) \, \partial_{\chi_i} \boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y}) \, \boldsymbol{Y} \;\; = \;\; \boldsymbol{0} \tag{91}
$$

with

$$
\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} \hat{J} & 0 & 0 & 0 & 0 \\ \hat{J}v_1 & \hat{J}\rho & 0 & 0 & 0 \\ \hat{J}v_2 & 0 & \hat{J}\rho & 0 & 0 \\ \hat{J}v_3 & 0 & 0 & \hat{J}\rho & 0 \\ \hat{J}E & \hat{J}\rho v_1 & \hat{J}\rho v_2 & \hat{J}\rho v_3 & \hat{J}\rho e_{,p} \end{bmatrix}, \tag{92}
$$

$$
\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\hat{J}g_1 & 0 & 0 & 0 & 0 \\ -\hat{J}g_2 & 0 & 0 & 0 & 0 \\ -\hat{J}g_3 & 0 & 0 & 0 & 0 \\ -\hat{J}s & -\hat{J}\rho g_1 & -\hat{J}\rho g_2 & -\hat{J}\rho g_3 & 0 \end{bmatrix}, \tag{93}
$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|c|c|c|c} \hat{J}w_i & \rho\ \mathrm{cof}\hat{F}_{1i} & \rho\ \mathrm{cof}\hat{F}_{2i} & \rho\ \mathrm{cof}\hat{F}_{3i} & 0 \\ \hline \hat{J}w_i v_1 & \begin{array}{l}\hat{J}\rho w_i + \\ \rho v_1 \mathrm{cof}\hat{F}_{1i}\end{array} & \rho v_1 \mathrm{cof}\hat{F}_{2i} & \rho v_1 \mathrm{cof}\hat{F}_{3i} & \mathrm{cof}\hat{F}_{1i} \\ \hline \hat{J}w_i v_2 & \rho v_2 \mathrm{cof}\hat{F}_{1i} & \begin{array}{l}\hat{J}\rho w_i + \\ \rho v_2 \mathrm{cof}\hat{F}_{2i}\end{array} & \rho v_2 \mathrm{cof}\hat{F}_{3i} & \mathrm{cof}\hat{F}_{2i} \\ \hline \hat{J}w_i v_3 & \rho v_3 \mathrm{cof}\hat{F}_{1i} & \rho v_3 \mathrm{cof}\hat{F}_{2i} & \begin{array}{l}\hat{J}\rho w_i + \\ \rho v_3 \mathrm{cof}\hat{F}_{3i}\end{array} & \mathrm{cof}\hat{F}_{3i} \\ \hline \hat{J}w_i E & \begin{array}{l}\hat{J}\rho w_i v_1 + \\ (\rho E + p \\ \ -\rho^2 e_{,\rho})\mathrm{cof}\hat{F}_{1i}\end{array} & \begin{array}{l}\hat{J}\rho w_i v_2 + \\ (\rho E + p \\ \ -\rho^2 e_{,\rho})\mathrm{cof}\hat{F}_{2i}\end{array} & \begin{array}{l}\hat{J}\rho w_i v_3 + \\ (\rho E + p \\ \ -\rho^2 e_{,\rho})\mathrm{cof}\hat{F}_{3i}\end{array} & \begin{array}{l}\hat{J}\rho w_i e_{,p} + \\ \mathrm{cof}\hat{F}_{ki} v_k\end{array} \end{array}\right] \tag{94}$$

Remark 11 *This choice leads to Jacobians of the Euler fluxes which are not invariant if considered separately. By inspection, it is easy to realize that there is a large number of terms which contain components of the velocity vector* $\boldsymbol{v}$*. Therefore, a single Euler flux Jacobian or an arbitrary combination of Euler flux Jacobians are not necessarily invariant. This is precisely the situation for the perturbation to the test function* $-(\hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}^h) \cdot \hat{\boldsymbol{\tau}} = (\hat{\boldsymbol{A}}_0^T\ \partial_t|_{\boldsymbol{\chi}} \hat{\boldsymbol{W}}^h + \hat{\boldsymbol{A}}_i^T \partial_{\chi_i} \hat{\boldsymbol{W}}^h)\hat{\boldsymbol{\tau}}$*, which lacks invariance properties, with potentially very negative consequences on the overall stability of the formulation.*

Remark 12 *Although in principle it is possible to develop a tensor* $\hat{\boldsymbol{\tau}}$ *which would make the perturbation to the test function Galilean invariant, it should be evident to the reader that, in practice, the current structure of the Jacobians makes this task extremely difficult.*

Remark 13 *Obviously, also the approximation to* $\boldsymbol{Y}'$ *is not invariant.*

8.1.2 A new Galilean invariant approach

The previous approach is not the only way to derive a quasi-linear form of the Euler equations. Starting from (87)–(89), additional algebraic manipulations can be performed. Let us therefore remove the mass conservation equation terms from the momentum and total energy equations, and the kinetic energy equation from the total energy equation. Hence, the following system of equations in *advective* form is obtained:

$$0 = \hat{J}\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}w_j \frac{\partial \rho}{\partial \chi_j} + \rho\ \mathrm{cof}\hat{F}_{ij}\frac{\partial v_i}{\partial \chi_j} \tag{95}$$

$$0 = \hat{J}\rho\left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial v_i}{\partial \chi_j} + \frac{\partial p}{\partial \chi_j}\ \mathrm{cof}\hat{F}_{ij} - \hat{J}\rho g_i \tag{96}$$

$$0 = \hat{J}\rho\left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho w_j \frac{\partial e}{\partial \chi_j} + \frac{\partial v_i}{\partial \chi_j} p\ \mathrm{cof}\hat{F}_{ij} - \hat{J}\rho s \tag{97}$$

Thus,

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} \hat{J} & 0 & 0 & 0 & 0 \\ 0 & \hat{J}\rho & 0 & 0 & 0 \\ 0 & 0 & \hat{J}\rho & 0 & 0 \\ 0 & 0 & 0 & \hat{J}\rho & 0 \\ 0 & 0 & 0 & 0 & \hat{J}\rho e_{,p} \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\hat{J}g_1 & 0 & 0 & 0 & 0 \\ -\hat{J}g_2 & 0 & 0 & 0 & 0 \\ -\hat{J}g_3 & 0 & 0 & 0 & 0 \\ -\hat{J}s & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{98}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \left[\begin{array}{c|c|c|c|c} \hat{J}w_i & \rho \operatorname{cof}\hat{F}_{1i} & \rho \operatorname{cof}\hat{F}_{2i} & \rho \operatorname{cof}\hat{F}_{3i} & 0 \\ \hline 0 & \hat{J}\rho w_i & 0 & 0 & \operatorname{cof}\hat{F}_{1i} \\ \hline 0 & 0 & \hat{J}\rho w_i & 0 & \operatorname{cof}\hat{F}_{2i} \\ \hline 0 & 0 & 0 & \hat{J}\rho w_i & \operatorname{cof}\hat{F}_{3i} \\ \hline 0 & (p - \rho^2 e_{,\rho}) \operatorname{cof}\hat{F}_{1i} & (p - \rho^2 e_{,\rho}) \operatorname{cof}\hat{F}_{2i} & (p - \rho^2 e_{,\rho}) \operatorname{cof}\hat{F}_{3i} & \hat{J}\rho w_i e_{,p} \end{array}\right] \tag{99}$$

Remark 14 *Each of the generalized advective matrices developed respects the principle of Galilean invariance, since they are function of $\hat{\boldsymbol{F}}$, $\hat{J}$, $\boldsymbol{w}$, p, ρ, $e_{,p}$, and $e_{,\rho}$, all invariant quantities.*

Remark 15 *One can think about the proposed approach as being "minimalist". In fact it produces the minimal number of entries in the Jacobians while still retaining the generalized advective structure of the quasi-linear form, now reduced to the mass conservation equation, the advective form of the momentum equation, and the advective form of the internal energy equation.*

Remark 16 *With respect to the standard Jacobians, the Galilean invariant Jacobians require 77 fewer terms to be computed. The compact sparsity pattern is clearly noticeable in (98)–(99). It is unusual and quite remarkable that a consistent approach leads to a significant reduction in the computational cost.*

Remark 17 *In appendix A, the previous discussion is extended to the case of pressure primitive variables, and density-internal energy variables. The conclusions that can be drawn in these two cases are virtually identical to the present case of density-pressure variables.*

8.1.3 The Lagrangian limit

The Lagrangian limit is very instructive in understanding the issues related to lack of Galilean invariance.

Standard, non-invariant approach

$$\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ Jv_1 & \rho_0 & 0 & 0 & 0 \\ Jv_2 & 0 & \rho_0 & 0 & 0 \\ Jv_3 & 0 & 0 & \rho_0 & 0 \\ JE & \rho_0 v_1 & \rho_0 v_2 & \rho_0 v_3 & \rho_0 e_{,p} \end{bmatrix}, \tag{100}$$

$$\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -Jg_1 & 0 & 0 & 0 & 0 \\ -Jg_2 & 0 & 0 & 0 & 0 \\ -Jg_3 & 0 & 0 & 0 & 0 \\ -Js & -\rho_0 g_1 & -\rho_0 g_2 & -\rho_0 g_3 & 0 \end{bmatrix}, \tag{101}$$

where $\rho_0 = \rho\ J$ is the initial density distribution, and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|c|c|c|c} 0 & \rho \operatorname{cof} F_{1i} & \rho \operatorname{cof} F_{2i} & \rho \operatorname{cof} F_{3i} & 0 \\ \hline 0 & \rho v_1 \operatorname{cof} F_{1i} & \rho v_1 \operatorname{cof} F_{2i} & \rho v_1 \operatorname{cof} F_{3i} & \operatorname{cof} F_{1i} \\ \hline 0 & \rho v_2 \operatorname{cof} F_{1i} & \rho v_2 \operatorname{cof} F_{2i} & \rho v_2 \operatorname{cof} F_{3i} & \operatorname{cof} F_{2i} \\ \hline 0 & \rho v_3 \operatorname{cof} F_{1i} & \rho v_3 \operatorname{cof} F_{2i} & \rho v_3 \operatorname{cof} F_{3i} & \operatorname{cof} F_{3i} \\ \hline 0 & \begin{array}{l}(\rho E + p \\ \quad -\rho^2 e_{,\rho}) \operatorname{cof} F_{1i}\end{array} & \begin{array}{l}(\rho E + p \\ \quad -\rho^2 e_{,\rho}) \operatorname{cof} F_{2i}\end{array} & \begin{array}{l}(\rho E + p \\ \quad -\rho^2 e_{,\rho}) \operatorname{cof} F_{3i}\end{array} & \operatorname{cof} \hat{F}_{ki} v_k \end{array}\right] \tag{102}$$

Remark 18 *A large number of terms are multiplied by the velocity components, with the potential for very dangerous consequences, since now a standard implementation of the SUPG operator would generate nodal forces depending on the observer. As documented in [16] and Figure 1, simulations of even mild shocks could not be successfully completed with this approach.*

Galilean invariant approach

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ 0 & \rho_0 & 0 & 0 & 0 \\ 0 & 0 & \rho_0 & 0 & 0 \\ 0 & 0 & 0 & \rho_0 & 0 \\ 0 & 0 & 0 & 0 & \rho_0 e_{,p} \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -Jg_1 & 0 & 0 & 0 & 0 \\ -Jg_2 & 0 & 0 & 0 & 0 \\ -Jg_3 & 0 & 0 & 0 & 0 \\ -Js & 0 & 0 & 0 & 0 \end{bmatrix} \tag{103}$$

where $\rho_0 = \rho\ J$, and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \begin{bmatrix} 0 & \rho \operatorname{cof} F_{1i} & \rho \operatorname{cof} F_{2i} & \rho \operatorname{cof} F_{3i} & 0 \\ 0 & 0 & 0 & 0 & \operatorname{cof} F_{1i} \\ 0 & 0 & 0 & 0 & \operatorname{cof} F_{2i} \\ 0 & 0 & 0 & 0 & \operatorname{cof} F_{3i} \\ 0 & (p-\rho^2 e_{,\rho}) \operatorname{cof} F_{1i} & (p-\rho^2 e_{,\rho}) \operatorname{cof} F_{2i} & (p-\rho^2 e_{,\rho}) \operatorname{cof} F_{3i} & 0 \end{bmatrix} \tag{104}$$

Remark 19 *None of the entries in (103)–(104) depends on $\boldsymbol{v}$. The Lagrangian formulation presented here is slightly different from the one in [17], in which the algebraic constraint $\rho_0 = \rho\ J$ is enforced strongly in the equations. This amounts to removing the first row and column from (103)–(104). Using a diagonal $\hat{\boldsymbol{\tau}}$ tensor, the simulations in [17] never suffered from instabilities, even in the most demanding implosion computations, with shock strengths exceeding Mach 10,000,000.*

8.1.4 A simple example: one-dimensional, Lagrangian gas dynamics

The effect of lack of Galilean invariance on the perturbation to the test function can be easily appreciated in a simple one-dimensional example. Let us consider the general definition of the $\hat{\boldsymbol{\tau}}$ tensor given in [6, 7]:

$$\hat{\boldsymbol{\tau}} = \hat{\boldsymbol{A}}_0^{-1} \left(\boldsymbol{C}^2 + \left(\frac{\partial \xi_0}{\partial t} \right)^2 \boldsymbol{I} + \frac{\partial \xi_i}{\partial X_j} \frac{\partial \xi_i}{\partial X_k} \boldsymbol{A}_j \boldsymbol{A}_k \right)^{-1/2} \tag{105}$$

where $\boldsymbol{A}_i = \hat{\boldsymbol{A}}_i \hat{\boldsymbol{A}}_0^{-1}$, $\boldsymbol{C} = \hat{\boldsymbol{C}} \hat{\boldsymbol{A}}_0^{-1}$ and ξ_i are the coordinates in the parent domain of each element, and ξ_0 refers to the time axis.

Remark 20 *It is not the intention of the author to critisize a particular reference in the available litarature. The large majority of stabilization operators adopts similar expressions, so it is fair to say that the following example is prototypical of current stabilization approaches to compressible flows.*

In one dimension,

$$\frac{\partial \xi_0}{\partial t} = \frac{2}{\Delta t} \tag{106}$$

$$\frac{\partial \xi_i}{\partial X_j}\frac{\partial \xi_i}{\partial X_k}\boldsymbol{A}_j\boldsymbol{A}_k = \left(\frac{2}{\Delta X}\right)^2 \boldsymbol{A}_1^2 \tag{107}$$

Substituting (100)–(102) into (105), and recalling that in the current space-time formulation $\partial_t|_{\boldsymbol{\chi}}\hat{\boldsymbol{W}} = \mathbf{0}$, it can be obtained:

$$\begin{aligned} -(\hat{\mathcal{L}}^*_{SH}\hat{\boldsymbol{W}}^h)\cdot\hat{\boldsymbol{\tau}} &= (\hat{\boldsymbol{A}}_0^T\,\partial_t|_{\boldsymbol{\chi}}\hat{\boldsymbol{W}}^h + \hat{\boldsymbol{A}}_i^T\partial_{\chi_i}\hat{\boldsymbol{W}}^h)\hat{\boldsymbol{\tau}} \\ &= \partial_X\hat{\boldsymbol{W}}^h\hat{\boldsymbol{A}}_1\hat{\boldsymbol{\tau}} \end{aligned} \tag{108}$$

where, for an ideal gas,

$$\hat{\boldsymbol{A}}_1\hat{\boldsymbol{\tau}} = \begin{pmatrix} -\frac{v}{J}\beta & \frac{\beta}{J} & 0 \\ \frac{v^2(-3+\gamma)\beta}{2J} & \frac{-v(-2+\gamma)\beta}{J} & \frac{(\gamma-1)\beta}{J} \\ \frac{v\left(-2p\gamma+v^2\left(2-3\gamma+\gamma^2\right)\rho\right)\beta}{2(\gamma-1)\rho J} & \frac{\left(2p\gamma+v^2\left(-3+5\gamma-2\gamma^2\right)\rho\right)\beta}{2(\gamma-1)\rho J} & \frac{v(\gamma-1)\beta}{J} \end{pmatrix} \tag{109}$$

$$\beta = \frac{\Delta t}{2\sqrt{1+\alpha^2}}, \quad \alpha = \frac{c_s\Delta t}{\Delta x}, \quad \Delta x = J\Delta X, \quad c_s = \sqrt{\gamma\frac{p}{\rho}} \tag{110}$$

Depending on the value of v, the perturbation to the test function can assume a wide range of values (v can also be negative, so that sign inversions can occur, particularly problematic to stability). It is clear that this approach leads to observer-dependent stabilization operators. Instead, in the case of the Galiean invariant approach,

$$\hat{\boldsymbol{A}}_1\hat{\boldsymbol{\tau}} = \begin{pmatrix} 0 & \frac{\beta}{J} & 0 \\ 0 & 0 & \frac{\gamma-1}{J}\beta \\ 0 & \frac{\beta c_s^2}{(\gamma-1)J} & 0 \end{pmatrix} \tag{111}$$

independent of the velocity v.

8.1.5 The Eulerian limit

The standard approach is widely documented in the literature for Eulerian meshes, and will be shown to be inconsistent with the Galilean principle.

Standard, non-invariant approach

$$
\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & \rho & 0 & 0 & 0 \\ v_2 & 0 & \rho & 0 & 0 \\ v_3 & 0 & 0 & \rho & 0 \\ E & \rho v_1 & \rho v_2 & \rho v_3 & \rho e_{,p} \end{bmatrix}, \tag{112}
$$

$$
\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix}, \tag{113}
$$

and, for $i = 1, 2, 3,$

$$
\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|c|c|c|c} v_i & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & 0 \\ \hline v_i v_1 & \rho(v_i + v_1\delta_{1i}) & \rho v_1 \delta_{2i} & \rho v_1 \delta_{3i} & \delta_{1i} \\ \hline v_i v_2 & \rho v_2 \delta_{1i} & \rho(v_i + v_2\delta_{2i}) & \rho v_2 \delta_{3i} & \delta_{2i} \\ \hline v_i v_3 & \rho v_3 \delta_{1i} & \rho v_3 \delta_{2i} & \rho(v_i + v_3\delta_{3i}) & \delta_{3i} \\ \hline v_i E & \begin{array}{l} \rho v_i v_1 + \\ (\rho E + p \\ -\rho^2 e_{,\rho})\delta_{1i} \end{array} & \begin{array}{l} \rho v_i v_2 + \\ (\rho E + p \\ -\rho^2 e_{,\rho})\delta_{2i} \end{array} & \begin{array}{l} \rho v_i v_3 + \\ (\rho E + p \\ -\rho^2 e_{,\rho})\delta_{3i} \end{array} & \begin{array}{l} \rho v_i e_{,p} \\ + v_i \\ \end{array} \end{array}\right] \tag{114}
$$

Remark 21 *Although non-invariant, the previous Jacobians and their variations with different sets of variables are currently used in the large majority of SUPG-stabilized finite element methods for compressible flow applications, with potentially very dangerous consequences on the reliability of the results.*

Remark 22 *As a justification for the inconsistencies found in the literature to date, it is virtually impossible to discern whether a velocity term transforms correctly, if only the Eulerian form of the equations is available, since $\boldsymbol{w} = \hat{\boldsymbol{F}}^{-1}(\boldsymbol{v} - \hat{\boldsymbol{v}}) = \boldsymbol{I}(\boldsymbol{v} - \boldsymbol{0}) = \boldsymbol{v}$. The reverse approach is needed, in which first a consistent ALE formulation is developed and then the Eulerian equations are derived as a limit.*

Galilean invariant approach

$$
\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & \rho e_{,p} \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 & 0 & 0 & 0 & 0 \\ -g_2 & 0 & 0 & 0 & 0 \\ -g_3 & 0 & 0 & 0 & 0 \\ -s & 0 & 0 & 0 & 0 \end{bmatrix} \tag{115}
$$

and, for $i = 1, 2, 3,$

$$
\hat{\boldsymbol{A}}_i^{(Gal)} = \begin{bmatrix} v_i & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & 0 \\ 0 & \rho v_i & 0 & 0 & \delta_{1i} \\ 0 & 0 & \rho v_i & 0 & \delta_{2i} \\ 0 & 0 & 0 & \rho v_i & \delta_{3i} \\ 0 & (p - \rho^2 e_{,\rho})\delta_{1i} & (p - \rho^2 e_{,\rho})\delta_{2i} & (p - \rho^2 e_{,\rho})\delta_{3i} & \rho v_i e_{,p} \end{bmatrix} \tag{116}
$$

8.2 Conservation variables

Conservation variables are widely used in SUPG methods for aircraft design applications. Therefore, they are a very good candidate for the invariance analysis. Unfortunately, casting the residuals in advective form is not a successful strategy in deriving invariant Euler flux Jacobians. The reason for this problem has to be traced to the particular structure of the conservation variables, which combine kinematic and thermodynamic quantities in their definition. As a starting point in the derivations, the following identity will become very useful:

$$
\begin{aligned}
\mathbf{0} &= \partial_t|_{\boldsymbol{\chi}}\hat{\boldsymbol{U}} + \partial_{\chi_i}\hat{\boldsymbol{G}}_i + \hat{\boldsymbol{Z}} \\
&= \partial_t|_{\boldsymbol{\chi}}(\hat{J}\boldsymbol{U}) + \partial_{\chi_i}(w_i\hat{J}\boldsymbol{U} + \hat{\boldsymbol{G}}_i^L) + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \partial_{\chi_i}(\hat{F}_{ij}^{-1}c_j\hat{J}\boldsymbol{U} + \hat{\boldsymbol{G}}_i^L) + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \partial_{\chi_i}(\hat{J}\hat{F}_{ji}^{-T}c_j\boldsymbol{U} + \hat{\boldsymbol{G}}_i^L) + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \partial_{\chi_i}(\mathrm{cof}\hat{F}_{ji}c_j\boldsymbol{U} + \hat{\boldsymbol{G}}_i^L) + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \boldsymbol{U}\mathrm{cof}\hat{F}_{ji}\partial_{\chi_i}c_j + \mathrm{cof}\hat{F}_{ji}c_j\partial_{\chi_i}\boldsymbol{U} + \partial_{\chi_i}\hat{\boldsymbol{G}}_i^L + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{c} + \hat{J}w_i\partial_{\chi_i}\boldsymbol{U} + \partial_{\chi_i}\hat{\boldsymbol{G}}_i^L + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} + \hat{J}w_i\partial_{\chi_i}\boldsymbol{U} + \partial_{\chi_i}\hat{\boldsymbol{G}}_i^L + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}w_i\partial_{\chi_i}\boldsymbol{U} + \boldsymbol{U}\mathrm{cof}\hat{F}_{ij}\partial_{\chi_j}v_i + \partial_{\chi_i}\hat{\boldsymbol{G}}_i^L + \hat{\boldsymbol{Z}} \\
&= \hat{J}\partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{w}\cdot\nabla_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{J}\boldsymbol{U}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} + \partial_{\chi_i}\hat{\boldsymbol{G}}_i^L + \hat{\boldsymbol{Z}}
\end{aligned}
\tag{117}
$$

where the term $\partial_{\chi_i}\hat{\boldsymbol{G}}_i^L$ can be readily computed, using again the Piola identity:

$$
\partial_{\chi_i}\hat{\boldsymbol{G}}_i^L = \partial_{\chi_i}\begin{bmatrix} 0 \\ p\ \mathrm{cof}\hat{F}_{1i} \\ p\ \mathrm{cof}\hat{F}_{2i} \\ p\ \mathrm{cof}\hat{F}_{3i} \\ v_j p\ \mathrm{cof}\hat{F}_{ji} \end{bmatrix} = \begin{bmatrix} 0 \\ \partial_{\chi_i}p\ \mathrm{cof}\hat{F}_{1i} \\ \partial_{\chi_i}p\ \mathrm{cof}\hat{F}_{2i} \\ \partial_{\chi_i}p\ \mathrm{cof}\hat{F}_{3i} \\ (p\ \partial_{\chi_i}v_j + v_j\partial_{\chi_i}p)\ \mathrm{cof}\hat{F}_{ji} \end{bmatrix}
\tag{118}
$$

The quasi-linear form of the system of equations for conservation variables can be completed by deriving expressions for $\partial_{\chi_j}v_i$, and $\partial_{\chi_j}p$. For the sake of simplicity, and without lack of generality on the conclusions, the case of an ideal gas will be considered, for which:

$$
\begin{aligned}
v_i &= \frac{U_{i+1}}{U_1} && (119)\\
\frac{\partial v_i}{\partial \chi_j} &= \frac{1}{U_1}\frac{\partial U_{i+1}}{\partial \chi_j} - \frac{U_{i+1}}{U_1^2}\frac{\partial U_1}{\partial \chi_j} && (120)\\
p &= (\gamma-1)\left(U_5 - \frac{1}{2U_1}\sum_{k=1} U_{k+1}^2\right) && (121)\\
\frac{\partial p}{\partial \chi_j} &= (\gamma-1)\left(\frac{\partial U_5}{\partial \chi_j} + \frac{1}{2}\sum_{k=1}^{3}\left(\frac{U_{k+1}}{U_1}\right)^2 \frac{\partial U_1}{\partial \chi_j} - \sum_{k=1}^{3}\frac{U_{k+1}}{U_1}\frac{\partial U_{k+1}}{\partial \chi_j}\right) && (122)\\
e &= \frac{U_5}{U_1} - \frac{1}{2}\sum_{k=1}\left(\frac{U_{k+1}}{U_1}\right)^2 && (123)\\
\frac{\partial e}{\partial \chi_j} &= \frac{1}{U_1}\frac{\partial U_5}{\partial \chi_j} - \frac{U_5}{U_1^2}\frac{\partial U_1}{\partial \chi_j} - \frac{U_{k+1}}{U_1^2}\frac{\partial U_{k+1}}{\partial \chi_j} + \sum_{k=1}^{3}\frac{U_{k+1}U_{k+1}}{U_1^3}\frac{\partial U_1}{\partial \chi_j} && (124)\\
\kappa &= \frac{1}{2}\sum_{k=1}\left(\frac{U_{k+1}}{U_1}\right)^2 && (125)
\end{aligned}
$$

where $\kappa = \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}$ is the kinetic energy per unit mass.

8.2.1 The "standard", non-invariant approach

The quasi-linear vector form reads

$$
\hat{\boldsymbol{A}}_0\ \partial_t|_{\boldsymbol{\chi}}\boldsymbol{U} + \hat{\boldsymbol{A}}_i(\boldsymbol{U})\ \partial_{\chi_i}\boldsymbol{U} + \hat{\boldsymbol{C}}(\boldsymbol{U})\ \boldsymbol{U} \quad = \quad \boldsymbol{0} \qquad (126)
$$

with

$$
\hat{\boldsymbol{A}}_0^{(NG)} = \hat{J}\boldsymbol{I}_{5\times 5}, \qquad \hat{\boldsymbol{C}}^{(NG)} = \hat{J}\begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ -g_1 & 0 & 0 & 0 & 0\\ -g_2 & 0 & 0 & 0 & 0\\ -g_3 & 0 & 0 & 0 & 0\\ -s & -g_1 & -g_2 & -g_3 & 0 \end{bmatrix}, \qquad (127)
$$

and, for $i = 1, 2, 3,$

$$
\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{l|l|l|l|l}
\hat{J}w_i - \frac{U_i^{\mathrm{cof}}}{U_1} & \mathrm{cof}\,\hat{F}_{1i} & \mathrm{cof}\,\hat{F}_{2i} & \mathrm{cof}\,\hat{F}_{3i} & 0 \\
\hline
-\frac{U_2 U_i^{\mathrm{cof}}}{U_1^2} & \hat{J}w_i + \frac{U_2}{U_1}\mathrm{cof}\,\hat{F}_{1i} & \frac{U_2}{U_1}\mathrm{cof}\,\hat{F}_{2i} & \frac{U_2}{U_1}\mathrm{cof}\,\hat{F}_{3i} & (\gamma-1)\mathrm{cof}\,\hat{F}_{1i} \\
+(\gamma-1)\kappa\,\mathrm{cof}\,\hat{F}_{1i} & -\frac{(\gamma-1)U_2}{U_1}\mathrm{cof}\,\hat{F}_{1i} & -\frac{(\gamma-1)U_3}{U_1}\mathrm{cof}\,\hat{F}_{1i} & -\frac{(\gamma-1)U_4}{U_1}\mathrm{cof}\,\hat{F}_{1i} & \\
\hline
-\frac{U_3 U_i^{\mathrm{cof}}}{U_1^2} & \frac{U_3}{U_1}\mathrm{cof}\,\hat{F}_{1i} & \hat{J}w_i + \frac{U_3}{U_1}\mathrm{cof}\,\hat{F}_{2i} & \frac{U_3}{U_1}\mathrm{cof}\,\hat{F}_{3i} & (\gamma-1)\mathrm{cof}\,\hat{F}_{2i} \\
+(\gamma-1)\kappa\,\mathrm{cof}\,\hat{F}_{2i} & -\frac{(\gamma-1)U_2}{U_1}\mathrm{cof}\,\hat{F}_{2i} & -\frac{(\gamma-1)U_3}{U_1}\mathrm{cof}\,\hat{F}_{2i} & -\frac{(\gamma-1)U_4}{U_1}\mathrm{cof}\,\hat{F}_{2i} & \\
\hline
-\frac{U_4 U_i^{\mathrm{cof}}}{U_1^2} & \frac{U_4}{U_1}\mathrm{cof}\,\hat{F}_{1i} & \frac{U_4}{U_1}\mathrm{cof}\,\hat{F}_{2i} & \hat{J}w_i + \frac{U_4}{U_1}\mathrm{cof}\,\hat{F}_{3i} & (\gamma-1)\mathrm{cof}\,\hat{F}_{3i} \\
+(\gamma-1)\kappa\,\mathrm{cof}\,\hat{F}_{3i} & -\frac{(\gamma-1)U_2}{U_1}\mathrm{cof}\,\hat{F}_{3i} & -\frac{(\gamma-1)U_3}{U_1}\mathrm{cof}\,\hat{F}_{3i} & -\frac{(\gamma-1)U_4}{U_1}\mathrm{cof}\,\hat{F}_{3i} & \\
\hline
-\frac{U_5 U_i^{\mathrm{cof}}}{U_1^2} & \frac{U_5}{U_1}\mathrm{cof}\,\hat{F}_{1i} & \frac{U_5}{U_1}\mathrm{cof}\,\hat{F}_{2i} & \frac{U_5}{U_1}\mathrm{cof}\,\hat{F}_{3i} & \hat{J}w_i \\
+\frac{(\gamma-1)\kappa U_i^{\mathrm{cof}}}{U_1} & -\frac{(\gamma-1)U_2 U_i^{\mathrm{cof}}}{U_1^2} & -\frac{(\gamma-1)U_3 U_i^{\mathrm{cof}}}{U_1^2} & -\frac{(\gamma-1)U_4 U_i^{\mathrm{cof}}}{U_1^2} & +\frac{(\gamma-1)U_i^{\mathrm{cof}}}{U_1} \\
-\frac{pU_i^{\mathrm{cof}}}{U_1^2} & +\frac{p}{U_1}\mathrm{cof}\,\hat{F}_{1i} & +\frac{p}{U_1}\mathrm{cof}\,\hat{F}_{2i} & +\frac{p}{U_1}\mathrm{cof}\,\hat{F}_{3i} &
\end{array}\right] \tag{128}
$$

where $U_i^{\mathrm{cof}} = \sum_{k=1}^{3} U_{k+1}\mathrm{cof}\hat{F}_{ki}$.

Remark 23 *It is easy to realize that the Euler flux Jacobians contain a large number of terms dependent on the velocity components $v_i = U_{i+1}/U_1$. Therefore, a single Euler flux Jacobian or an arbitrary combination of Euler flux Jacobians are not necessarily invariant.*

8.2.2 Non-invariant Jacobians for the advective form

Due to the structure of the conservation variable vector, casting the system of equations in the advective form (95)–(97) does not yield invariant Jacobians. In fact,

$$
\hat{\boldsymbol{A}}_0^{(Adv)} = \hat{J}\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
-\frac{U_2}{U_1} & 1 & 0 & 0 & 0 \\
-\frac{U_3}{U_1} & 0 & 1 & 0 & 0 \\
-\frac{U_4}{U_1} & 0 & 0 & 1 & 0 \\
-\frac{U_5}{U_1}+2\kappa & -\frac{U_2}{U_1} & -\frac{U_3}{U_1} & -\frac{U_4}{U_1} & 1
\end{bmatrix}, \quad
\hat{\boldsymbol{C}}^{(Adv)} = \hat{J}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
-g_1 & 0 & 0 & 0 & 0 \\
-g_2 & 0 & 0 & 0 & 0 \\
-g_3 & 0 & 0 & 0 & 0 \\
-s & 0 & 0 & 0 & 0
\end{bmatrix}, \tag{129}
$$

and, for $i = 1, 2, 3,$

$$
\hat{\boldsymbol{A}}_i^{(Adv)} = \left[\begin{array}{l|l|l|l|l}
\hat{J}w_i - \frac{U_i^{\mathrm{cof}}}{U_1} & \mathrm{cof}\,\hat{F}_{1i} & \mathrm{cof}\,\hat{F}_{2i} & \mathrm{cof}\,\hat{F}_{3i} & 0 \\
\hline
-\hat{J}w_i\frac{U_2}{U_1} & \hat{J}w_i & -\frac{(\gamma-1)U_3}{U_1}\mathrm{cof}\,\hat{F}_{1i} & -\frac{(\gamma-1)U_4}{U_1}\mathrm{cof}\,\hat{F}_{1i} & (\gamma-1)\mathrm{cof}\,\hat{F}_{1i} \\
+(\gamma-1)\kappa\,\mathrm{cof}\,\hat{F}_{1i} & -\frac{(\gamma-1)U_2}{U_1}\mathrm{cof}\,\hat{F}_{1i} & & & \\
\hline
-\hat{J}w_i\frac{U_3}{U_1} & -\frac{(\gamma-1)U_2}{U_1}\mathrm{cof}\,\hat{F}_{2i} & \hat{J}w_i & -\frac{(\gamma-1)U_4}{U_1}\mathrm{cof}\,\hat{F}_{2i} & (\gamma-1)\mathrm{cof}\,\hat{F}_{2i} \\
+(\gamma-1)\kappa\,\mathrm{cof}\,\hat{F}_{2i} & & -\frac{(\gamma-1)U_3}{U_1}\mathrm{cof}\,\hat{F}_{2i} & & \\
\hline
-\hat{J}w_i\frac{U_4}{U_1} & -\frac{(\gamma-1)U_2}{U_1}\mathrm{cof}\,\hat{F}_{3i} & -\frac{(\gamma-1)U_3}{U_1}\mathrm{cof}\,\hat{F}_{3i} & \hat{J}w_i & (\gamma-1)\mathrm{cof}\,\hat{F}_{3i} \\
+(\gamma-1)\kappa\,\mathrm{cof}\,\hat{F}_{3i} & & & -\frac{(\gamma-1)U_4}{U_1}\mathrm{cof}\,\hat{F}_{3i} & \\
\hline
\hat{J}w_i\left(-\frac{U_5}{U_1}+2\kappa\right) & -\hat{J}w_i\frac{U_2}{U_1} & -\hat{J}w_i\frac{U_3}{U_1} & -\hat{J}w_i\frac{U_4}{U_1} & \hat{J}w_i \\
-\frac{pU_i^{\mathrm{cof}}}{U_1^2} & +\frac{p}{U_1}\mathrm{cof}\,\hat{F}_{1i} & +\frac{p}{U_1}\mathrm{cof}\,\hat{F}_{2i} & +\frac{p}{U_1}\mathrm{cof}\,\hat{F}_{3i} &
\end{array}\right] \tag{130}
$$

Remark 24 *Even the advective form of the equations generates Jacobians which are not invariant, since they contain components of the material velocity vector* $\boldsymbol{v}$*. This, by all means, does not imply that it is not possible to cast the quasi-linear form of the ALE Euler equations in invariant form, but a more complicated structure might be required. For the time being, this form is still to be found.*

Remark 25 *Similar considerations may apply in the case of the entropy variables, which possess a structure very similar to the one of conservation variables.*

9 Summary

An extended invariance analysis of stabilized methods for compressible and incompressible flows has been presented in the general ALE context. It was shown that most of the stabilization operators designed to date for compressible flow applications on Eulerian meshes *do not* satisfy the principle of Galilean invariance. It has been argued that this is both a physical and numerical flaw, since a non-invariant Petrov-Galerkin test space can have direct consequences on the stability properties of SUPG methods.

Given the disastrous results documented in [16] for the Lagrangian limit, it is the opinion of the author that *at least* a "reasonable doubt" on the correctness of the stabilization techniques lacking invariance has to be raised. It was shown that, for the density-pressure, the pressure primitive, and the density-internal energy vairables, a simple manipulation of the quasi-linear form of the equations of motion leads to Galilean invariant formulations. This approach also has the advantage of a significant reduction in the computational cost of the stabilization operator. The reliability of the new approach under severe conditions was proven in [17], where diagonal $\boldsymbol{\tau}$ tensors were used to stabilize shocks of strength in excess of Mach 10,000,000.

More work is needed to develop Galilean invariant SUPG operators for the conservation and entropy variables. Also, additional examples, in which lack of Galilean invariance leads to catastrophic results, need to be found and investigated, to broaden the discussion and further confirm the importance of the issue.

As a final comment, when considering complex compressible flow applications, conformity with physics principles in the design of stabilization and subgrid-scale operators appears to be one of the *not so many* guidelines available to the scientist. In this context, the price to be paid by neglecting the *Galilean sanity check* may be much greater than expected. The effects of invariance inconsistencies are usually difficult to isolate and track *a posteriori*, in large-scale industrial implementations.

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Appendix

A Quasi-linear forms and invariance: additional sets of solution variables

A.1 Pressure primitive variables

A quasi-linear form of the Euler equations using pressure primitive variables ($\boldsymbol{Y} = [e \ \boldsymbol{v}^T \ p]^T$) is now derived. In the literature, the temperature T is tipically used in place of the internal energy e (see, in the case of ideal gases, Hauke and Hughes [6, 7], and Hauke [5]). However, the classical expressions for the Euler Jacobians can be recovered noticing that, for an ideal gas, $e = c_v T$, where c_v is the specific heat for an isocoric thermodynamic transformation. Also in the case of pressure primitive variables, the derivations in (117) become very useful, since the ALE geometric terms are not under the derivative symbol. The following manipulations will also be used:

$$\partial_\bullet \boldsymbol{U} = \partial_\bullet \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ \rho E \end{bmatrix} = \partial_\bullet \left(\rho \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ E \end{bmatrix} \right) = \rho \begin{bmatrix} 0 \\ \partial_\bullet v_1 \\ \partial_\bullet v_2 \\ \partial_\bullet v_3 \\ v_k \partial_\bullet v_k + \partial_\bullet e \end{bmatrix} + \partial_\bullet \rho \begin{bmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ E \end{bmatrix} \tag{131}$$

where either $\partial_\bullet = \partial_t|_{\boldsymbol{\chi}}$ or $\partial_\bullet = \partial_{\chi_j}$, and $\partial_\bullet \rho = \rho_{,e} \partial_\bullet e + \rho_{,p} \partial_\bullet p$, with $\rho_{,e} = \left.\frac{\partial \rho}{\partial e}\right|_p$, and $\rho_{,p} = \left.\frac{\partial \rho}{\partial p}\right|_e$.

A.1.1 The "standard", non-invariant approach

Defining $\tilde{f}_\rho = \rho(e,p)/e$, the quasi-linear vector form reads

$$\hat{\boldsymbol{A}}_0 \, \partial_t|_{\boldsymbol{\chi}} \boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y}) \, \partial_{\chi_i} \boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y}) \, \boldsymbol{Y} = \boldsymbol{0} \tag{132}$$

with

$$\hat{\boldsymbol{A}}_0^{(NG)} = \hat{J} \begin{bmatrix} \rho_{,e} & 0 & 0 & 0 & \rho_{,p} \\ v_1 \rho_{,e} & \rho & 0 & 0 & v_1 \rho_{,p} \\ v_2 \rho_{,e} & 0 & \rho & 0 & v_2 \rho_{,p} \\ v_3 \rho_{,e} & 0 & 0 & \rho & v_3 \rho_{,p} \\ E \rho_{,e} + \rho & \rho v_1 & \rho v_2 & \rho v_3 & E \rho_{,p} \end{bmatrix}, \tag{133}$$

$$\hat{\boldsymbol{C}}^{(NG)} = \hat{J} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_2 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_3 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -s \tilde{f}_\rho & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix}, \tag{134}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|c|c|c|c} \hat{J} w_i \rho_{,e} & \rho \operatorname{cof} \hat{F}_{1i} & \rho \operatorname{cof} \hat{F}_{2i} & \rho \operatorname{cof} \hat{F}_{3i} & \hat{J} w_i \rho_{,p} \\ \hline \hat{J} w_i v_1 \rho_{,e} & \hat{J} \rho w_i + \rho v_1 \operatorname{cof} \hat{F}_{1i} & \rho v_1 \operatorname{cof} \hat{F}_{2i} & \rho v_1 \operatorname{cof} \hat{F}_{3i} & \hat{J} w_i v_1 \rho_{,p} + \operatorname{cof} \hat{F}_{1i} \\ \hline \hat{J} w_i v_2 \rho_{,e} & \rho v_2 \operatorname{cof} \hat{F}_{1i} & \hat{J} \rho w_i + \rho v_2 \operatorname{cof} \hat{F}_{2i} & \rho v_2 \operatorname{cof} \hat{F}_{3i} & \hat{J} w_i v_2 \rho_{,p} + \operatorname{cof} \hat{F}_{2i} \\ \hline \hat{J} w_i v_3 \rho_{,e} & \rho v_3 \operatorname{cof} \hat{F}_{1i} & \rho v_3 \operatorname{cof} \hat{F}_{2i} & \hat{J} \rho w_i + \rho v_3 \operatorname{cof} \hat{F}_{3i} & \hat{J} w_i v_3 \rho_{,p} + \operatorname{cof} \hat{F}_{3i} \\ \hline \hat{J} w_i E \rho_{,e} + \hat{J} w_i \rho & \hat{J} \rho w_i v_1 + (\rho E + p) \operatorname{cof} \hat{F}_{1i} & \hat{J} \rho w_i v_2 + (\rho E + p) \operatorname{cof} \hat{F}_{2i} & \hat{J} \rho w_i v_3 + (\rho E + p) \operatorname{cof} \hat{F}_{3i} & \hat{J} E w_i \rho_{,p} + \operatorname{cof} \hat{F}_{ki} v_k \end{array}\right] \tag{135}$$

Remark 26 *These definitions of the Jacobians of the Euler fluxes are not invariant, since they contain components of the velocity vector* $\boldsymbol{v}$*. The conclusions to be drawn are therefore virtually identical to the previous case of density-pressure variables.*

A.1.2 Galilean invariant approach

Casting the Euler equations in advective form, the following set of invariant flux Jacobians is readily obtained:

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \hat{J} \begin{bmatrix} \rho_{,e} & 0 & 0 & 0 & \rho_{,p} \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ \rho & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \hat{J} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_2 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_3 \tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -s \tilde{f}_\rho & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{136}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \left[\begin{array}{c|c|c|c|c} \hat{J} w_i \rho_{,e} & \rho \operatorname{cof} \hat{F}_{1i} & \rho \operatorname{cof} \hat{F}_{2i} & \rho \operatorname{cof} \hat{F}_{3i} & \hat{J} w_i \rho_{,p} \\ \hline 0 & \hat{J} \rho w_i & 0 & 0 & \operatorname{cof} \hat{F}_{1i} \\ \hline 0 & 0 & \hat{J} \rho w_i & 0 & \operatorname{cof} \hat{F}_{2i} \\ \hline 0 & 0 & 0 & \hat{J} \rho w_i & \operatorname{cof} \hat{F}_{3i} \\ \hline \hat{J} \rho w_i & p \operatorname{cof} \hat{F}_{1i} & p \operatorname{cof} \hat{F}_{2i} & p \operatorname{cof} \hat{F}_{3i} & 0 \end{array}\right] \tag{137}$$

A.1.3 The Lagrangian limit

Standard, non-invariant approach

$$\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} J\rho_{,e} & 0 & 0 & 0 & J\rho_{,p} \\ Jv_1\rho_{,e} & \rho_0 & 0 & 0 & Jv_1\rho_{,p} \\ Jv_2\rho_{,e} & 0 & \rho_0 & 0 & Jv_2\rho_{,p} \\ Jv_3\rho_{,e} & 0 & 0 & \rho_0 & Jv_3\rho_{,p} \\ JE\rho_{,e}+\rho_0 & \rho_0 v_1 & \rho_0 v_2 & \rho_0 v_3 & JE\rho_{,p} \end{bmatrix}, \tag{138}$$

$$\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -Jg_1\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -Jg_2\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -Jg_3\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -Js\tilde{f}_\rho & -\rho_0 g_1 & -\rho_0 g_2 & -\rho_0 g_3 & 0 \end{bmatrix}, \tag{139}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|l|l|l|l} 0 & \rho\ \mathrm{cof}F_{1i} & \rho\ \mathrm{cof}F_{2i} & \rho\ \mathrm{cof}F_{3i} & 0 \\ \hline 0 & \rho v_1 \mathrm{cof}F_{1i} & \rho v_1 \mathrm{cof}F_{2i} & \rho v_1 \mathrm{cof}F_{3i} & \mathrm{cof}F_{1i} \\ \hline 0 & \rho v_2 \mathrm{cof}F_{1i} & \rho v_2 \mathrm{cof}F_{2i} & \rho v_2 \mathrm{cof}F_{3i} & \mathrm{cof}F_{2i} \\ \hline 0 & \rho v_3 \mathrm{cof}F_{1i} & \rho v_3 \mathrm{cof}F_{2i} & \rho v_3 \mathrm{cof}F_{3i} & \mathrm{cof}F_{3i} \\ \hline 0 & (\rho E+p)\mathrm{cof}F_{1i} & (\rho E+p)\mathrm{cof}F_{2i} & (\rho E+p)\mathrm{cof}F_{3i} & \mathrm{cof}\hat{F}_{ki} v_k \end{array}\right] \tag{140}$$

Galilean invariant approach

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} J & 0 & 0 & 0 & 0 \\ 0 & \rho_0 & 0 & 0 & 0 \\ 0 & 0 & \rho_0 & 0 & 0 \\ 0 & 0 & 0 & \rho_0 & 0 \\ \rho_0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -Jg_1\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -Jg_2\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -Jg_3\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -Js\tilde{f}_\rho & 0 & 0 & 0 & 0 \end{bmatrix} \tag{141}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \begin{bmatrix} 0 & \rho\ \mathrm{cof}F_{1i} & \rho\ \mathrm{cof}F_{2i} & \rho\ \mathrm{cof}F_{3i} & 0 \\ 0 & 0 & 0 & 0 & \mathrm{cof}F_{1i} \\ 0 & 0 & 0 & 0 & \mathrm{cof}F_{2i} \\ 0 & 0 & 0 & 0 & \mathrm{cof}F_{3i} \\ 0 & p\ \mathrm{cof}F_{1i} & p\ \mathrm{cof}F_{2i} & p\ \mathrm{cof}F_{3i} & 0 \end{bmatrix} \tag{142}$$

A.1.4 The Eulerian limit

Standard, non-invariant approach

$$\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} \rho_{,e} & 0 & 0 & 0 & \rho_{,p} \\ v_1\rho_{,e} & \rho & 0 & 0 & v_1\rho_{,p} \\ v_2\rho_{,e} & 0 & \rho & 0 & v_2\rho_{,p} \\ v_3\rho_{,e} & 0 & 0 & \rho & v_3\rho_{,p} \\ E\rho_{,e}+\rho & \rho v_1 & \rho v_2 & \rho v_3 & E\rho_{,p} \end{bmatrix}, \tag{143}$$

$$\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_2\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_3\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -s\tilde{f}_\rho & -\rho g_1 & -\rho g_2 & -\rho g_3 & 0 \end{bmatrix}, \tag{144}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{l|l|l|l|l} v_i\rho_{,e} & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & v_i\rho_{,p} \\ \hline v_iv_1\rho_{,e} & \rho(v_i+v_1\delta_{1i}) & \rho v_1\delta_{2i} & \rho v_1\delta_{3i} & \delta_{1i} + v_iv_1\rho_{,p} \\ \hline v_iv_2\rho_{,e} & \rho v_2\delta_{1i} & \rho(v_i+v_2\delta_{2i}) & \rho v_2\delta_{3i} & \delta_{2i} + v_iv_2\rho_{,p} \\ \hline v_iv_3\rho_{,e} & \rho v_2\delta_{1i} & \rho v_3\delta_{2i} & \rho(v_i+v_3\delta_{3i}) & \delta_{3i} + v_iv_3\rho_{,p} \\ \hline v_i(E\rho_{,e}+\rho) & \rho v_iv_1+ & \rho v_iv_2+ & \rho v_iv_3+ & v_iE\rho_{,p} \\ & (\rho E+p)\delta_{1i} & (\rho E+p)\delta_{2i} & (\rho E+p)\delta_{3i} & +v_i \end{array}\right] \tag{145}$$

Galilean invariant approach

$$\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} \rho_{,e} & 0 & 0 & 0 & \rho_{,p} \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ \rho & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -g_1\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_2\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -g_3\tilde{f}_\rho & 0 & 0 & 0 & 0 \\ -s\tilde{f}_\rho & 0 & 0 & 0 & 0 \end{bmatrix} \tag{146}$$

and, for $i = 1, 2, 3$,

$$\hat{\boldsymbol{A}}_i^{(Gal)} = \begin{bmatrix} v_i\rho_{,e} & \rho\delta_{1i} & \rho\delta_{2i} & \rho\delta_{3i} & v_i\rho_{,p} \\ 0 & \rho v_i & 0 & 0 & \delta_{1i} \\ 0 & 0 & \rho v_i & 0 & \delta_{2i} \\ 0 & 0 & 0 & \rho v_i & \delta_{3i} \\ \rho v_i & p\,\delta_{1i} & p\,\delta_{2i} & p\,\delta_{3i} & 0 \end{bmatrix} \tag{147}$$

A.2 Density-internal energy variables

It is of interest, especially for the community developing hydrocodes, the set of variables $\boldsymbol{Y} = [\rho\ \boldsymbol{v}^T\ e]^T$). The derivations of the non-invariant and invariant Jacobians for the quasi-linear vector form

$$\hat{\boldsymbol{A}}_0\ \partial_t|_{\boldsymbol{\chi}}\boldsymbol{Y} + \hat{\boldsymbol{A}}_i(\boldsymbol{Y})\ \partial_{\chi_i}\boldsymbol{Y} + \hat{\boldsymbol{C}}(\boldsymbol{Y})\ \boldsymbol{Y} \quad = \quad \boldsymbol{0} \tag{148}$$

are reported below. Also in this case, the invariant approach was obtained using the advective form of the Euler equations.

A.2.1 The "standard", non-invariant approach

$$\hat{\boldsymbol{A}}_0^{(NG)} = \begin{bmatrix} \hat{J} & 0 & 0 & 0 & 0 \\ \hat{J}v_1 & \hat{J}\rho & 0 & 0 & 0 \\ \hat{J}v_2 & 0 & \hat{J}\rho & 0 & 0 \\ \hat{J}v_3 & 0 & 0 & \hat{J}\rho & 0 \\ \hat{J}E & \hat{J}\rho v_1 & \hat{J}\rho v_2 & \hat{J}\rho v_3 & \hat{J}\rho \end{bmatrix}, \tag{149}$$

$$
\hat{\boldsymbol{C}}^{(NG)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\hat{J}g_1 & 0 & 0 & 0 & 0 \\ -\hat{J}g_2 & 0 & 0 & 0 & 0 \\ -\hat{J}g_3 & 0 & 0 & 0 & 0 \\ -\hat{J}s & -\hat{J}\rho g_1 & -\hat{J}\rho g_2 & -\hat{J}\rho g_3 & 0 \end{bmatrix}, \tag{150}
$$

and, for $i = 1, 2, 3$,

$$
\hat{\boldsymbol{A}}_i^{(NG)} = \left[\begin{array}{c|c|c|c|c}
\hat{J}w_i & \rho\ \mathrm{cof}\hat{F}_{1i} & \rho\ \mathrm{cof}\hat{F}_{2i} & \rho\ \mathrm{cof}\hat{F}_{3i} & 0 \\ \hline
\begin{array}{l} p_{,\rho}\,\mathrm{cof}\hat{F}_{1i} \\ +\hat{J}w_i v_1 \end{array} & \begin{array}{l} \rho v_1 \mathrm{cof}\hat{F}_{1i} \\ +\hat{J}\rho w_i \end{array} & \rho v_1 \mathrm{cof}\hat{F}_{2i} & \rho v_1 \mathrm{cof}\hat{F}_{3i} & p_{,e}\,\mathrm{cof}\hat{F}_{1i} \\ \hline
\begin{array}{l} p_{,\rho}\,\mathrm{cof}\hat{F}_{2i} \\ +\hat{J}w_i v_2 \end{array} & \rho v_2 \mathrm{cof}\hat{F}_{1i} & \begin{array}{l} \rho v_2 \mathrm{cof}\hat{F}_{2i} \\ +\hat{J}\rho w_i \end{array} & \rho v_2 \mathrm{cof}\hat{F}_{3i} & p_{,e}\,\mathrm{cof}\hat{F}_{2i} \\ \hline
\begin{array}{l} p_{,\rho}\,\mathrm{cof}\hat{F}_{3i} \\ +\hat{J}w_i v_3 \end{array} & \rho v_3 \mathrm{cof}\hat{F}_{1i} & \rho v_3 \mathrm{cof}\hat{F}_{2i} & \begin{array}{l} \rho v_3 \mathrm{cof}\hat{F}_{3i} \\ +\hat{J}\rho w_i \end{array} & p_{,e}\,\mathrm{cof}\hat{F}_{3i} \\ \hline
\begin{array}{l} \hat{J}w_i E+ \\ p_{,\rho}\mathrm{cof}\hat{F}_{ki} v_k \end{array} & \begin{array}{l} \hat{J}\rho w_i v_1 + \\ (\rho E + p)\mathrm{cof}\hat{F}_{1i} \end{array} & \begin{array}{l} \hat{J}\rho w_i v_2 + \\ (\rho E + p)\mathrm{cof}\hat{F}_{2i} \end{array} & \begin{array}{l} \hat{J}\rho w_i v_3 + \\ (\rho E + p)\mathrm{cof}\hat{F}_{3i} \end{array} & \begin{array}{l} \hat{J}\rho w_i + \\ p_{,e}\mathrm{cof}\hat{F}_{ki} v_k \end{array}
\end{array} \right] \tag{151}
$$

A.2.2 Galilean invariant approach

$$
\hat{\boldsymbol{A}}_0^{(Gal)} = \begin{bmatrix} \hat{J} & 0 & 0 & 0 & 0 \\ 0 & \hat{J}\rho & 0 & 0 & 0 \\ 0 & 0 & \hat{J}\rho & 0 & 0 \\ 0 & 0 & 0 & \hat{J}\rho & 0 \\ 0 & 0 & 0 & 0 & \hat{J}\rho \end{bmatrix}, \quad \hat{\boldsymbol{C}}^{(Gal)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\hat{J}g_1 & 0 & 0 & 0 & 0 \\ -\hat{J}g_2 & 0 & 0 & 0 & 0 \\ -\hat{J}g_3 & 0 & 0 & 0 & 0 \\ -\hat{J}s & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{152}
$$

and, for $i = 1, 2, 3$,

$$
\hat{\boldsymbol{A}}_i^{(Gal)} = \left[\begin{array}{c|c|c|c|c}
\hat{J}w_i & \rho\ \mathrm{cof}\hat{F}_{1i} & \rho\ \mathrm{cof}\hat{F}_{2i} & \rho\ \mathrm{cof}\hat{F}_{3i} & 0 \\ \hline
p_{,\rho}\ \mathrm{cof}\hat{F}_{1i} & \hat{J}\rho w_i & 0 & 0 & p_{,e}\ \mathrm{cof}\hat{F}_{1i} \\ \hline
p_{,\rho}\ \mathrm{cof}\hat{F}_{2i} & 0 & \hat{J}\rho w_i & 0 & p_{,e}\ \mathrm{cof}\hat{F}_{2i} \\ \hline
p_{,\rho}\ \mathrm{cof}\hat{F}_{3i} & 0 & 0 & \hat{J}\rho w_i & p_{,e}\ \mathrm{cof}\hat{F}_{3i} \\ \hline
0 & p\ \mathrm{cof}\hat{F}_{1i} & p\ \mathrm{cof}\hat{F}_{2i} & p\ \mathrm{cof}\hat{F}_{3i} & \hat{J}\rho w_i
\end{array} \right] \tag{153}
$$

B Proof of the Reynolds transport theorem

Theorem 1 *Let Ω be a material domain displacing with velocity $\boldsymbol{v}$. Let Γ be the boundary of Ω, and f a scalar. Let $\hat{\Omega}$ be the inverse image of Ω through the invertible mapping $\hat{\boldsymbol{\varphi}}$, that is, $\Omega = \hat{\boldsymbol{\varphi}}(\hat{\Omega})$. Then, the following formula holds:*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} f\ d\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\hat{\Omega}} f\hat{J}\ d\hat{\Omega} = \int_{\hat{\Omega}} \left.\frac{\partial(f\hat{J})}{\partial t}\right|_{\boldsymbol{\chi}} d\hat{\Omega} + \int_{\hat{\Gamma}=\partial\hat{\Omega}} f\boldsymbol{w}\cdot\hat{\boldsymbol{n}}\ \hat{J}\ d\hat{\Gamma} \tag{154}
$$

Proof:

Ω is a material domain, and deforms according to the material velocity $\boldsymbol{v}$. Espressing the integral over $\hat{\Omega}$ in the Lagrangian frame, with $\hat{\Omega} = \boldsymbol{\psi}(\Omega_0)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\hat{\Omega}} f \hat{J} \; d\hat{\Omega} \;\; = \;\; \int_{\Omega_0 = \boldsymbol{\psi}^{-1}(\hat{\Omega})} \left. \frac{\partial (\hat{f} \tilde{J})}{\partial t} \right|_{\boldsymbol{X}} d\Omega_0 = \int_{\Omega_0} \dot{\hat{f}} \tilde{J} + \hat{f} \dot{\tilde{J}} \; d\Omega_0 \tag{155}$$

where $\tilde{J} = \det(\partial_{\boldsymbol{X}} \boldsymbol{\psi}(\boldsymbol{X}, t))$, $\boldsymbol{\chi} = \boldsymbol{\psi}(\boldsymbol{X}, t)$, and $\hat{f} = \hat{J} f$. Let us focus on the term $\dot{\tilde{J}}$:

$$\begin{aligned}
\dot{\tilde{J}} \;\; &= \;\; \left. \frac{\partial}{\partial t} \right|_{\boldsymbol{X}} \det(\partial_{\boldsymbol{X}} \boldsymbol{\psi}(\boldsymbol{X}, t)) \\
&= \;\; \tilde{J} \; \mathrm{tr} \left(\left(\left. \frac{\partial}{\partial t} \right|_{\boldsymbol{X}} \partial_{\boldsymbol{X}} \boldsymbol{\psi}(\boldsymbol{X}, t) \right) (\partial_{\boldsymbol{X}} \boldsymbol{\psi}(\boldsymbol{X}, t))^{-1} \right) \\
&= \;\; \tilde{J} \; \mathrm{tr} \left(\left(\left. \frac{\partial}{\partial t} \right|_{\boldsymbol{X}} \frac{\partial \boldsymbol{\psi}}{\partial \boldsymbol{X}} \right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\chi}} \right) \\
&= \;\; \tilde{J} \; \mathrm{tr} \left(\left(\frac{\partial}{\partial \boldsymbol{X}} \left. \frac{\partial \boldsymbol{\psi}}{\partial t} \right|_{\boldsymbol{X}} \right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\chi}} \right) \\
&= \;\; \tilde{J} \; \mathrm{tr} \left(\frac{\partial \boldsymbol{w}}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\chi}} \right) \\
&= \;\; \tilde{J} \; \nabla_{\boldsymbol{\chi}} \cdot \boldsymbol{w}
\end{aligned} \tag{156}$$

Now,

$$\begin{aligned}
\dot{\hat{f}} \tilde{J} + \hat{f} \dot{\tilde{J}} \;\; &= \;\; \dot{\hat{f}} \tilde{J} + \tilde{J} \; \nabla_{\boldsymbol{\chi}} \cdot \boldsymbol{w} \\
&= \;\; \tilde{J} \left(\left. \frac{\partial \hat{f}}{\partial t} \right|_{\boldsymbol{\chi}} + \boldsymbol{w} \cdot \nabla_{\boldsymbol{\chi}} \hat{f} + \nabla_{\boldsymbol{\chi}} \cdot \boldsymbol{w} \right) \\
&= \;\; \tilde{J} \left(\left. \frac{\partial \hat{f}}{\partial t} \right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot (\boldsymbol{w} \hat{f}) \right)
\end{aligned} \tag{157}$$

Mapping back to the domain $\hat{\Omega}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\hat{\Omega}} f \hat{J} \; d\hat{\Omega} \;\; = \;\; \int_{\hat{\Omega}} \left(\left. \frac{\partial (f \hat{J})}{\partial t} \right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot (\boldsymbol{w} \hat{f}) \right) \; d\hat{\Omega} \tag{158}$$

A straighforward application of the Gauss divergence theorem yields (154). $\square$