AN IMPROVED CONVERGENCE BOUND FOR AGGREGATION-BASED DOMAIN DECOMPOSITION PRECONDITIONERS

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Abstract. In this paper we present a two-level overlapping domain decomposition preconditioner for the finite element discretization of elliptic problems in two and three dimensions. The computational domain is partitioned into overlapping subdomains, and a coarse space correction, based on aggregation techniques, is added. Our definition of the coarse space does not require the introduction of a coarse grid. We consider a set of assumptions on the coarse basis functions to bound the condition number of the resulting preconditioned system. These assumptions only involve geometrical quantities associated with the aggregates and the subdomains. We prove that the condition number using the two-level additive Schwarz preconditioner is $\mathcal{O}(H/\delta + H_0/\delta)$, where H and H_0 are the diameter of the subdomains and the aggregates, respectively, and δ is the overlap among the subdomains and the aggregates. This extends the bounds presented in [10, 16, 17]. Numerical experiments on a model problem are reported to illustrate the performance of the proposed preconditioner.

1. Introduction. In this paper we consider the iterative solution of the linear system arising from the discretization of the scalar Poisson problem

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^d, d \in \{2,3\}$. The finite element discretization leads to a large sparse linear system, which is typically solved using the conjugate gradient method [7]. As the condition number of the associated matrix is $\mathcal{O}(h^{-2})$, where h is the mesh size of the triangulation, a preconditioner needs to be employed [15]. In this paper we explore preconditioners based on domain decomposition techniques. These methods share many features and are often competitors with multigrid methods [2, 20]. A domain decomposition preconditioner is naturally suited for parallel computations as the general principle is to divide the problem into subproblems or subdomains and assign each subdomain to a different processor. The key idea associated with obtaining scalable convergence is the projection of the operator onto an additional coarse space.

Here, we consider Schwarz procedures. In its simplest form, the Schwarz preconditioner is defined by subdividing the computational domain Ω into M overlapping subdomains Ω_i, then solving local Dirichlet-type problems on each Ω_i. It is well-known that this procedure, referred to as the one-level Schwarz method, is non-scalable. In fact, the information exchange among the subdomains is only through the overlapping regions, while for elliptic problems the domain of dependence is global. A scalable

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[4]This work was partially supported by the DOE NNSA ASC program and the DOE Office of Science MICS program at Sandia National Laboratory. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

method may be recovered by the addition of a suitable coarse operator, i.e. a projection of the original problem is solved on an additional coarse space. This auxiliary problem furnishes a way to efficiently transfer information to all subdomains in each step. As in a multigrid setting, the coarse problem is used to correct the "smooth" (low frequency) part of the error, whereas the local preconditioner is used to relax the "high-frequency" ones.

A typical approach to define the coarse operator is to resort to finite-element approximation on a coarse triangulation. For structured meshes of complex geometric domain, it is a relatively easy task, while for unstructured meshes this may be problematic or computationally expensive, especially for 3D computations. A possible way to overcome the difficulties induced by the definition of a coarse triangulation, without losing the power of two-level methods, is to resort to aggregation procedures. This is done as follows. First, the PDE problem is discretized on the fine grid. Then, the fine grid nodes are grouped in contiguous sets, called *aggregates*. The basis functions of the coarse space are formed by summing up the finite element basis functions of the nodes in each aggregate.

Aggregation procedures have been presented in literature by various authors. An aggregation technique was first introduced in 1951 by Leontief [12, Chapter 9] and then extensively used in Economics; see [13] and the references therein. For multigrid applications, smoothed aggregation techniques are used, for example, in [1, 22], where investigations of the smoothed aggregation properties have been reported. In the framework of domain decomposition methods, the focus is mainly on two-level methods. Results are presented in [14] for the shallow water equations and 2D potential flows, in [5] for 3D potential flow computations, in [8] for groundwater flows, in [9] for multiphase flows, in [11] for discontinuous Galerkin approximation of advection-diffusion problems, and in [18, 19] for the 3D compressible Euler Equations on unstructured tetrahedral grids.

For symmetric coercive problems, it is possible to estimate the condition number of the preconditioned operator using the abstract Schwarz theory of Dryja and Widlund [4]. The contribution of this paper is to present a convergence bound that depends linearly on $(H + H_0)/\delta$, where H is the size of the subdomains, H_0 is the size of the aggregates, and δ is the amount of overlap among the subdomains and the aggregates. This extends the bound presented in [10], where the authors assume $H = H_0$ (or more precisely, that aggregates and subdomains coincide).

This paper is organized as follows. In Section 2 we outline the model problem and its discretization. Section 3 presents the one level and two level Schwarz preconditioners, and introduces the aggregation technique. Section 4 presents the improved convergence bound for aggregation. Section 5 reports numerical results, for 2D and 3D domains. In Section 6 conclusions are presented.

2. Model Problem. Let us consider the weak form of the Poisson problem (1.1), whose finite-element approximation reads

$$\begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h, \end{array} \tag{2.1}$$

with $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega$, $f(v_h) = \int_\Omega f v_h \, d\Omega$. By assumption, $f \in L_2(\Omega)$, and $V_h \subset H_0^1(\Omega)$ is the space of linear finite element functions defined on a conforming,

quasi-uniform triangulation $\mathcal{T}_h$ of Ω. Existence of a unique solution of (2.1) is shown, for instance, in [15].

The algebraic formulation of problem (2.1) reads

$$A\mathbf{u} = \mathbf{f}, \tag{2.2}$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $\mathbf{u} \in \mathbb{R}^n$ the solution vector, and $\mathbf{f} \in \mathbb{R}^n$ the discretization of the force term f.

3. Schwarz Preconditioners. The one level Schwarz preconditioner for the iterative solution of (2.2) can be defined as follows. First, the computational domain Ω is partitioned into M overlapping subdomains Ω_i. On each Ω_i we have, by construction, $\mathrm{diam}(\Omega_i) = \mathcal{O}(H)$. Then, the preconditioner reads

$$B' = \sum_{i=1}^{M} B_i, \tag{3.1}$$

where $B_i = R_i^T A_i^{-1} R_i$, $A_i = R_i A R_i^T$, and R_i^T is the discrete representation of the interpolation from the subspace V_i (of functions of V_h whose support is contained in Ω_i) to the space V_h.

Preconditioner (3.1) is not scalable and the resulting condition number can be bounded by

$$\kappa(B'A) \le C\frac{1}{H\delta} \tag{3.2}$$

(see for instance [3]). Here and in the following, C (with or without subscripts) denotes a constant independent of H, h, the overlap among the subdomains δ, and H_0.

To avoid the term $1/H$ in (3.2), a further correction term B_0 can be added. The resulting preconditioner reads

$$B'' = B_0 + B' = \sum_{i=0}^{M} B_i, \tag{3.3}$$

where $B_0 = R_0^T A_0^{-1} R_0$ is a projection on a space V_0, which is "coarse" in the sense that it contains a limited number of degrees of freedom, to make the "exact" inversion of A_0 computationally acceptable—that is, $n_0 = \dim(V_0) \ll n$. Preconditioner (3.3) is usually referred to as two-level additive Schwarz preconditioner.

The properties (and the parallel performance) of two-level Schwarz preconditioners will strongly depend on the definition of the coarse space V_0. When possible, the coarse space V_0 may be itself embedded into V_h. Often, the coarse operator represents the discretization of the continuous problem on a very coarse mesh. In this case, the following result holds.

LEMMA 3.1. *Consider the additive two-level overlapping Schwarz method, where the overlap is uniform of width $\mathcal{O}(\delta)$, the coarse grid space V_0 corresponds to the finite-element functions on elements of $\mathcal{O}(H)$, and $V_0 \subset V_h$. Then*

$$\kappa(B''A) \le C\left(1 + \frac{H}{\delta}\right), \tag{3.4}$$

where C is a constant independent of h, H and δ.

Proof. See [20]. □

Alternatively, V_0 can be defined be resorting to aggregation procedures. This is done as follows. First, the differential problem is discretized on the fine grid, to form the matrix A. Then, the fine grid nodes are grouped into non-overlapping sets of contiguous nodes, called *aggregates*. This decomposition is usually obtained using a graph partitioning algorithm. The number of aggregates n_0 represents the dimension of the coarse space, since each aggregate will be given a unique coarse grid function. Using these aggregates, a simple procedure can be followed to define a restriction operator $R_0^\star$ (we will report one possible strategy to define $R_0^\star$ in Section 5). This operator is usually referred to as tentative restriction. Then, a smoother is applied to $R_0^\star$, in order to reduce the energy of the basis functions and increase their overlap. This defines the final restriction operator R_0. This second step consists of applying a smoother S to produce the final operator R_0 via $R_0 = SR_0^\star$.

Possible choices for S are, for example, a simple Richardson smoother [11, 10], or a recursive Richardson smoother [1, 23]. This construction of R_0 is sometimes referred to as *smoothed aggregation* (SA). If $S = I$ (i.e. $R_0 = R_0^\star$), the procedure is usually referred to as *non-smoothed aggregation*. Both $R_0^\star$ and S are selected so that $R_0 \in \mathbb{R}^{n_0 \times n}$.

The linear operator R_0 defines the basis function of the coarse space as follows. Let φ_j be a generic basis function of V_h; then, a generic coarse space basis function Φ_i can be written as

$$\Phi_i(x) = \sum_j R_0(i,j)\varphi_j(x),$$

and the final aggregates are defined as $\Theta_i = \text{supp}(\Phi_i)$ (where $\text{supp}(\Phi_i)$ refers to the support of Φ_i). We define, $H_0 = \max_{j=1,\ldots,n_0} \text{diam}(\Theta_j)$, and δ_0 is the amount of overlap among the aggregates.

REMARK 1. *$V_0 \subset V_h$, since V_0 is defined such that each element of V_0 is actually composed by a linear combination of elements of V_h.*

Once R_0 is computed, the coarse matrix is defined as $A_0 = R_0 A R_0^T$. More details on two-level methods based on aggregation can be found in [16, 17] and the references therein. We note that, although this paper focuses on two-level preconditioners, similar ideas can be used to define multilevel preconditioners, see instance [22, 21] and the references therein.

4. Convergence Estimate for Aggregation. Let $\{\Theta_i, i = 1, \ldots, n_0\}$ be a set of aggregates, defined as outlined in Section 3, and $\{\Phi_i, i = 1, \ldots, n_0\}$ a set of basis functions that satisfy $\text{supp}(\Phi_i) = \Theta_i$.

In order to prove our theoretical estimates, we need to introduce aggregates corresponding to the boundary nodes. Those aggregates and their corresponding basis functions are needed in the proof to define a partition of the unity over the entire computation domain. These additional aggregates do not need to be implemented in practice, since Dirichlet boundary nodes can be eliminated from linear system (2.2) before its solution.

Let ω be the set of aggregates corresponding to internal nodes, and γ be the set of aggregates constructed by aggregating boundary nodes. For each $i \in \gamma$, we introduce an additional function $\Phi_i \in V_h$, so that $\sum_{i \in \omega \cup \gamma} \Phi_i(x) = 1,\ \forall x \in \overline{\Omega}$.

The following properties will be used to characterize the aggregates and the basis functions of the coarse space.

PROPERTY 1 (partition). *There are three constants C_1, C_2 and C_3 so that, for each aggregate $\Theta_i, i \in \omega \cup \gamma$, we have:*
a. $\mathrm{diam}(\Theta_i) \leq C_1 H_0$*;*
b. the Lebesgue measure $|\Theta_i|$ of Θ_i satisfies $|\Theta_i| \leq C_2 H_0^d$ if $i \in \omega$, and $|\Theta_i| \leq C_3 H_0^{d-1}\delta_0$ if $i \in \gamma$;
c. the overlap among the aggregates is of order δ_0.

Property 1 states that the aggregates have diameter of comparable size H_0 and are shape-regular. The following Property 2 requires a certain regularity on the coarse space basis function Φ_i.

PROPERTY 2 (coarse space). *There exists a constant $C > 0$ such that the basis functions $\{\Phi_i\}, i \in \omega \cup \gamma$ of the coarse space satisfy*
a. $\|\Phi_i\|^2_{L^\infty(\Omega)} \leq C$*;*
b. $\|\nabla \Phi_i\|^2_{L^\infty(\Omega)} \leq C/\delta_0^2$*;*
c. $\sum_{i \in \omega \cup \gamma} \Phi_i(x) = 1, \quad \forall x \in \bar{\Omega}$*;*

Properties 2.a and 2.b require a certain point-wise regularity, while property 2.c states that the $\{\Phi_i\}$ forms a partition of the unity on the entire domain (verified by construction).

From property 2 it easily follows that

$$|\Phi_i|^2_{H^1(\Omega)} \leq C \frac{H_0^{d-1}}{\delta_0} \tag{4.1}$$

$$||\Phi_i||^2_{L^2(\Omega)} \leq C H_0^d. \tag{4.2}$$

REMARK 2. *A non-negative function Φ_i, which is constant in the interior of Θ_i and decreases to zero with $\|\nabla \Phi_i\|_{L^\infty(\Omega)} \leq 1/\delta_0$ in a layer of width δ_0, satisfies property 2. In particular, the non-smoothed case defined by equation (5.1), defines a set of functions $\{\Phi_i\}$ which satisfies property 2 with $\delta_0 = h$.*

Also, let us define the operator $Q_0 : V_h \to V_0$ by

$$\begin{aligned} Q_0 u &= \textstyle\sum_{i \in \omega \cup \gamma} \alpha_i \Phi_i, \\ \alpha_i &= \begin{cases} \frac{1}{|\Theta_i|} \int_{\Theta_i} u(\mathbf{x}) d\Omega & i \in \omega \\ 0 & i \in \gamma \end{cases} \end{aligned} \tag{4.3}$$

LEMMA 4.1. *If properties 1 and 2 are satisfied, then there exists a constant $C > 0$ independent of H_0, δ_0 and h such that $\forall u_h \in V_h$*
a. $||u - Q_0 u||^2_{L^2(\Omega)} \leq C H_0^2 |u|^2_{H^1(\Omega)}$*,*
b. $|Q_0 u|^2_{H^1(\Omega)} \leq C \frac{H_0}{\delta_0} |u|^2_{H^1(\Omega)}$*.*

Proof. See [16, 17]. □

In the following we will use the following Lemma.

LEMMA 4.2. *Let $\Omega_i \subset \mathbb{R}^d, d = 2, 3$, be a rectangle of diameter H, and let Γ_{δ_i} be a strip along its boundary of width $\delta > 0$. Then, for any function $u \in H^1(\Omega_i)$,*

$$\|u\|^2_{L^2(\Gamma_{\delta_i})} \leq C\delta^2 \left[\left(1 + \frac{H}{\delta}\right) |u|^2_{H^1(\Omega_i)} + \frac{1}{H\delta} \|u\|^2_{L^2(\Omega_i)} \right]. \tag{4.4}$$

Proof. See, for instance, [4]. □

For completeness, we also report the convergence bounds presented in [10] and in [17].

THEOREM 4.3. *Let Properties 1 (with $H_0 = H$ and $\delta_0 = \delta$) and 2 hold. Then, there exists a constant $C > 0$ such that*

$$\kappa(B''A) \leq C\left(1 + \frac{H}{\delta}\right).$$

Proof. See [10, Lemma 7]. □

THEOREM 4.4. *Let the properties 1 and 2 hold. Then there exists $C > 0$ such that*

$$\kappa(B''A) \leq C\left(1 + \frac{H_0}{\delta_0}\right)\left(1 + \frac{H}{\delta}\right). \tag{4.5}$$

Proof. See [17]. □

Theorem 4.4 states that, for a given decomposition into subdomains, the condition number can be reduced by increasing the number of the aggregates. However, Theorem 4.4 is not-optimal, as in the case $H = H_0, \delta = \delta_0$ the bound in the condition number reads

$$\kappa(B''A) \leq C\left(1 + \frac{H}{\delta}\right)^n, \tag{4.6}$$

with $n = 2$, while Theorem 4.3 states that (4.6) holds with $n = 1$. The following Theorem, which is the main result of this paper, addresses this problem.

THEOREM 4.5 (aggregation coarse space). *Let properties 1 and 2 hold. Then, for the additive two-level overlapping Schwarz method, when the overlap is uniform of width $\mathcal{O}(\delta)$, $\delta_0 = \delta$, each aggregate is completely contained in only one subdomain, $H \geq H_0$, and $V_0 = \mathrm{span}\{\Phi_i\}$, there exist $C > 0, C_0 > 0$ independent of h, δ, H, H_0, such that*

$$\kappa(B''A) \leq C\frac{H}{\delta}\left(1 + C_0\frac{H_0}{H}\right). \tag{4.7}$$

Proof. Define $\mathcal{I}_h : V \to V_h$ such that

$$\mathcal{I}_h(u) = \sum_{i=1}^{n} u(x_i)\varphi_i,$$

where φ_i is a generic basis function of the mesh, and the x_i's are the mesh nodal points.

Let $\omega_i = \{j \in \omega \cup \gamma \mid \mathrm{supp}(\Phi_j) \subseteq \Omega_i\}$. Let $V_i = H_0^1(\Omega_i) \cap V_h,\ i = 1, \ldots, M$. A function $u \in V_h$ can be decomposed as $u = u_0 + \sum_{i=1}^{M} u_i$, with

$$u_0 = Q_0 u, \qquad u_i = \sum_{j \in \omega_i} \mathcal{I}_h\left(\Phi_j\left(u - \alpha_j\right)\right), \quad i = 1, \ldots, M,$$

with Q_0 defined as in (4.3), where

$$u_0 = \sum_{i \in \omega \cup \gamma} \Phi_i \alpha_i = \sum_{i=1}^{M} \sum_{j \in \omega_i} \Phi_j \alpha_j + \sum_{i \in \gamma} \Phi_i \alpha_i,$$

corresponding to the n_0 aggregates plus the additional n_γ functions on the boundary. This is a proper decomposition, since

$$\begin{aligned}\sum_{i=0}^{M} u_i &= \sum_{j\in\omega\cup\gamma} \Phi_j\alpha_j + \sum_{i=1}^{M}\sum_{j\in\omega_i} \mathcal{I}_h\left(\Phi_j u - \Phi_j\alpha_j\right)\\ &= \sum_{i=1}^{M}\sum_{j\in\omega_i} \left\{\Phi_j\alpha_j + \Phi_j u - \Phi_j\alpha_j\right\}\\ &= \sum_{j} \Phi_j u = u,\end{aligned}$$

since $\sum_i \Phi_i(x) = 1$, $\forall x \in \bar{\Omega}$, owing to property 2 and $u_i \in V_i$. Define $\Psi_i = \sum_{j\in\omega_i} \Phi_j$. (Note that Ψ_i defines a partition of unity.) Using the inverse inequality, we get, over a generic element $K \subset \Omega_i$,

$$\begin{aligned}|u_i|^2_{H^1(K)} &= |\sum_{j\in\omega_i} \mathcal{I}_h\left(\Phi_j u - \Phi_j\alpha_j\right)|^2_{H^1(K)}\\ &= |\sum_{j\in\omega_i} \mathcal{I}_h\Phi_j u - \sum_{j\in\omega_i} \mathcal{I}_h\Phi_j\alpha_j|^2_{H^1(K)}\\ &\le |\Psi_i u|^2_{H^1(K)} + |\sum_{j\in\omega_i} \Phi_j\alpha_j|^2_{H^1(K)}. \qquad (4.8)\end{aligned}$$

Let $\bar{\Psi}_{K,i}$ indicate the average of Ψ_i over the element K. With this notation, and using property 2 and the fact that $\delta = \mathcal{O}(h)$, the first term at the right-hand side of (4.8) can be bounded as follows:

$$\begin{aligned}|\Psi_i u|^2_{H^1(K)} &= \left|\bar{\Psi}_{K,i}u + \left(\Psi_i - \bar{\Psi}_{K,i}\right)u\right|^2_{H^1(K)} \qquad (4.9)\\ &\le 2\left|\bar{\Psi}_{K,i}u\right|^2_{H^1(K)} + 2\left|\left(\Psi_i - \bar{\Psi}_{K,i}\right)u\right|^2_{H^1(K)}\\ &\le 2\left|u\right|^2_{H^1(K)} + C\delta^{-2}\|\left(\Psi_i - \bar{\Psi}_{K,i}\right)u\|^2_{L^2(K)}\\ &\le 2\left|u\right|^2_{H^1(K)} + C\delta^{-2}\|\Psi_i - \bar{\Psi}_{K,i}\|^2_{L^\infty(K)}\|u\|^2_{L^2(K)}. \qquad (4.10)\end{aligned}$$

The second term of (4.10) has a non-zero contribution only near the boundaries of Ω_i, given by summing over all the elements of Ω_i and using lemma 4.2,

$$\begin{aligned}|\Psi_i u|^2_{H^1(\Omega_i)} &\le C\left|u\right|^2_{H^1(\Omega_i)} + C\delta^{-2}\left|u\right|^2_{L^2(\Gamma_{\delta_i})}\\ &= C\left|u\right|^2_{H^1(\Omega_i)} + C\left(1+\frac{H}{\delta}\right)\left|u\right|^2_{H^1(\Omega_i)}\\ &\le C\left(1+\frac{H}{\delta}\right)\left|u\right|^2_{H^1(\Omega_i)}.\end{aligned}$$

By manipulating the second term at the right-hand side of (4.8), we obtain

$$\sum_{i=1}^{M}\sum_{K\in\Omega_i}\left|\sum_{j\in\omega_i}\Phi_j\alpha_j\right|^2_{H^1(K)} \le C\left|Q_0 u\right|^2_{H^1(\Omega)}.$$

Finally, using Lemma 4.1.b and $\delta_0 = \delta$, we obtain

$$\begin{aligned}\sum_{i=0}^{M} |u_i|^2_{H^1(\Omega)} &= |u_0|^2_{H^1(\Omega)} + \sum_{i=1}^{M} |u_i|^2_{H^1(\Omega)} \\ &\leq C_0 \frac{H_0}{\delta} |u|^2_{H^1(\Omega)} + C\left(1 + \frac{H}{\delta}\right) |u|^2_{H^1(\Omega)} \\ &\leq \left(C_0 \frac{H_0}{\delta} + C\frac{H}{\delta}\right) |u|^2_{H^1(\Omega)} .\end{aligned}$$

Inequality (4.7) follows from the standard theory for abstract Schwarz methods (see for instance [20] and reference therein). □

Figure 4.1 shows an example of coarse basis functions that satisfy (on the left), and do not satisfy (on the right) the hypothesis of Theorem 4.5.

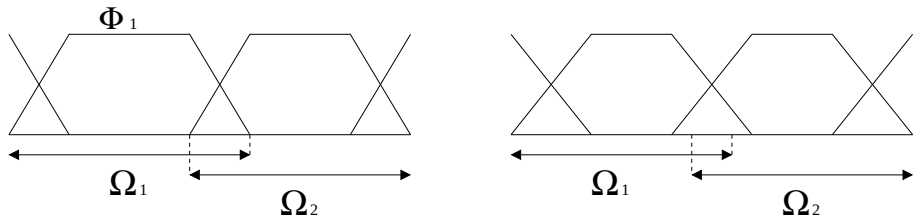

FIG. 4.1. *Example of coarse basis functions that satisfy (left) and do not satisfy (right) the hypothesis of Theorem 4.5.*

REMARK 3. *Theorem 4.5 states that, for a given decomposition into subdomains, the condition number can be reduced by decreasing the diameter of the aggregates. This is of particular advantage in situation where the decomposition into subdomains (and henceforth H) is fixed. This may occur, for instance, if the Schwarz preconditioner is used in a parallel environment, and each subdomain is assigned to a different processor.*

5. Numerical Results. We report in this section some numerical results for our model problem (2.1), with $\Omega = (0,1) \times (0,1)$ and $\Omega = (0,1) \times (0,1) \times (0,1)$. The mesh is constructed by dividing Ω into n^2 equal squares, subdivided into triangles, or n^3 cubes, subdivided into tetrahedra. Thus, we obtain a triangulation with $h = \frac{1}{n}$. As regards the decomposition into subdomains, we consider overlapping squares (or cubes) Ω_i of area $H^d, d = 2, 3$. We use linear finite elements, and solve the linear system (2.2) by the preconditioned conjugate gradient method.

As described in Section 3, the first step of aggregation requires a simple procedure to define the restriction operator $R_0^\star$, in such a way that each aggregate will eventually correspond to a "degree of freedom" of the coarse operator. To define the aggregates, we decompose the set of nodes into non-overlapping sets, $\vartheta_j^\star, j = 1, \ldots, n_0$. The $n_0 \times n$ boolean matrix $R_0^\star$ is constructed in such a way that each row corresponds to an aggregate and each column to a node, as follows:

$$R_0^\star(i,j) = \begin{cases} 1 & \text{if } j \in \vartheta_i^\star \\ 0 & \text{if } j \notin \vartheta_i^\star. \end{cases} \tag{5.1}$$

$R_0^\star$ can be viewed as a simple grid transfer operator corresponding to piecewise constant interpolation; see Remark 2.

REMARK 4. *To define (5.1), we construct aggregates with minimal overlap $\delta_0 = h$. Alternatively, non-overlapping aggregates can be defined as non-overlapping partitions of Ω; see [10].*

For non-smoothed aggregation, the aggregates (whose shape is approximately a square for $d = 2$ and a cube for $d = 3$) are defined as described by Equation (5.1). The corresponding preconditioner is indicated by B''_{nsa}. For smoothed aggregation, the following Richardson smoother is employed:

$$S = \left(I - \frac{\varpi}{\lambda_M} A \right), \tag{5.2}$$

where λ_M is the maximum eigenvalue of A (estimated using 10 steps of a conjugate gradient method), and $\varpi = 4/3$ is the damping parameter. The corresponding preconditioner is indicated by B''_{sa}.

The tables report the estimated condition number for the preconditioned system; see for instance [6]. The stopping criteria is 10^{-12} on the relative residual.

A comparison of the condition numbers obtained using a geometric coarse grid and aggregation is reported in [17] where it is shown that, for the considered 2D model problem, aggregation procedures result in condition numbers about four times larger than using a geometric coarse grid. In this paper, we present results concerning aggregation schemes only. Let $\chi = H_0/H$ indicate the ratio between the diameter of the aggregates and the diameter of the subdomains. $\chi = 1$ means that all the nodes in each (minimal overlap) subdomains have been assigned to the same aggregate, while larger values of χ indicates that the coarse space has been enriched by using more aggregates on each subdomain.

Tables 5.1 and 5.3 report the estimated condition number for non-smoothed aggregation, using $\delta = \delta_0 = h$ (minimal overlap), for different values of χ. From Theorem 4.5, the condition number in this particular case has a linear dependency on h^{-1}. This is confirmed by the tables. For example, in two-dimensions with $\chi = 1$ and $H = 1/10$, the condition number increases from 110.35 to 220.01 when h decreases from 1/240 to 1/480.

Tables 5.2 and 5.4 use $\delta = h$ and $\delta_0 = 3h$. Although the proof of Theorem 4.5 requires $\delta_0 = \delta$, these results suggest that, at least for this problem, this condition may be relaxed. Especially for small values of χ, $\kappa(B''_{sa}A)$ is significantly smaller than $\kappa(B''_{nsa}A)$. (Should $\delta = \delta_0 = 3h$, then $\kappa(B''_{sa}A) = \kappa(B''_{nsa}A)\ /3$.) Therefore, it may be convenient, in a parallel environment, to use smoothed aggregation even if the subdomains share minimal overlap.

Theorem 4.5 also states that the quantity

$$\frac{\kappa(B''A)\ \delta}{H} \tag{5.3}$$

depends linearly on χ. Figure 5.1 displays graphically the quantity (5.3) for data in Tables 5.1 (figure on the left) and 5.2 (figure on the right) with respect to χ. The figure also reports the least-square approximation of the considered data. The scatter in the data is primarily due to the presence of the Dirichlet nodes which make the linear system easier to solve for large values of h and small values of H. While there is some scatter, the overall trend of the data closely matches the asymptotic behavior predicted by the theory.

6. Conclusions. We have presented an improved convergence bound for two level additive domain decomposition preconditioners based on aggregation. This

		$\chi = 1$	$\chi = 1/2$	$\chi = 1/4$	$\chi = 1/8$
$h^{-1} = 120$	$H^{-1} = 4$	103.94	70.08	38.64	–
	$H^{-1} = 6$	81.88	50.31	33.28	–
	$H^{-1} = 8$	66.41	40.32	27.62	–
	$H^{-1} = 10$	–	–	–	–
$h^{-1} = 180$	$H^{-1} = 4$	155.67	101.57	61.87	34.63
	$H^{-1} = 6$	122.41	78.46	43.05	29.37
	$H^{-1} = 8$	99.69	59.39	37.66	24.95
	$H^{-1} = 10$	82.95	45.92	25.35	–
$h^{-1} = 240$	$H^{-1} = 4$	207.63	144.71	84.06	44.26
	$H^{-1} = 6$	163.01	99.71	58.09	39.90
	$H^{-1} = 8$	132.15	82.65	45.76	30.24
	$H^{-1} = 10$	110.35	62.71	35.57	–
$h^{-1} = 360$	$H^{-1} = 4$	311.11	195.71	135.98	70.37
	$H^{-1} = 6$	244.24	162.47	90.01	46.74
	$H^{-1} = 8$	197.94	118.94	69.23	36.60
	$H^{-1} = 10$	164.33	105.37	58.47	–
$h^{-1} = 480$	$H^{-1} = 4$	–	–	–	–
	$H^{-1} = 6$	325.49	206.15	124.88	62.37
	$H^{-1} = 8$	263.50	171.13	92.97	47.96
	$H^{-1} = 10$	220.01	141.09	74.44	39.91

TABLE 5.1

2D Laplace problem. Estimated condition number using B''_{nsa} ($\delta = \delta_0 = h$). $\chi = H_0/H$.

		$\chi = 1$	$\chi = 1/2$	$\chi = 1/4$	$\chi = 1/8$
$h^{-1} = 120$	$H^{-1} = 4$	73.76	40.03	20.14	–
	$H^{-1} = 6$	51.22	26.02	12.55	–
	$H^{-1} = 8$	38.63	19.52	11.70	–
	$H^{-1} = 10$	–	–	–	–
$h^{-1} = 180$	$H^{-1} = 4$	111.24	58.62	32.03	14.55
	$H^{-1} = 6$	77.47	41.29	20.33	9.36
	$H^{-1} = 8$	59.68	30.52	14.65	10.09
	$H^{-1} = 10$	47.74	23.67	15.80	9.22
$h^{-1} = 240$	$H^{-1} = 4$	150.60	85.59	44.78	21.10
	$H^{-1} = 6$	105.53	53.70	27.34	13.06
	$H^{-1} = 8$	80.18	43.26	20.79	9.84
	$H^{-1} = 10$	64.31	31.96	–	8.80
$h^{-1} = 360$	$H^{-1} = 4$	226.59	119.54	64.48	34.28
	$H^{-1} = 6$	158.85	89.74	49.59	21.07
	$H^{-1} = 8$	120.15	61.05	22.86	14.81
	$H^{-1} = 10$	95.37	54.32	27.51	13.61
$h^{-1} = 480$	$H^{-1} = 4$	–	–	–	–
	$H^{-1} = 6$	213.22	115.59	65.38	29.60
	$H^{-1} = 8$	158.63	91.45	46.92	21.11
	$H^{-1} = 10$	129.60	76.55	34.69	16.60

TABLE 5.2

2D Laplace problem. Estimated condition number using B''_{sa}. $\chi = H_0/H$.

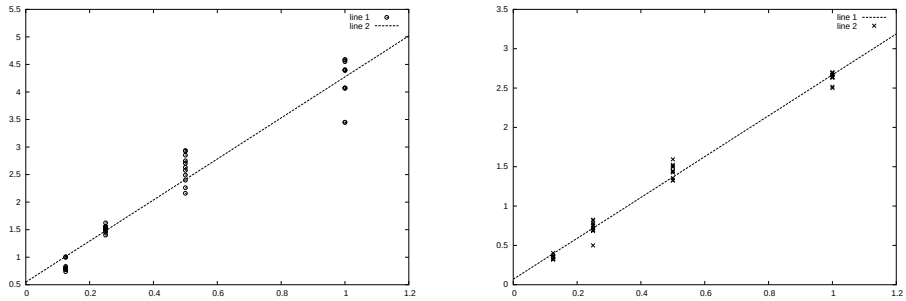

FIG. 5.1. *Graphical representation of data contained in Table 5.1 (left) and Table 5.2 (right). The x-axis reports the values of $\chi = H_0/H$; the y-axis reports $\kappa\delta/H$. The dots represent the data in the table, the line is the least-square approximation.*

		$\chi = 1$	$\chi = 1/2$	$\chi = 1/4$
$h^{-1} = 36$	$H^{-1} = 2$	42.02	32.92	23.15
	$H^{-1} = 3$	39.51	26.82	19.89
	$H^{-1} = 4$	35.34	22.68	16.63
	$H^{-1} = 5$	–	–	–
$h^{-1} = 48$	$H^{-1} = 2$	55.79	43.44	30.54
	$H^{-1} = 3$	52.43	36.21	23.18
	$H^{-1} = 4$	46.85	30.04	21.60
	$H^{-1} = 5$	41.25	27.36	17.61
$h^{-1} = 72$	$H^{-1} = 2$	–	–	–
	$H^{-1} = 3$	78.38	54.61	35.36
	$H^{-1} = 4$	69.90	46.40	28.71
	$H^{-1} = 5$	–	–	–
$h^{-1} = 96$	$H^{-1} = 2$	–	–	–
	$H^{-1} = 3$	104.37	75.31	47.37
	$H^{-1} = 4$	92.99	61.20	37.61
	$H^{-1} = 5$	–	–	–

TABLE 5.3
3D Laplace problem. Estimated condition number using B''_{nsa}. $\chi = H_0/H$.

bound generalizes other results presented in literature. Numerical results are reported to confirm the proposed estimate.

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		$\chi = 1$	$\chi = 1/2$	$\chi = 1/4$
	$H^{-1} = 2$	–	–	14.76
$h^{-1} = 36$	$H^{-1} = 3$	34.11	18.54	9.53
	$H^{-1} = 4$	28.22	13.92	7.70
	$H^{-1} = 5$	–	–	–
	$H^{-1} = 2$	52.72	34.92	20.12
$h^{-1} = 48$	$H^{-1} = 3$	45.56	24.67	13.53
	$H^{-1} = 4$	37.90	18.82	9.76
	$H^{-1} = 5$	30.56	17.06	8.33
	$H^{-1} = 2$	–	–	–
$h^{-1} = 72$	$H^{-1} = 3$	68.86	39.46	21.67
	$H^{-1} = 4$	57.32	30.91	15.66
	$H^{-1} = 5$	–	–	–
	$H^{-1} = 2$	–	–	–
$h^{-1} = 96$	$H^{-1} = 3$	91.85		
	$H^{-1} = 4$	76.66	40.49	22.34
	$H^{-1} = 5$	–	–	–

TABLE 5.4

3D Laplace problem. Estimated condition number using B''_{sa}. $\chi = H_0/H$.

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