Successes and Challenges in the control, estimation/ID, and forecasting of chaotic fluid systems

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First main point:

Accurate models for numerical simulation are not necessarily the right models for control design.

We will illustrate by describing three types of models which we believe are quite suitable for control design in high-dimensional flow systems.

Outline of Part 1 (Modeling issues)

Background: Adjoint- and Riccati-based noncooperative analysis.

- Linear control of nonlinear systems.
- Regularization is key! (Bartosz Protas)

A. A time-periodic approach for jet control. (Catalin Trenchea)

- Underlying assumption: **approximate time periodicity**.
- Natural simplification $\Rightarrow$ tractable adjoint optimization.

B. A spatially localized approach for b.l. control. (Markus Högberg)

- Underlying assumption: **(nearly) parallel flow**.
- Truncated convolution kernels $\Rightarrow$ overlapping decentralized control.

C. A nonlocalized approach for b.l. control. (Patricia Cathalifaud)

- Underlying assumption: **parabolic evolution of system in** x
- Sacrifice localization in $x \Rightarrow$ global perspective on system evolution.

– Background –

Essentials of adjoint-based and Riccati-based noncooperative analysis

1 Adjoint-based analysis

State equation:

$$\boxed{\begin{array}{ll} E\dot{\mathbf{q}} = N(\mathbf{q}, \mathbf{f}, \phi \quad) & \text{on} \quad 0 < t < T \\ \mathbf{q} = \mathbf{q}_0 & \text{at} \quad t = 0 \end{array}}$$

with: $\mathbf{q}$ = state, $\mathbf{f}$ = external force, ϕ = control .

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with: $\mathbf{q} =$ state, $\mathbf{f} =$ external force, $\phi =$ control .

Perturbation equation:

$$\left\{ \begin{array}{ll} \mathcal{L}\,\mathbf{q}' = B_\phi \phi' & \text{on} \quad 0 < t < T \\ \mathbf{q}' = 0 & \text{at} \quad t = 0 \end{array} \right\} \Rightarrow$$

Small perturbations ϕ' to control ϕ

cause small perturbation $\mathbf{q}'$ to state $\mathbf{q}$.

$\mathcal{L}\,\mathbf{q}' \triangleq \left(E\frac{d}{dt} - A\right)\mathbf{q}'$ is the linearization of the state eqn about the trajectory $\mathbf{q}(\phi \quad)$.

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Cost function (minimize w.r.t. ϕ):

$$\boxed{\mathcal{J} = \frac{1}{2}\int_0^T (\mathbf{q}^* Q \mathbf{q} + \ell^2 \phi^* \phi \qquad)\, dt} \quad \Rightarrow \quad \mathcal{J}' = \int_0^T (\mathbf{q}^* Q \mathbf{q}' + \ell^2 \phi^* \phi' \qquad)\, dt.$$

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Statement of adjoint identity. Define inner product $\langle \mathbf{r}, \mathbf{q}' \rangle = \int_0^T \mathbf{r}^* \mathbf{q}' \, dt$. Then:

$$\boxed{\langle \mathbf{r}, \mathcal{L} \mathbf{q}' \rangle = \langle \mathcal{L}^* \mathbf{r}, \mathbf{q}' \rangle + \mathbf{b}}$$

with: $\mathbf{r} =$ adjoint, $\mathcal{L}^* \mathbf{r} = \left(-E^* \frac{d}{dt} - A^* \right) \mathbf{r}$, $\mathbf{b} = \mathbf{r}^* E \mathbf{q}' \Big|_{t=T} - \mathbf{r}^* E \mathbf{q}' \Big|_{t=0}$.

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Extraction of gradients. Combining equations, we have:

$$\langle \mathbf{r}, B_\phi \phi' + B_\psi \psi' \rangle = \langle Q\mathbf{q}, \mathbf{q}' \rangle \quad \Rightarrow \quad \int_0^T \mathbf{q}^* Q \mathbf{q}' \, dt = \int_0^T \mathbf{r}^* (B_\phi \phi' + B_\psi \psi') \, dt.$$

$$\mathcal{J}' = \int_0^T \left[\left(B_\phi^* \mathbf{r} + \ell^2 \phi\right)^* \phi' + \left(B_\psi^* \mathbf{r} - \gamma^2 \psi\right)^* \psi' \right] dt \triangleq \int_0^T \left[\left(\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi}\right)^* \phi' + \left(\frac{\mathcal{D}\mathcal{J}}{\mathcal{D}\psi}\right)^* \psi' \right] dt$$

As ϕ' and ψ' are arbitrary, $\boxed{\dfrac{\mathcal{D}\mathcal{J}}{\mathcal{D}\phi} = B_\phi^* \mathbf{r} + \ell^2 \phi}$ and $\boxed{\dfrac{\mathcal{D}\mathcal{J}}{\mathcal{D}\psi} = B_\psi^* \mathbf{r} - \gamma^2 \psi}$.

2 Riccati-based analysis

Characterization of saddle point. The control ϕ which minimizes $\mathcal{J}$ and the disturbance ψ which maximizes $\mathcal{J}$ are given by

$$\frac{\mathscr{D}\mathcal{J}}{\mathscr{D}\phi} = 0, \quad \frac{\mathscr{D}\mathcal{J}}{\mathscr{D}\psi} = 0 \quad \Rightarrow \quad \phi = -\frac{1}{\ell^2} B_\phi^* \mathbf{r}, \quad \psi = \frac{1}{\gamma^2} B_\psi^* \mathbf{r}.$$

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Combined matrix form. Combining the perturbation and adjoint eqns at the saddle point determined above, assuming $E = I$, gives:

$$\begin{array}{r} \text{Perturbation equation} \rightarrow \\ \text{Adjoint equation} \rightarrow \end{array} \quad \begin{bmatrix} \dot{\mathbf{q}}' \\ \mathbf{r} \end{bmatrix} = \begin{bmatrix} A & \overbrace{-\frac{1}{\ell^2} B_\phi B_\phi^* + \frac{1}{\gamma^2} B_\psi B_\psi^*}^{\text{control and disturbance}} \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} \mathbf{q}' \\ \mathbf{r} \end{bmatrix}$$

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Solution Ansatz. Relate perturbation $\mathbf{q}' = \mathbf{q}'(t)$ and adjoint $\mathbf{r} = \mathbf{r}(t)$:

$$\boxed{\mathbf{r} = X\mathbf{q}'}, \quad \text{where } X = X(t).$$

Riccati equation. Inserting solution ansatz into the combined matrix form to eliminate $\mathbf{r}$ and combining rows to eliminate $\dot{\mathbf{q}}'$ gives:

$$\left[-\dot{X} = A^* X + XA + X\left(\frac{1}{\gamma^2} B_\psi B_\psi^* - \frac{1}{\ell^2} B_\phi B_\phi^*\right) X + Q\right] \mathbf{q}'.$$

As this equation is valid for all $\mathbf{q}'$, it follows that:

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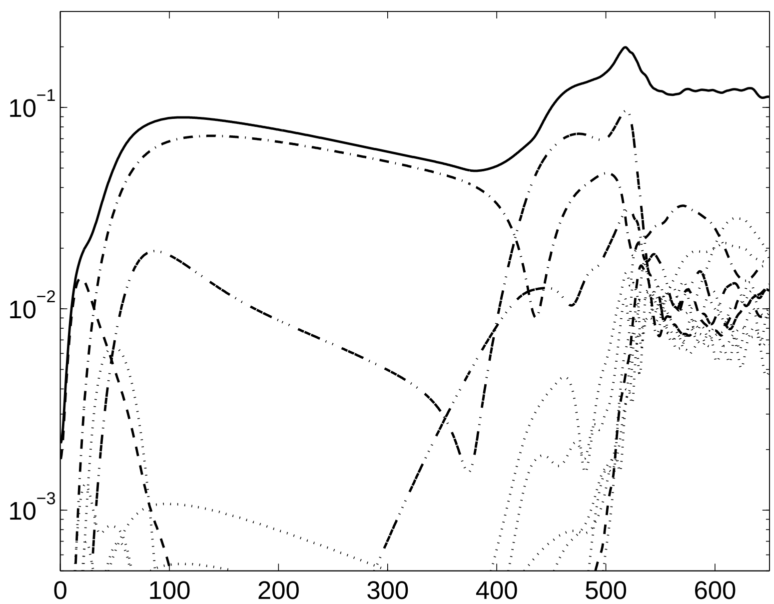
Note that, by the characterization of the saddle point, we have

$$\psi = \frac{1}{\gamma^2}B_\psi^*\mathbf{r} \quad \text{and} \quad \boxed{\phi = K\mathbf{q}' \quad \text{where} \quad K = -\frac{1}{\ell^2}B_\phi^* X}.$$

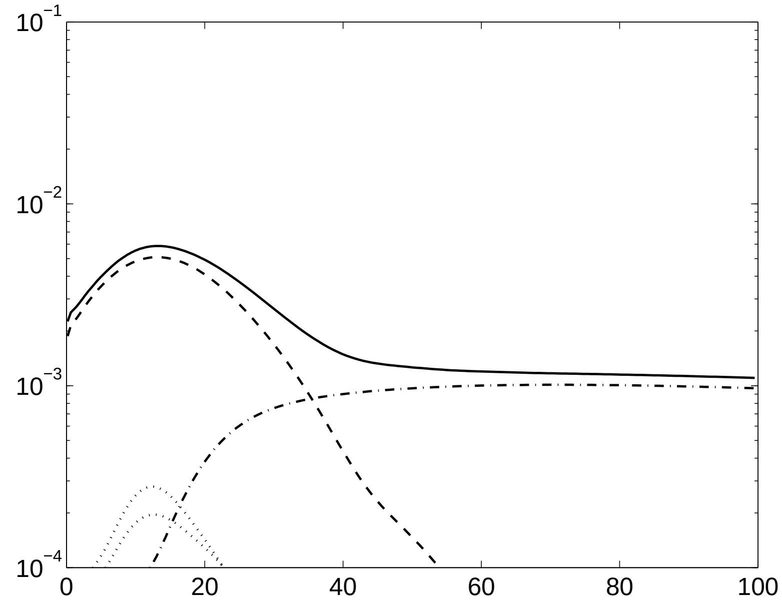
This is referred to as the finite-horizon $\mathcal{H}_\infty$ control solution, and may be solved for linear time-varying (LTV) systems.

Oblique-wave channel flow transition benchmarks

Feedback control increases the threshold energy of the initial perturbation which leads to transition by up to 100x.
(Högberg, B, Henningson 2002, *JFM*).



No control (transition).



With contol (relaminarization).

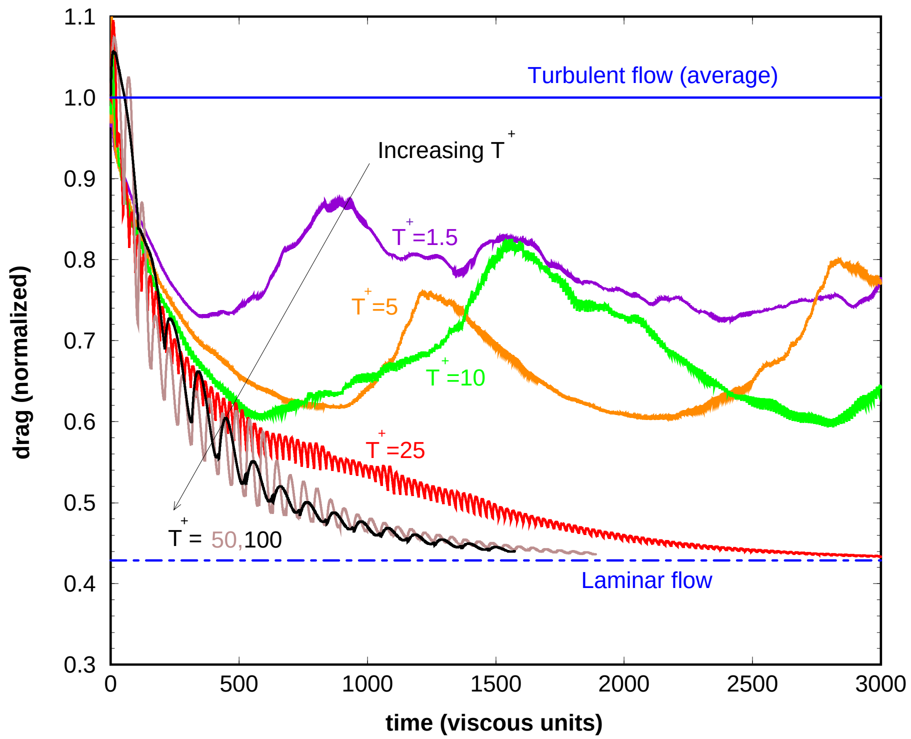
Plotted are: (——) the total flow perturbation energy in all wavenumber components other than $\{0,0\}$, as well as the energy in: (----) the $\{1,1\}$ component, (–·–) the $\{0,2\}$ component, (–··–) the $\{0,4\}$ component, (–···–) the $\{1,2\}$ component, and (·······) the other components.

Turbulent channel flow benchmarks

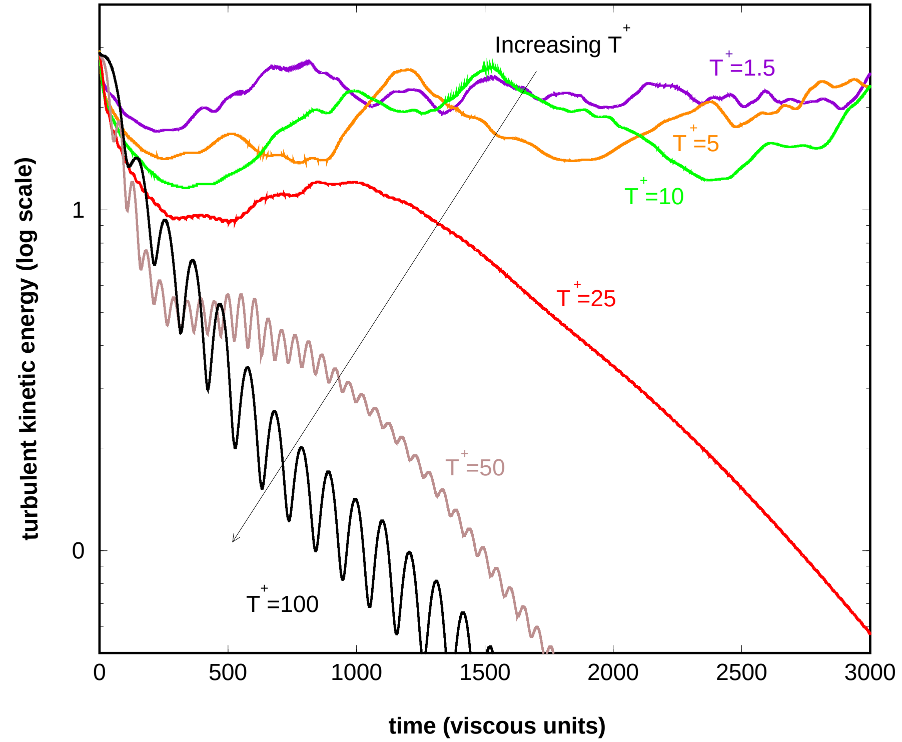
($Re_\tau = 100$ DNS, $64 \times 65 \times 64$ grid, $4\pi \times 2 \times 4\pi/3$ box).

Adjoint-based receding-horizon model predictive control.

(B, Moin, & Temam 2001, *JFM*).



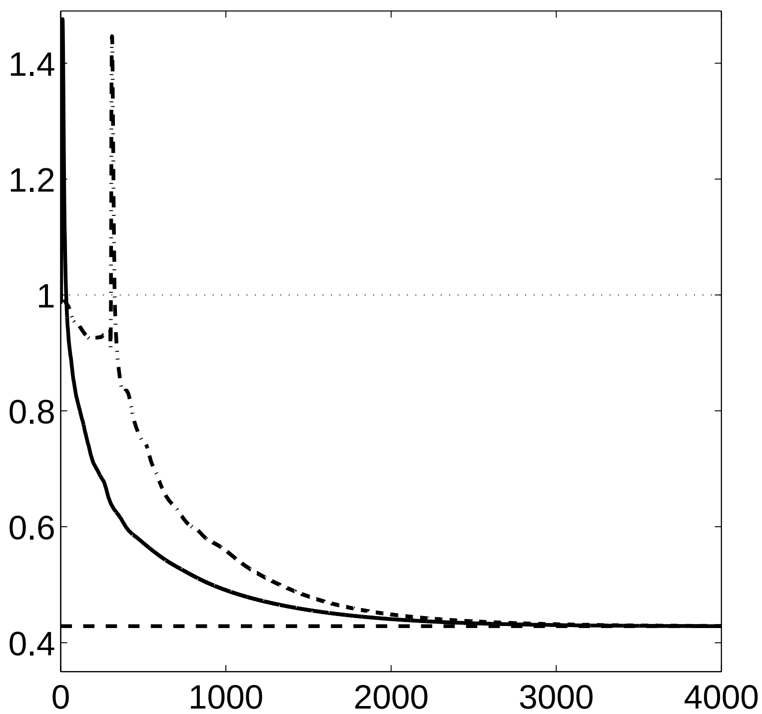
History of drag.



History of TKE.

Riccati-based constant-gain linear state feedback.

(Högberg & B, *PoF*).



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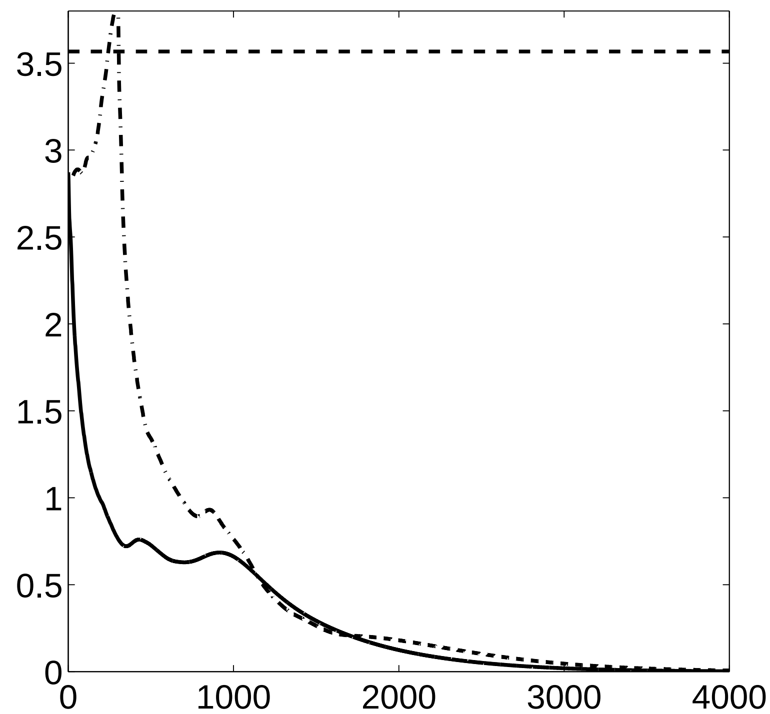
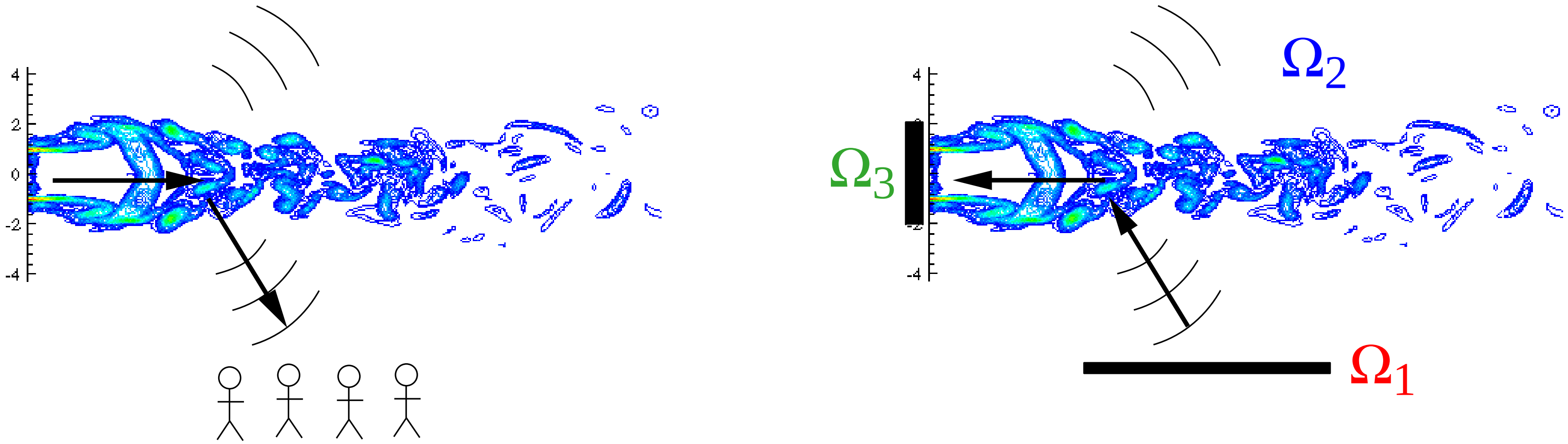


History of TKE.

Regularization issues in adjoint analysis

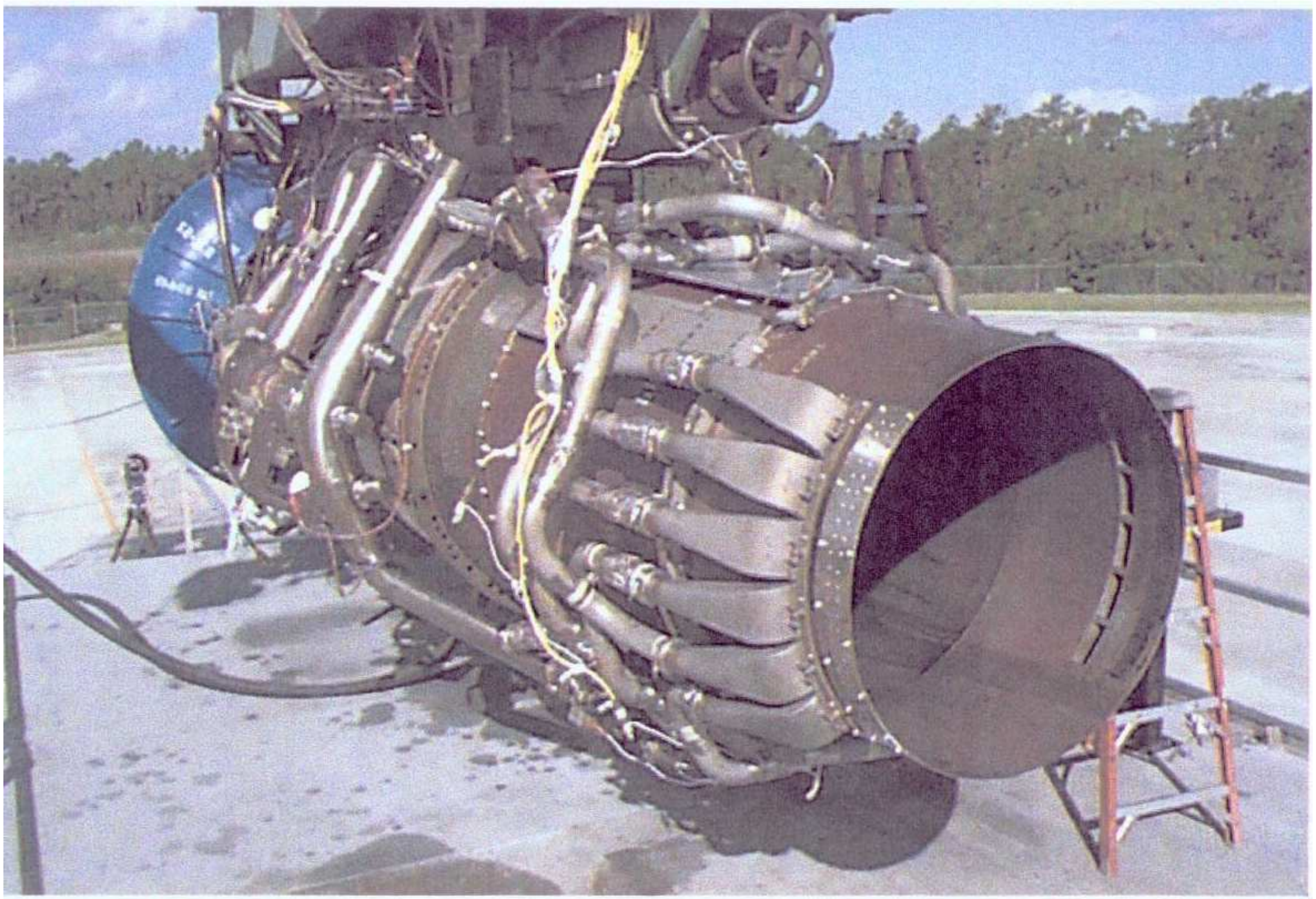


Cost function domain: Ω_1, Flow domain: Ω_2, Control domain Ω_3.

Three components of optimization problem:

1. Statement of cost function: inner product on Ω_1.
2. Statement of adjoint identity: inner product on Ω_2.
3. Identification of gradient: inner product on Ω_3.

Prototype implementation for jet control



Once control strategy is optimized, actuators which are an order of magnitude smaller should also be effective.

– Method A –
A time-periodic approach for systems dominated by vortex shedding

Artificial assumption of *spatial* periodicity is fundamental
to the numerical simulation of near-wall turbulence.

For certain systems dominated by vortex shedding, artificial assumption
of *time* periodicity might be similarly useful.

Results in 4D stationary system amenable to multigrid analysis.
$\Rightarrow$ can leverage prior guesses in iterative control optimization algorithm!

Nonuniqueness of solutions.
Q: Which solution should we design our controls for?

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Q: Which solution should we design our controls for?
The worst one, of course! $\Rightarrow$ a noncooperative optimization!

Summary of Method A

New mathematical analysis necessary to set the stage for adjoint-based optimization. The central problem is that the manifold of solutions of exact problem is discrete (there may be only a finite number of solutions) so gradient-based methods based on adjoints are useless.

Algorithm: Solve an approximate problem first which is continuous. Control the level of approximation, gradually refining it to determine the solution of the optimization problem you are interested in!

See B & Trenchea (*AIAA Paper*, 2003) for details.

– Method B –
A spatially localized feedback calculation based on the parallel flow assumption

3D O.S./Squire control problem **solved in Fourier space**, where decoupling simplifies to several 1D problems.

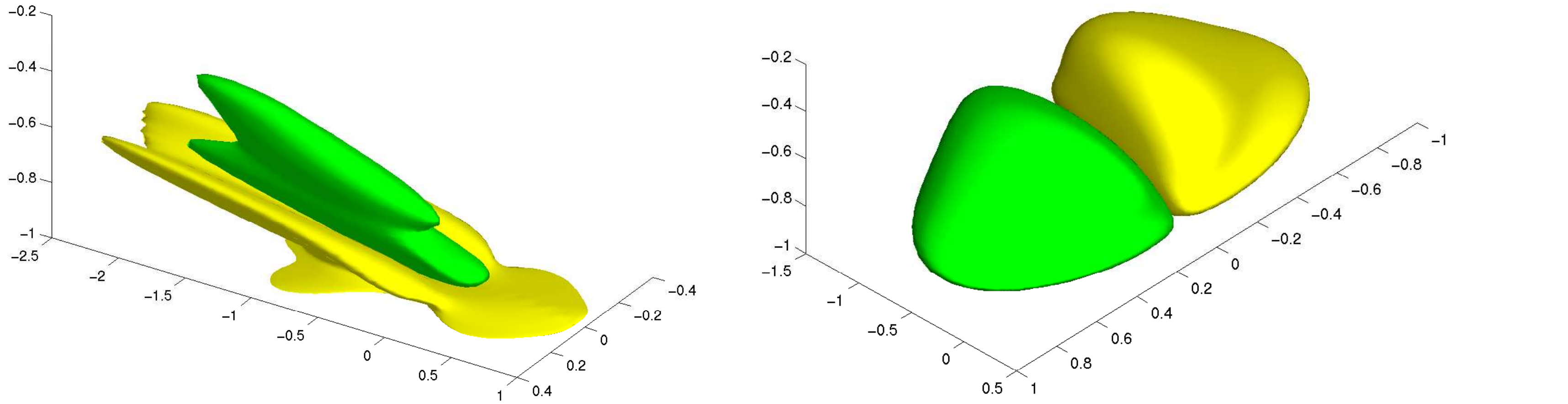
Result **inverse transformed to physical space**, yields localized convolution kernels.

Procedure highly sensitive to nuances of control formulation and numerical discretization:

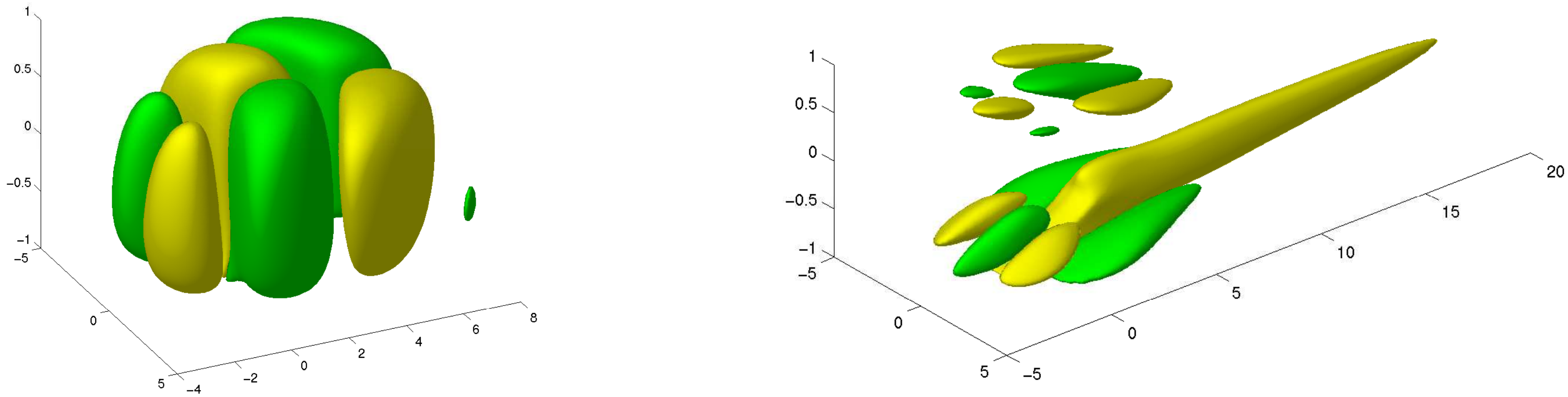
– Lifting of controls results in penalization of $\dot{\phi}$ in cost function.

– Spurious eigenvalues must be addressed.

Spatial localization of convolution kernels



Controller gain relating the state v (left) and ω (right) inside the domain to the control forcing term $\dot{\phi}(0,-1,0)$ on the wall.



Estimator gain relating the wall measurement $\Delta y(0,-1,0)$ to the forcing of the estimator model for v (left) and ω (right) inside the domain.

- Kernels independent of the box size in which they were computed, so long as the computational box is sufficiently large.

$\Rightarrow$ Nonphysical assumption of spatial periodicity is relaxed.

- Kernels are well-resolved with grid resolutions appropriate for the simulation of the physical system of interest.

$\Rightarrow$ Grid independent.

- Kernels eventually decay exponentially, and may be truncated to any desired degree of precision.

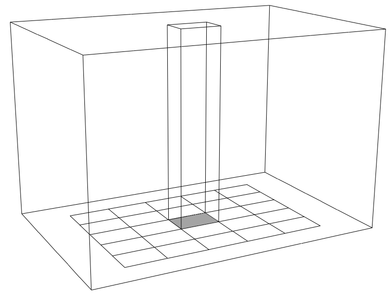
$\Rightarrow$ Truncated kernels are *spatially compact* with *finite support*.

- Kernel structure is physically tenable, but not imposed *a priori*:

$\Rightarrow$ *Control* convolution kernels angle away from the actuator *upstream*.

$\Rightarrow$ *Estimation* convolution kernels extend well *downstream* of sensor.

Overlapping decentralized implementation of feedback



Experimental apparatus

actuator inputs on tile $\{i, j\}$ *only*

sensor measurements on tile $\{i, j\}$ *only*

Localized physical-space compensation:
estimation of state above tile $\{i, j\}$,
computation of control on tile $\{i, j\}$

$\Longleftrightarrow$ $\Longleftrightarrow$

Decentralized logic circuit replicated on each tile

Communication with neighboring tiles about nearby sensor measurements and state estimates

Summary of Method B

Approach is not plagued by the communication bottleneck experienced by schemes which require centralized computation of FFTs and iFFTs.

Communication between tiles is limited in spatial extent by size of the truncated kernels. $\Rightarrow$ **By replication, can extend system to large arrays of sensors and actuators.**

For spatially-invariant systems, logic on each tile is identical. Extension to gradually evolving flows (Blasius, Falkner-Skan-Cooke, ...) is easy.

See B & Liu, *JFM* 1998, and Högberg, B, & Henningson, *JFM* 2002.

– Method C –
A spatially nonlocal feedback calculation based on the parabolic flow assumption

Fourier transforms still used to decouple problem in z, but we now use a boundary-layer assumption to simplify the problem in x.

We have a finite-horizon linear system discretized in direction (y), but the system evolves in **space** (x), not time (t).

Control problem solved by marching a Riccati equation in **space**.

There is **no causality constraint** for systems which evolve in space:

> **downstream measurements may be used to update upstream controls**.

Some new tools are needed to solve this problem correctly.

Summary of Method C

The control responds to upstream and downstream disturbances

- Discrete formulation of the state equation (Delta formulation).
- Augmentation of the state vector by the disturbances entering the downstream system (to cast in the standard *causal* framework).

Regularity of the control is insured

- Augmentation of the state vector by the current control variable.
- Simplification to standard discrete form by a change of variables.

Estimator responds to upstream and downstream measurements

- Derivation of a smoothing problem from a classical filtering problem.

See B & Cathalifaud, *Systems & Control Letters* 2004, for details.

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- Sacrifice localization in $x \Rightarrow$ global perspective on system evolution.