A Unifying Framework for Robust and Stochastic Optimization Models and Methods

John R. Birge

The University of Chicago Graduate School of Business

www.ChicagoGSB.edu/fac/john.birge

Themes

- Goals for robust and stochastic optimization can align
- Different approaches may lead to the same outcomes
- Some cases create apparent paradoxes between the approaches (that can be resolved with a consistent framework)
- Methods/results from each approach may useful

Outline

- Traditional views
- Overall framework
- Consistent interpretation
- Paradoxes, pitfalls, and resolutions
- Converging methods and results
- Conclusions and revisions

Traditional Views

<u>Stochastic optimization (SO)</u>

$$min_{x2\,X}\, E_P[f(x,\xi)]$$

where P is a (known) prob. measure on ξ.

Issues: What are f, P?

<u>Robust optimization (RO)</u>

$$min_{x2\,X}\, [max_{\xi 2 \Xi}\, g(x,\xi)]$$

where Ξ is the set of possible ξ.

Issues: What are g, Ξ?

Misinterpretations

- Objective functions:
  - f and g are the same in each model
- Probability distribution:
  - P, Ξ must be known with certainty
- Results are inconsistent with each rationality or behavior

Easy Form of Resolution

Make models look the same:

RO => SO: Let $f(x,\xi) = v \geq g(x,\xi)\ \forall \xi \in \Xi$

$$\min_{x\in X} E_P[f(x,\xi)] \Leftrightarrow \min_{x\in X} E_P[v | v \geq g(x,\xi) \forall \xi]$$

$$\Leftrightarrow \min_{x\in X} [\max_{\xi\in\Xi} g(x,\xi)]$$

SO => RO: Let Ξ be the domain of P, $g(x,\xi) =>$ $g(x, \mathrm{P}) = E_P[f(x,\xi)]$

$$\min_{x\in X} [\max_{P\in\Xi} g(x, P)] \Leftrightarrow \min_{x\in X} E_P[f(x,\xi)]$$

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What about Probabilistic Constraints?

- Prob./chance-constrained form:

$$\min_{x \in X,\, P[h(x,\xi) \leq 0] \geq \alpha} f(x)$$

- RO Form:

$$P(\Xi) \geq \alpha,\ g(x,\xi) = f(x)\delta_{\{\xi | h(x,\xi) \leq 0\}}$$

- SO Form:

$$f(x,\xi) = f(x)\delta_{\{x |\ P[h(x,\xi) \leq 0] \geq \alpha\}}$$

What is the True Goal?

- Maximize expected utility?

  $f(x,\xi) = - U(x,\xi)$, P given

- A robust form?

  $g(x,\xi) = U(x,\xi)$ and for $\underline{\boldsymbol{\xi}}(x)=argmax_{\xi \in \Xi}\, g(x,\xi)$

  $\underline{\boldsymbol{P}}(\underline{\boldsymbol{\xi}}(x))=1$

  Expected utility with $\underline{\boldsymbol{P}}$ that depends on x

- Can this be rational?

Toward a Consistent View: Competition

- Suppose (a) competitor(s) choose(s) $y(x,\xi)$ to maximize $c(x,y,\xi)$
- Formulation:

$$min_{x \in X} E_P[f(x,y,\xi)|y \in argmax\ c(x,y,\xi)]$$

- y fixed (or f independent of y) => SO
- $y=\xi \in \Xi$, $f(x,y,\xi)=c(x,y,\xi)=g(x,\xi)$ => RO
- SO assumes *irrelevant* adversary
- RO assumes *perfect* adversary

Paradoxes and Pitfalls

- Value of Information: “Blau’s dilemma”
- Suppose demand=$b=0$ w.p. 0.9 and 1 w.p. 0.1
- Problem:

min x s.t. P[x¸ b]¸ 0.9

Solution: $x^*=0$

With perfect information: $x^P=0$ w.p. 0.9 and 1 w.p. 0.1

EVPI = Exp. Value without Perfect Information – Exp. Value with Perfect Information

= 0 – 0.1 = -0.1 < 0

(Same may be true with EVSampleInformation)

For RO, let *Ξ = {b | P[b]¸ 0.9} = {0}*

Problems with “Paradox”

- Utility may depend on information level
  - With no information, 0.9 may be acceptable but not the same with more information
  - Cannot may direct comparisons in information value
- Not including role of competitor
  - Competitor may gain information as well
  - In this case, more information may not always be beneficial

Coherent and Rational Risk Measures

- *R* is a coherent risk measure if
  - *R* is convex and decreasing
  - $R(x(\xi)+a)=R(x(\xi)) + a, a$2 <
  - $R(\lambda\, x(\xi)) = \lambda\, R(x(\xi))$

Von Neumann-Morganstern (rational) utility (negative risk):

Complete, Transitive, Continuous, Monotonic, Substitutable (Independent)

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Resolving Utility Problems

Role in RO model

$R(x,\Xi) = Max_{\xi \in \Xi}\, g(x,\xi)$

may not have all the properties (unless interpreted differently)

Examples: $g(x,\xi)= \xi^T(x-b)$, $\Xi=\{ \xi \mid \xi^T \xi \leq \varepsilon^2\}$

$Max_{\xi \in \Xi}\, g(x,\xi)=\varepsilon \, ||x-b||$

Not coherent in x but ok in $||x-b||$

$g(x,\xi) = max\{ \xi \in \Xi \mid \xi \leq x\}=min\{ \xi^{max},x\}$

Not coherent when min is ξ^{max} *but ok if x*

Re-interpretation may be consistent with axioms

Problems with Other Forms: Mean-Variance

- Suppose objective is

$$Mean(f(x)) + \lambda Variance(f(x))$$

- vNM independence:

  Suppose $E(x1)=-1$, $Var(x1)=1$, $E(x2)=-1.5$, $Var(x2)=0.25$

  $R(x1)=0$, $R(x2)=-1.25 \Rightarrow x2$ Â $x1$

  Consider adding a to each with $E(a)=0, Var(a)=\alpha^2$, $Cov(x1,a)=-\alpha$, $Cov(x2,a)=0.5\alpha$; $E(x1+a)+Var(x1+a)=-1+1-2\alpha+\alpha^2$

  $E(x2+a)+Var(x2+a)=-1.5+0.25+\alpha+\alpha^2$

  $R(x1)-R(x2)=-3\alpha+1.25<0$ if $\alpha > 1.25/3 \Rightarrow x1+a$ Â $x2+a$

- Two-stage problem

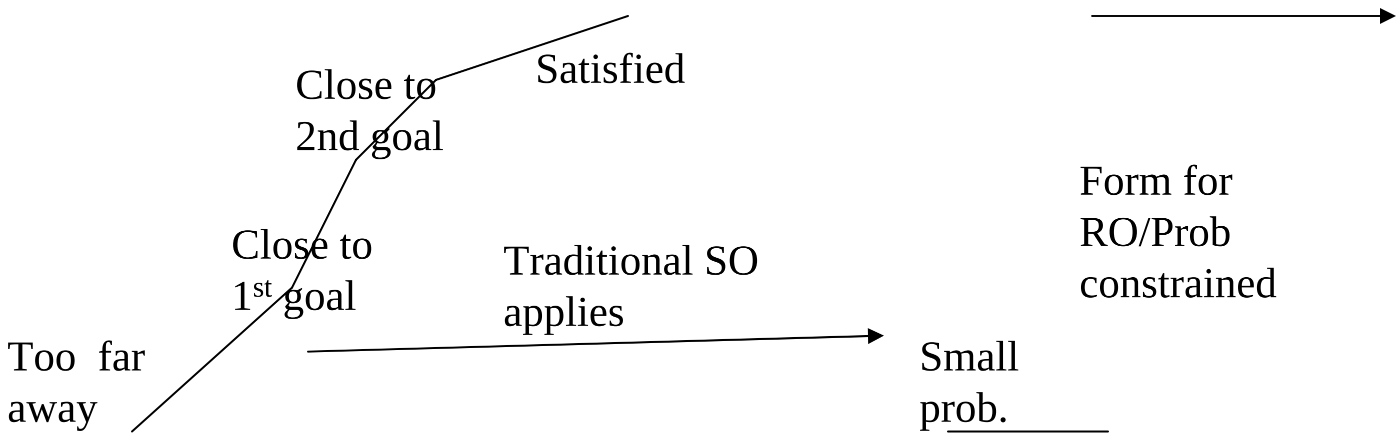
$$f(x,y,\xi) = c(x) + q(y(\xi))$$

Min of $Mean(f) + \lambda Variance(f)$ may not be have $f(x,y,\xi)=\min_y c(x) + q(y(x,\xi))$

- Resolution: fix utility as quadratic (or other)

Do Axioms Matter?

- What is observed? (Kahnemann-Tversky prospect theory)
  - Targets define utility
  - Preference depends on closeness to targets



Converging Models

- Both RO and SO models can apply for observed preferences
- Interpretation of a competitor brings them together
- Paradoxes generally concern mis-interpretations
- What about methods?

Convergent Methods

- Bounding methods for SO:

  Find P^* s.t. $E_{P^*}[f(x,\xi)] \leq (\geq)\ E_P[f(x,\xi)]$

  Equivalent to $Max(Min)_{P' \in \boldsymbol{P}}\ E_{P'}[f(x,\xi)]$
- Procedures:
  - Generalized programming (subproblems to generate weights on $\xi \in \Xi$)
  - Use of convexity properties
  - Finite support (but often non-convex subproblems)
- Direct interpretation for RO: Interpret Ξ as $\boldsymbol{P}$

Combining: When to Use What?

- Risk-neutral expectation
  - Repeated (often), Complete markets (after transformation) and discounting
  - Distribution from fundamentals
- Traditional expected utility
  - Can define function, incomplete market
- "Worst-case" robust or given probability
  - Little information, only survivability counts
- Competition and distribution domains
  - Allows consistent view from risk-neutral to "worst case"

Conclusions

- Traditional stochastic optimization and robust optimization can be viewed in same framework
- Can model decision problems in either framework
- Problems when mis-interpreting one situation to the other
- View of competition and distributions allows broad perspective