Robust Portfolio Selection

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Markets and portfolios

- Discrete time market with n assets
- Historical market returns $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N \in \mathbb{R}^n$
- Investment:
  - ϕ_i: fraction of wealth invested in asset i
  - $\phi \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} \phi_i = 1$

Goal: Portfolio that maximizes return and minimizes risk

- Have define risk and return
- Two standard methodologies
  - Parametric methods: e.g. $\mathbf{R}$ IID Normal
  - Non-parametric methods: Data rules!

Markowitz portfolio selection model

- Model class: Returns $\mathbf{R}$ are IID $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- Model selection: Maximum likelihood estimation
  - $\hat{\boldsymbol{\mu}} = \frac{1}{N}\sum_{t=1}^{N} \mathbf{R}_t$
  - $\hat{\boldsymbol{\Sigma}} = \frac{1}{N-1}\sum_{t=1}^{N}(\mathbf{R}_t - \hat{\boldsymbol{\mu}})(\mathbf{R}_t - \hat{\boldsymbol{\mu}})^T$
- Portfolio ϕ
  - return: Expected return $\boldsymbol{\mu}^T\phi$
  - risk: Variance $\phi^T\boldsymbol{\Sigma}\phi$
- Portfolio selection problem

$$\begin{array}{rl} \max & \boldsymbol{\mu}^T\phi - \lambda\phi^T\boldsymbol{\Sigma}\phi, \\ \text{s.t.} & \mathbf{1}^T\phi = 1. \end{array}$$

Sharpe portfolio selection model

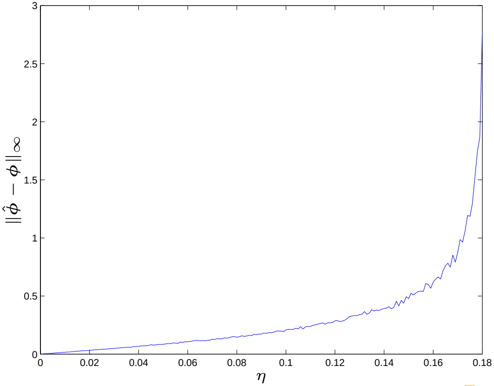
- $(n+1)$-th asset is riskless asset: return r_f
- Sharpe optimal portfolio

$$\left.\begin{array}{rl} \max & \frac{(\boldsymbol{\mu}-r_f\mathbf{1})^T\boldsymbol{\phi}}{\sqrt{\boldsymbol{\phi}^T\boldsymbol{\Sigma}\boldsymbol{\phi}}} \\ \text{s.t.} & \mathbf{1}^T\boldsymbol{\phi} = 1. \end{array}\right\} \quad \Rightarrow \quad \boldsymbol{\phi}^* = \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1})}$$

- Optimal portfolio: $\alpha e_{n+1} + (1-\alpha)\boldsymbol{\phi}^*$
- Great theoretical success: Capital Asset Pricing Model
- Unreliable practical performance: Parameter uncertainty!

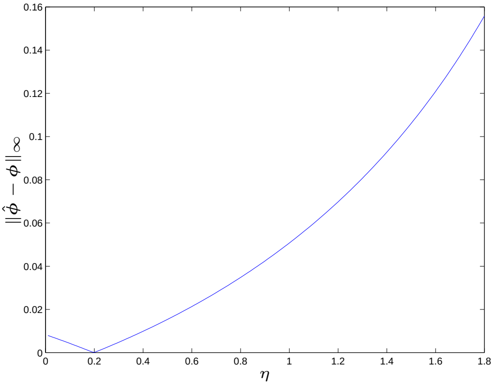
Instability of Sharpe optimal portfolio: error in $\boldsymbol{\mu}$

- Simple example $n = 2$
- Error in $\boldsymbol{\mu}$: $\eta = \frac{\|\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}\|}{\|\boldsymbol{\mu}\|}$
- Error in ϕ: $\|\hat{\phi} - \phi\|_\infty$
- Averaged over $N = 1000$ samples for each value of η

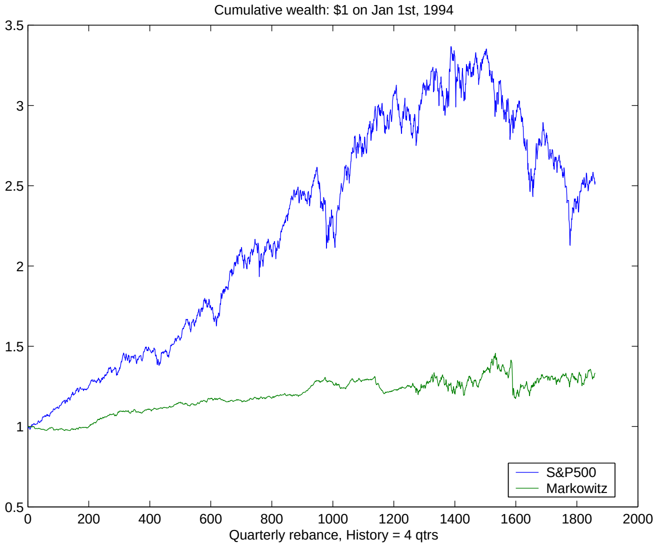


Instability of Sharpe optimal portfolio: error in $\boldsymbol{\Sigma}$

- Simple example $n = 2$
- Error in σ_1: $\eta = \frac{\|\hat{\sigma}_1 - \sigma_1\|}{\|\sigma_1\|}$
- Error in ϕ: $\|\hat{\phi} - \phi\|_\infty$
- Averaged over $N = 1000$ samples for each value of η



Sharpe optimal investment on S&P 500 assets



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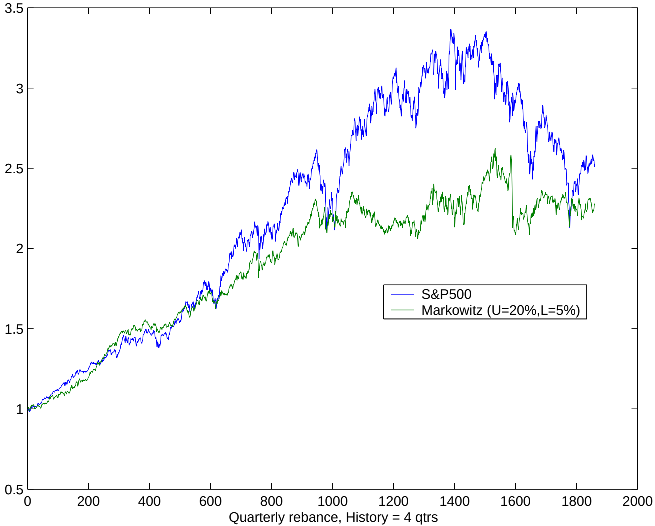

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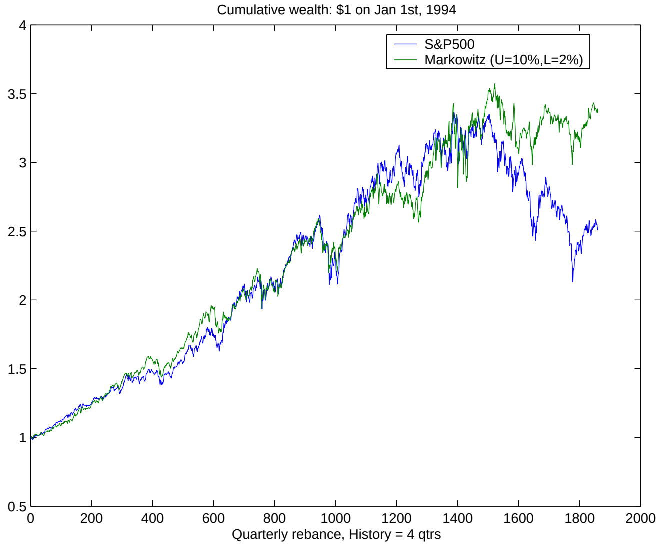

Methods to handle parameter uncertainty

- Solutions:
  - Bounds: Chopra (1993), Frost & Savarino (1988), Jagannathan & Ma (2003)
  - James-Stein estimates for the mean: Chopra et al (1993)
  - Bayesian estimation: Chopra (1993), Frost et al (1986), Black-Litterman
  - Resampling $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: Michaud (1989)
  - Stochastic programming: Ziemba & Mulvey (1998)

- These approaches
  - Do not model parameter uncertainty
  - Do not provide performance guarantees
  - Curse of dimensionality with stochastic programming

Simple robust portfolio selection

- Assume: covariance matrix $\boldsymbol{\Sigma}$ is known
- Mean return vector belongs to the uncertainty set

$$S_m = \left\{ \boldsymbol{\mu} : |\mu_i - \hat{\mu}_i| \leq \epsilon_i \right\}$$

- Minimax problem (Ben-Tal & Nmeirovski, El Ghaoui et. al.)

$$\begin{array}{rl} \max & \min_{\boldsymbol{\mu} \in S_m} \{\boldsymbol{\mu}^T \boldsymbol{\phi}\} - \lambda \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi} \\ \text{s.t.} & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

- Inner min: $\min_{\{\mu_i : |\mu_i - \hat{\mu}_i| \leq \epsilon_i\}} \{\mu_i \phi_i\} = (\hat{\mu}_i - \mathrm{sign}(\phi_i)\epsilon_i)\phi_i$
- First interpretation: modification of the return vector

$$\begin{array}{rl} \text{mmax} & (\hat{\boldsymbol{\mu}} - \mathrm{sign}(\phi)\epsilon)^T \boldsymbol{\phi} - \lambda \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi} \\ \text{s.t.} & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Simple robust portfolio selection

- Assume: covariance matrix $\boldsymbol{\Sigma}$ is known
- Mean return vector belongs to the uncertainty set

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- Inner min: $\min_{\{\mu_i : |\mu_i - \hat{\mu}_i| \leq \epsilon_i\}} \{\mu_i \phi_i\} = (\hat{\mu}_i - \mathrm{sign}(\phi_i) \epsilon_i) \phi_i$
- Second interpretation: modification of the covariance matrix

$$\begin{array}{rl} \max & \hat{\boldsymbol{\mu}}^T \boldsymbol{\phi} - \lambda \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi} - \boldsymbol{\phi}^T \, \mathbf{diag}(\frac{\epsilon_i}{|\phi_i|}) \boldsymbol{\phi} \\ \text{s.t.} & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Robust portfolio selection

- ω-confidence regions

$$
\begin{aligned}
S_\mu &= \left\{\boldsymbol{\mu} : \|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\| \leq c_\mu(\omega)\right\} \\
S_\sigma &= \left\{\boldsymbol{\Sigma} : \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}}(\mathbf{I} + \boldsymbol{\Delta})^{-1}\hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}}, \|\boldsymbol{\Delta}\| \leq c_\sigma(\omega)\right\}
\end{aligned}
$$

- Robust Sharpe optimal portfolio problem

$$
\begin{array}{ll}
\max & \min_{\boldsymbol{\mu} \in S_\mu, \boldsymbol{\Sigma} \in S_\mu} \left\{ \frac{(\boldsymbol{\mu} - r_f \mathbf{1})^T \boldsymbol{\phi}}{\sqrt{\boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}}} \right\} \\
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\text{s.t.} & \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{array}
$$

- Reformulation

$$
\begin{array}{ll}
\min & \max_{\boldsymbol{\Sigma}\in \mathcal{S}_\mu} \left\{\boldsymbol{\phi}^T \boldsymbol{\Sigma}\boldsymbol{\phi}\right\} \\
\text{s.t.} & \min_{\boldsymbol{\mu}\in \mathcal{S}_\mu} \left\{(\boldsymbol{\mu} - r_f \mathbf{1})^T \boldsymbol{\phi}\right\} \geq 1, \\
& \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{array}
$$

Robust portfolio selection

- ω-confidence regions

$$\begin{aligned} S_\mu &= \left\{\boldsymbol{\mu} : \|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\| \leq c_\mu(\omega)\right\} \\ S_\sigma &= \left\{\boldsymbol{\Sigma} : \boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}}(\mathbf{I} + \boldsymbol{\Delta})^{-1}\hat{\boldsymbol{\Sigma}}^{-\frac{1}{2}}, \|\boldsymbol{\Delta}\| \leq c_\sigma(\omega)\right\} \end{aligned}$$

- Robust Sharpe optimal portfolio problem

$$\begin{array}{ll} \max & \min_{\boldsymbol{\mu} \in S_\mu, \boldsymbol{\Sigma} \in S_\mu} \left\{ \frac{(\boldsymbol{\mu} - r_f \mathbf{1})^T \boldsymbol{\phi}}{\sqrt{\boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}}} \right\} \\ \text{s.t.} & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

- Reformulation

$$\begin{array}{ll} \min & \frac{1}{1 - c_{\sigma(\omega)}} \left\{ \boldsymbol{\phi}^T \hat{\boldsymbol{\Sigma}} \boldsymbol{\phi} \right\} \\ \text{s.t.} & (\hat{\boldsymbol{\mu}} - r_f \mathbf{1})^T \boldsymbol{\phi} - c_\mu(\omega) \|\boldsymbol{\phi}\| \geq 1, \\ & \mathbf{1}^T \boldsymbol{\phi} - \kappa = 0, \\ & \kappa \geq 0. \end{array}$$

- Second-order cone program: Efficiently solvable!

Theoretical properties of robust optimization

- Analog of Markowitz problem for ambiguous markets: Gilboa & Schmeidler (1989), Hansen & Sargent (2001)
- The 1-fund theorem extends: CAPM-type results possible
- Probabilistic guarantee on performance
  - pick confidence level ω
  - $\phi^*(\omega)$: solution of the robust max Sharpe ratio problem
  - $s^*(\omega)$: value of the robust max Sharpe ratio problem
  - Realized Sharpe ratio of $\phi^*(\omega) \geq s^*(\omega)$ with probability ω
- **Implementation**: How can we make this method scale ?

Large scale problems

- Factor model: $\mathbf{r} = \boldsymbol{\mu} + \mathbf{V}^T\mathbf{f} + \boldsymbol{\epsilon}$
  - mean return: $\boldsymbol{\mu} \in \mathbf{R}^n$ (n assets)
  - factor returns: $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \mathbf{F})$, $\mathbf{F} \in \mathbf{R}^{m \times m}$ (m factors)
  - factor loading matrix: $\mathbf{V} \in \mathbf{R}^{m \times n}$
  - residual returns: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$, $\mathbf{D} \succeq \mathbf{0}$
- Current strategy: Use point estimates of $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{F}, \mathbf{D})$

Large scale problems

- Robust factor model: $\mathbf{r} = \boldsymbol{\mu} + \mathbf{V}^T \mathbf{f} + \boldsymbol{\epsilon}$
  - mean return: $\boldsymbol{\mu} \in S_m = \{\boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\nu} : |\nu_i| \leq \gamma_i\}$
  - factor returns: $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \mathbf{F})$, $\mathbf{F}$ known and stable
  - factor loading matrix:
    $\mathbf{V} \in S_v = \left\{\mathbf{V} = \mathbf{V}_0 + \mathbf{W} : \|\mathbf{W}_i\|_g \stackrel{\Delta}{=} \sqrt{\mathbf{W}_i^T \mathbf{G} \mathbf{W}_i} \leq \rho_i\}\right\}$
  - residual returns: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$,
    $\mathbf{D} \in S_d = \{\mathbf{D} = \mathbf{diag}(\mathbf{d}) : \underline{d}_i \leq d_i \leq \overline{d}_i\}$
- Proposal: Solve a robust problem for the ω-confidence region

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- Proposal: Solve a robust problem for the ω-confidence region
- Steps in the robust recipe
  - Given return data $\{(\mathbf{r}^t, \mathbf{f}^t) : t = 1, \ldots, k\}$ and confidence level ω parametrize the uncertainty structure.
  - Given a particular choice of (S_d, S_m, S_v), solve a robust portfolio selection problem

Robust Sharpe ratio problems

- For a fixed $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{D})$ the market return

$$\mathbf{r} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{V}^T\mathbf{F}\mathbf{V} + \mathbf{D}) \quad \Rightarrow \quad r_\phi \sim \mathcal{N}(\boldsymbol{\mu}^T\phi, \phi^T(\mathbf{V}^T\mathbf{F}\mathbf{V} + \mathbf{D})\phi)$$

- Robust maximum Sharpe ratio portfolio selection:

$$\begin{array}{ll} \max & \min_{\{\boldsymbol{\mu}\in S_m, \mathbf{V}\in S_v, \mathbf{D}\in S_d\}} \frac{(\boldsymbol{\mu} - r_f\mathbf{1})^T\boldsymbol{\phi}}{\sqrt{\boldsymbol{\phi}^T(\mathbf{V}^T\mathbf{F}\mathbf{V}+\mathbf{D})\boldsymbol{\phi}}}, \\ \text{s.t.} & \mathbf{1}^T\boldsymbol{\phi} = 1. \end{array}$$

- Reformulation

$$\begin{array}{ll} \min & \max_{\{\mathbf{V}\in S_v, \mathbf{D}\in S_d\}} \{\phi^T(\mathbf{V}^T\mathbf{F}\mathbf{V} + \mathbf{D})\phi\}, \\ \text{s.t.} & \min_{\{\boldsymbol{\mu}\in S_m\}} \{(\boldsymbol{\mu} - r_f\mathbf{1})^T\phi\} \geq 1, \\ & \mathbf{1}^T\boldsymbol{\phi} - \kappa = 0, \\ & \kappa \geq 0. \end{array}$$

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Reformulating robust variance constraint

- $\phi^T(\mathbf{V}^T\mathbf{F}\mathbf{V}+\mathbf{D})\phi = \|\mathbf{F}^{\frac{1}{2}}\mathbf{V}\phi\|^2 + \phi^T\mathbf{D}\phi \leq \|\mathbf{F}^{\frac{1}{2}}\mathbf{V}\phi\|^2 + \phi^T\bar{\mathbf{D}}\phi$
- Worst case variance: all $\mathbf{W}_i$ must be aligned

$$\begin{aligned}
\max_{\mathbf{V}\in S_v} \left\|\mathbf{F}^{\frac{1}{2}}\mathbf{V}\phi\right\|^2 &= \max_{\{\mathbf{W}:\|\mathbf{W}_i\|_g\leq\rho_i\}} \left\|\mathbf{F}^{\frac{1}{2}}(\mathbf{V}_0\phi + \sum_{i=1}^{n}\phi_i\mathbf{W}_i)\right\|^2 \\
&= \max_{\{\mathbf{v}:\|\mathbf{v}\|_g\leq 1\}} \|\mathbf{F}^{\frac{1}{2}}(\mathbf{V}_0\phi + \rho\mathbf{v})\|^2, \quad \rho = \sum_i \rho_i\,|\phi_i|
\end{aligned}$$

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- $\mathcal{S}$-Lemma: duality for quadratic forms

$$\|F^{\frac{1}{2}}(\mathbf{V}_0\phi+\rho\mathbf{v})\|^2 \leq \nu \quad \forall\ \|\mathbf{v}\| \leq 1$$
$$\Updownarrow$$
$$\exists\ \tau \geq 0 : \mathbf{M} = \begin{bmatrix} \nu-\tau-\phi^T\mathbf{V}_0^T\mathbf{F}\mathbf{V}_0\phi & r\mathbf{F}\mathbf{V}_0\phi \\ r\phi^T\mathbf{F}\mathbf{V}_0 & \tau\mathbf{G}-r^2\mathbf{F} \end{bmatrix} \succeq \mathbf{0}$$

Reformulating robust variance constraint

- $\phi^T(\mathbf{V}^T\mathbf{F}\mathbf{V}+\mathbf{D})\phi = \|\mathbf{F}^{\frac{1}{2}}\mathbf{V}\phi\|^2 + \phi^T\mathbf{D}\phi \leq \|\mathbf{F}^{\frac{1}{2}}\mathbf{V}\phi\|^2 + \phi^T\bar{\mathbf{D}}\phi$
- Worst case variance: all $\mathbf{W}_i$ must be aligned

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$$\Updownarrow$$

$$\exists\ \tau \geq 0 : \mathbf{M} = \begin{bmatrix} \nu - \tau - \phi^T\mathbf{V}_0^T\mathbf{F}\mathbf{V}_0\phi & r\mathbf{F}\mathbf{V}_0\phi \\ r\phi^T\mathbf{F}\mathbf{V}_0 & \tau\mathbf{G} - r^2\mathbf{F} \end{bmatrix} \succeq \mathbf{0}$$

- Linear algebra manipulations: $\mathbf{M} \succeq \mathbf{0}$ if and only if

$$\begin{aligned} \mathbf{w} &= \mathbf{Q}^T\mathbf{V}_0\phi \\ \mathbf{1}^T\mathbf{t} &\leq \nu - \tau \\ w_i^2 &\leq t_i(1-\sigma\lambda_i), \quad \forall i \end{aligned}$$

Second-order cone constraints

Reformulation to SOCPs

- Collecting all the bits and pieces ...

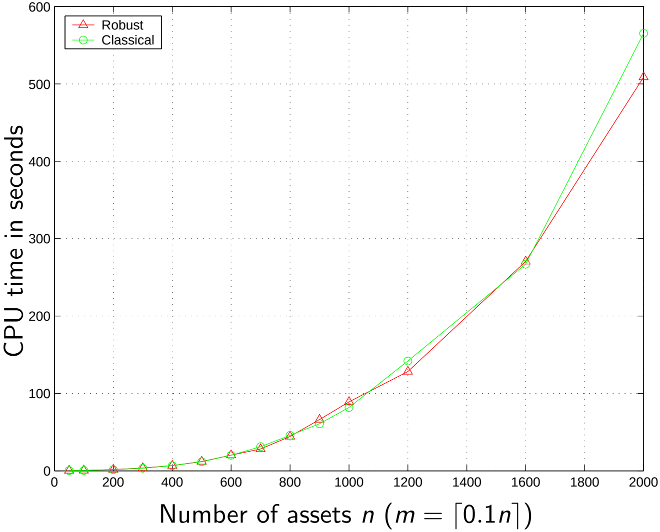
$$
\begin{array}{lrcl}
\min & \multicolumn{3}{l}{\tau + \mathbf{1}^T \mathbf{t} + \delta} \\
\text{s.t.} & \mathbf{w} & = & \mathbf{Q}^T \mathbf{V}_0 \phi \\
& r & = & \sum_{i=1}^{n} \rho_i \, |\phi_i| \\
& \mathbf{1}^T \phi & = & 1 \\
& \mathbf{1}^T \mathbf{t} & \leq & \nu - \tau \\
& w_i^2 & \leq & t_i(1 - \sigma \lambda_i), \quad i = 1, \ldots, m \\
& r^2 & \leq & \sigma \tau \\
& \sigma & \leq & \frac{1}{\lambda_{\max}(\mathbf{H})} \\
& \phi^T \mathbf{D} \phi & \leq & \delta
\end{array}
$$

- second-order cone program "comparable" to convex QPs.

Summary

- Markowitz-strategies sensitive to parameter perturbation
- Robust strategies correct this via *uncertainty sets*
- Uncertainty sets given by data and desired confidence level ω
- Resulting SOCP can be solved efficiently
- Robust investment strategy
  - Collect data: asset returns $\mathbf{r}$ and factor returns $\mathbf{f}$.
  - Compute the least squares estimates $\boldsymbol{\mu}_0$, $\mathbf{V}_0$ and $\mathbf{F}$
  - Choose a confidence level ω and define S_m, S_v, and S_d
  - Solve the SOCP corresponding to the robust problem of interest
- Our modifications
  - Replace original problem by the robust version.
  - Choice of ω is dictates risk averseness

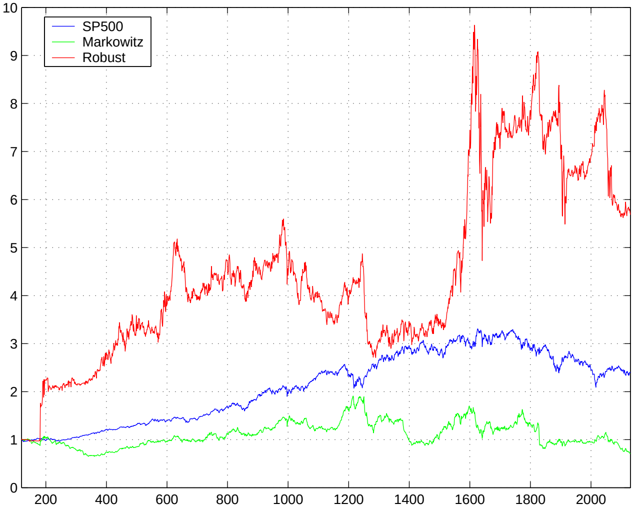
Computational time



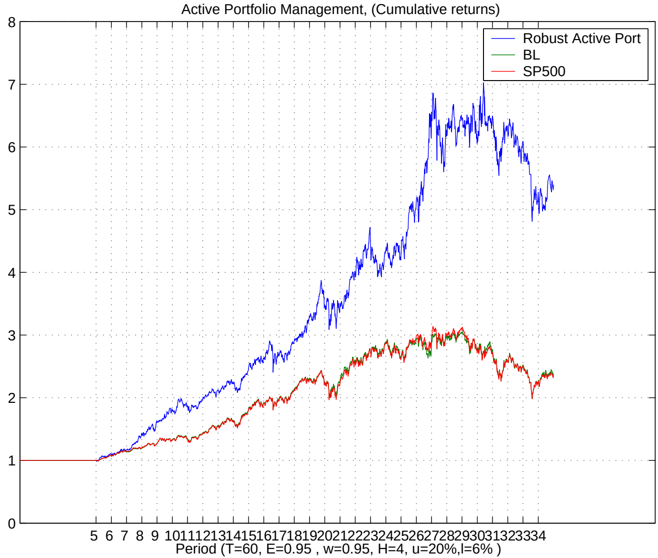
Performance on real market data

- Market period: December 31st, 1993 - March 26, 2003
- Assets: Constituents of S&P 500 index
- Factors: DJA, SPX, NDX, RUT, TYX + eigenvectors of $\mathbf{\Sigma}_R$
- Experimental procedure:
  - Investment period: $p = 60$ days
  - In each period, estimate $\mathbf{\Sigma}_R$ and keep eigenvectors corresponding to the top $E/100$ eigenvalues.
  - Estimate $\mathbf{V}_0$, $\boldsymbol{\mu}_0$, $\mathbf{G}$, $\boldsymbol{\rho}$ and γ over a $h = 4$ period history
  - Set $\overline{d}_i = s_i^2$ and $r_f =$ average T-bill rate
  - Compute robust (resp. Markowitz) portfolio ϕ_r^t (resp. ϕ_m^t) selected by robust (resp. Markowitz) Sharpe ratio problem
  - Re-balance ϕ_r^t and ϕ_m^t at the beginning of period .
- Out-of-sample testing but suffers from survivorship bias
- Myopic investment strategy

Plain vanilla robust portfolio selection



Robust index tracking and active portfolio selection



Multiperiod investment

- Return $\mathbf{R}_t = \boldsymbol{\pi} + \boldsymbol{\Pi}\mathbf{z}_t$
- Price $\mathbf{P}_t = \mathbf{P}_0 + \sum_{\tau=1}^{t} \mathbf{R}_\tau = \mathbf{P}_0 + \boldsymbol{\pi} t + \boldsymbol{\Pi}(\sum_{\tau=1}^{t} \mathbf{z}_\tau)$
- Constraints: $\mathbf{P}_0^T \phi_0 = 1, \quad \mathbf{P}_t^T \psi_t \leq 0,\ \psi_t = \phi_t - \phi_{t-1}$
- Objective: $\mathbf{P}_N^T \phi_N$
- Uncertainty structure: Bertsimas & Thiele

$$\mathcal{Z} = \left\{ (\mathbf{z}_1, \ldots, \mathbf{z}_N) : \| \sum_{k=1}^{\tau} \mathbf{z}_k \| \leq \sqrt{\tau}, \tau = 1, \ldots, N \right\}$$

- Robust optimization problem

$$\begin{array}{rl} \max & \min_{\mathbf{z} \in \mathcal{Z}} \left\{ \mathbf{P}_N(\mathbf{z})^T (\phi_0 + \sum_{t=1}^{N} \psi_t) \right\} \\ \text{s.t.} & \mathbf{P}_t(\mathbf{z})^T \psi_t \leq 0, \quad \mathbf{z} \in \mathcal{Z}, t = 1, \ldots, N, \\ & \mathbf{P}_0^T \phi_0 = 1. \end{array}$$

- An SOCP ... can be solved well.
- Make all decisions at time 0: typically conservative

Multiperiod investment: Adjustable robust methods

- $\psi_t = \psi_t^{(0)} + \sum_{\tau=1}^{t} \mathbf{F}_{t\tau} \mathbf{R}_\tau(\mathbf{z})$
- Robust constraint

$$\mathbf{P}_t(\mathbf{z})^T \psi_t = \mathbf{P}_t(\mathbf{z})^T \psi_t^{(0)} + \sum_{\tau=1}^{t} \mathbf{P}_t(\mathbf{z}) F_{t\tau} \mathbf{R}_\tau(\mathbf{z}) \leq 0, \quad \mathbf{z} \in \mathcal{Z}$$

- semidefinite constraint ... hard
- performance significantly superior

Robust mean-variance portfolio selection

- Robust Markowitz problem: explicitly models errors
  - The uncertainty set parametrized by confidence level ω
  - The robust optmization problem is an SOCP ... "easy"
  - No parameters need to be tuned
- Robust index tracking strategy
  - Works for capitalization weighted indices
  - Tracks the index with far fewer assets
  - The standard deviation is comparable to the index
- Robust active investment strategy
  - Does have some tunable parameters
  - Outperform S&P 500 by at least 60% using $\approx$ 70 stocks
  - Have a strategy to manage transaction costs
- Current focus
  - Better multiperiod models
  - Asset liability management