OPTIMAL AND ROBUST CONTROL OF THE NONLINEAR ALGEBRAIC GROWTH IN A BLASIUS BOUNDARY LAYER

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Algebraic instability in parallel flow

(Beware of Squire's theorem!)

Linearized Navier-Stokes equations (about $u = u_0(y)$):

$$\delta u_x + \delta v_y + \delta w_z = 0$$
$$\delta u_t + u_0 \delta u_x + u_{0y} \delta v + \delta p_x = \nu \Delta_2 \delta u$$
$$\delta v_t + u_0 \delta v_x + \delta p_y = \nu \Delta_2 \delta v$$
$$\delta w_t + u_0 \delta w_x + \delta p_z = \nu \Delta_2 \delta w$$

after Fourier expansion:

$$i\alpha U + i\beta W - V_y = 0$$
$$i(\omega - u_0 \alpha) U + u_{0y} V - i\alpha P = \nu \Delta_2 U$$
$$i(\omega - u_0 \alpha) V + P_y = \nu \Delta_2 V$$
$$i(\omega - u_0 \alpha) W - i\beta P = \nu \Delta_2 W$$

In the variables V and $\eta = i\alpha W - i\beta U$:

$$i(\omega - \alpha u_0)\Delta_2 V + i\alpha\, u_{0,yy} V = \nu \Delta_2 \Delta_2 V$$

(Orr - Sommerfeld)

$$i(\omega - \alpha u_0)\eta - \nu \Delta_2 \eta = i\beta\, u_{0,y} V$$

(Squire)

"There are no unstable eigenvalues in the three-dimensional problem that are not already present in the Orr-Sommerfeld equation" (Squire's theorem

Does eigenvalue analysis tell the whole story?

Inviscid approximation :

$$(\omega - \alpha u_0)\Delta_2 V + \alpha\, u_{0,YY} V = 0 \qquad \text{(Rayleigh)}$$

$$(\omega - \alpha u_0)\,\eta = i\beta\, u_{0,Y} V \qquad \text{(Landahl, etc.)}$$

Real eigenvalues of the Rayleigh equation are double real eigenvalues of the three-dimensional problem. Double real eigenvalues correspond to instability.

In the time domain :

$$\eta(Y_c, t) = e^{i\omega t} \int i\beta u_{0,Y} V \, dt \qquad (u_0(Y_c) = \omega/\alpha)$$

With viscosity included, the inviscid instability becomes a transient algebraic growth followed by exponential viscous decay.

(There also exists a two-dimension algebraic growth, described by Trefethen, Henningson, Reddy, etc.)

$$\boxed{\text{Is the algebraic growth still transient in a non-parallel boundary layer?}}$$

TRANSIENT GROWTH

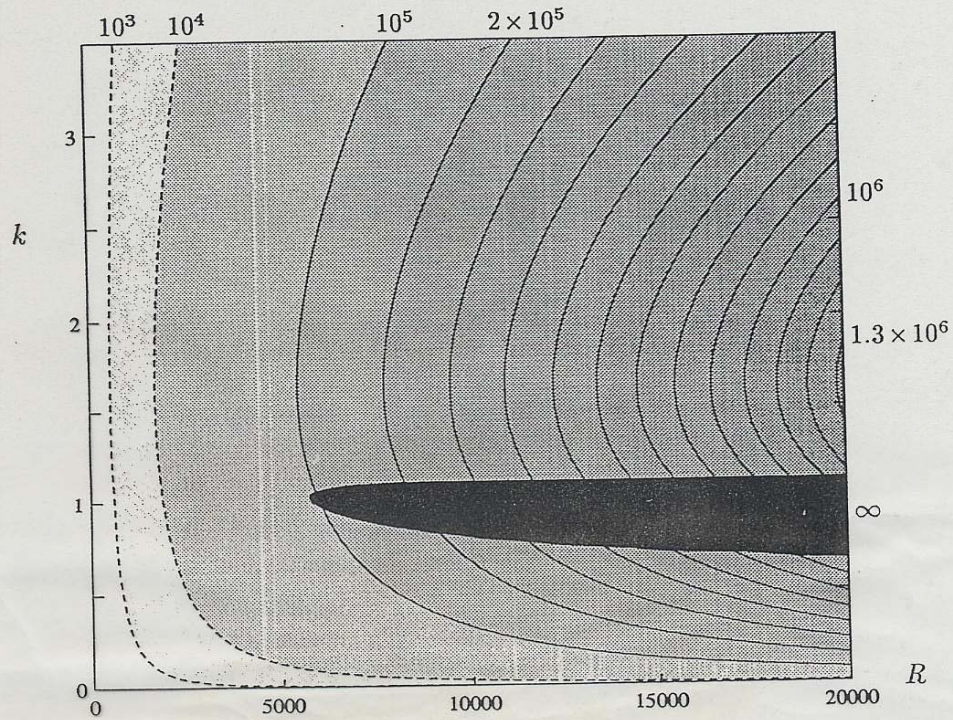


Fig. 2. Maximum possible amplification of 3D perturbations in linearized Poiseuille flow as a function of Reynolds number R and x-z wave number magnitude $k=\sqrt{\alpha^2+\beta^2}$. In the black region, with leftmost point $R=5772$, unstable eigenmodes exist and unbounded amplification is possible. The contours outside that region correspond to finite amplification factors of 10^3, 10^4 (dashed), and $1\times10^5, 2\times10^5, \ldots, 1.3\times10^6$. For example, amplification by a factor of 1000 is possible for all $R\geq549$. In the laboratory, transition to turbulence is observed at $R\approx1000$. The analogous picture for Couette flow looks qualitatively similar except that there is no black region.

Initial work: adjoint method for stability analysis.

Receptivity of quasi parallel TS waves
(Hill 1995) 1d adjoint

Receptivity of Görtler vortices
(Luchini & Bottaro 1998) 2d adjoint

Transient growth of b.l. perturbations
(Luchini 1997 - 2000
Andersson, Berggren, Henningson 1997 - 1999)
2d adjoint optimization

Optimal control of b.l. perturbations
(Cathalifaud & Luchini 2000
Zuccher, Luchini & Bottaro 2001 - 2002)
2d adjoint nonlinear control

Many other authors:

- shape optimization,
- weather predictions,
- non-fluid-dynamic control problems.

ADJOINT

$$\underline{x}_{i+1} = \underline{\underline{A}}\,\underline{x}_i$$

$$\underline{x}_0 \rightarrow \boxed{A} \rightarrow \boxed{A} \rightarrow \boxed{A} \rightarrow \boxed{A} \rightarrow \boxed{A} \rightarrow \boxed{A} \rightarrow \boxed{A} \rightarrow \underline{x}_n$$

→ DIRECT →

← ADJOINT ←

$$\underline{v} = \frac{\partial \Phi}{\partial \underline{x}_0} \leftarrow \boxed{A} \leftarrow \boxed{A} \leftarrow \boxed{A} \leftarrow \boxed{A} \leftarrow \boxed{A} \leftarrow \boxed{A} \leftarrow \boxed{A} \leftarrow \underline{\varphi} = \frac{\partial \Phi}{\partial \underline{x}_n}$$

- The adjoint runs backwards in time
- Computation time is comparable

OPTIMIZATION:

Once the sensitivity is known, optimization just requires a gradient algorithm.

Algebraic growth in a boundary layer ("Klebanoff mode")

3D, low-frequency fluctuations in boundary-layer thickness observed in a wind tunnel by Klebanoff (1962)

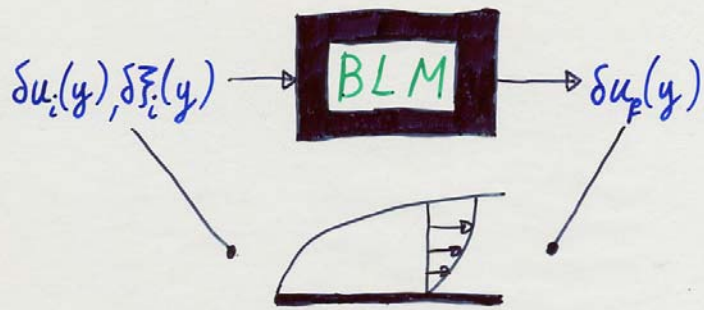
(and, actually, by G. I. Taylor 1938)

Never explained before algebraic growth was considered.

It occurs at a Reynolds number where all modes are stable.

"Bypass transition"

"BLACK BOX" APPROACH TO OPTIMIZATION



$\left(\delta\xi = (u_0 \delta w)_y - i\alpha u_0 \delta v \text{ is the initial condition appropriate to the leading-edge singularity}\right)$

$$g = \frac{E_f}{E_i} = \frac{\int_0^\infty |\delta u_f|^2 + \frac{1}{R}\left(|\delta v_f|^2 + |\delta w_f|^2\right) dy}{\int_0^\infty |\delta u_i|^2 + \frac{1}{R}\left(|\delta v_i|^2 + |\delta w_i|^2\right) dy} = MAX$$

$$R \to \infty: \quad g = R \frac{\int_0^\infty |\delta u_f|^2 dy}{\int_0^\infty |\delta v_i|^2 + |\delta w_i|^2 dy} \qquad (\delta u_i = 0)$$

$$\int_0^\infty |\delta u_f|^2 dy = \delta u_f^* \underline{\underline{Q}}_2 \, \delta u_f$$

$$\int_0^\infty |\delta v_i|^2 + (\delta w_i|^2 dy = \delta\xi_i^* \underline{\underline{Q}}_1 \, \delta\xi$$

$$\delta u_f = \underline{\underline{U}} \, \delta\xi_i$$

$$g = \boxed{\frac{\delta\xi_i^* \underline{\underline{U}}^+ \underline{\underline{Q}}_2 \underline{\underline{U}} \, \delta\xi_i}{\delta\xi_i^* Q_1 \, \delta\xi_i}}$$

Boundary-layer model of the Klebanoff mode.

Since the Klebanoff mode is of low frequency and large longitudinal extension, it must be amenable to a steady, three-dimensional $\boxed{\text{boundary-layer model}}$:

$$
\text{BLM:}\quad
\begin{aligned}
&\delta u_x + \delta v_y + i\alpha\,\delta w = 0\\
&u_0\,\delta u_x + v_0\,\delta v_y + u_{0,x}\,\delta u + u_{0,y}\,\delta v = \delta u_{yy} - \alpha^2\delta u\\
&u_0\,\delta v_x + v_0\,\delta v_y + v_{0,x}\,\delta u + v_{0,y}\,\delta v + \delta p_y = \delta v_{yy} - \alpha^2\delta v\\
&u_0\,\delta w_x + v_0\,\delta w_y + i\alpha\,\delta p = \delta w_{yy} - \alpha^2\delta w
\end{aligned}
$$

$$
u = u_d/U_\infty,\quad v = v_d R^{1/2}/U_\infty,\quad w = w_d R^{1/2}/U_\infty
$$
$$
x = x_d/L,\quad y = y_d R^{1/2}/L,\quad \alpha = \alpha_d L/R^{1/2}
$$

Two-dimensional solutions of BLM are stable.
(Libby & Fox 1964)

Three-dimensional solutions of BLM are unstable for small α. (Luchini 1996).

Since there is no Reynolds number in BLM, the instability is Reynolds number-independent

Rayleigh quotient

$$g = \frac{\delta\xi_i \, \underline{\underline{U}}^+ \underline{\underline{Q}}_2 \underline{\underline{U}} \, \delta\xi_i}{\delta\xi_i \, \underline{\underline{Q}}_1 \, \delta\xi_i}$$

$\Rightarrow$ g_{MAX} is the largest eigenvalue of

$$\underline{\underline{Q}}_1^{-1} \underline{\underline{U}}^+ \underline{\underline{Q}}_2 \underline{\underline{U}}$$

Simple direct iteration:

$$\delta\xi_i^{(n+1)} = \underline{\underline{Q}}_1^{-1} \underline{\underline{U}}^+ \underline{\underline{Q}}_2 \underline{\underline{U}} \, \delta\xi^{(n)}$$

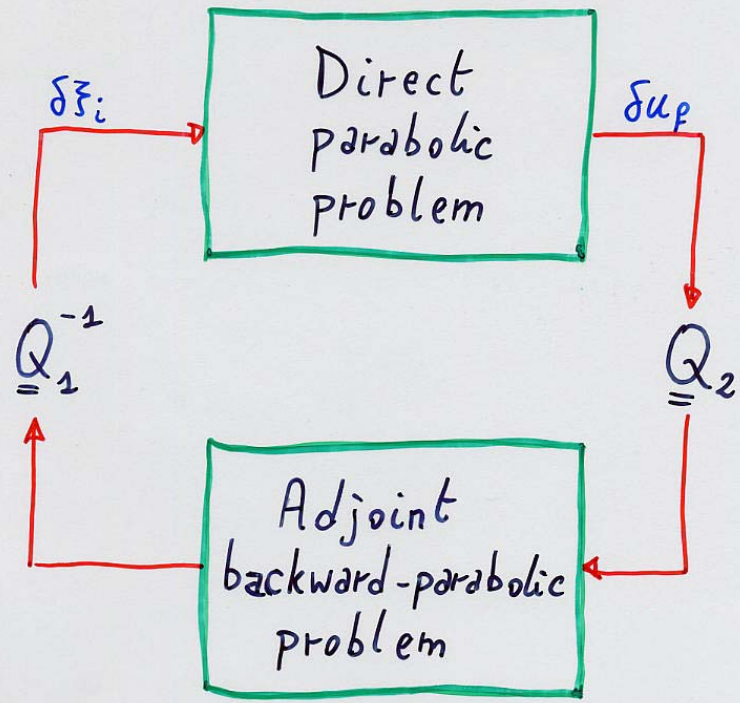
convenient if all the intermediate products are economically available.

$\underline{\underline{U}} \, \delta\xi_i^{(n)}$: output of a numerical simulation with initial condition $\delta\xi_i^{(n)}$.

$\underline{\underline{Q}}_2$: identity matrix.

$\underline{\underline{Q}}_1^{-1}$: $\alpha^2 - \frac{\partial^2}{\partial y^2}$ (Laplacian)

$\underline{\underline{U}}^+$: backward simulation of the adjoint differential problem (or difference problem) (Luchini & Bottaro, Görtler vortices: a backward-in-time approach to the receptivity problem. APS 1996)



- Guaranteed (and, actually, quick) convergence
- Gives both optimal perturbation and induced disturbance
- Green blocks common to the receptivity calculation of Luchini & Bottaro (1996)

Optimal perturbation

(alfa = 0.45)

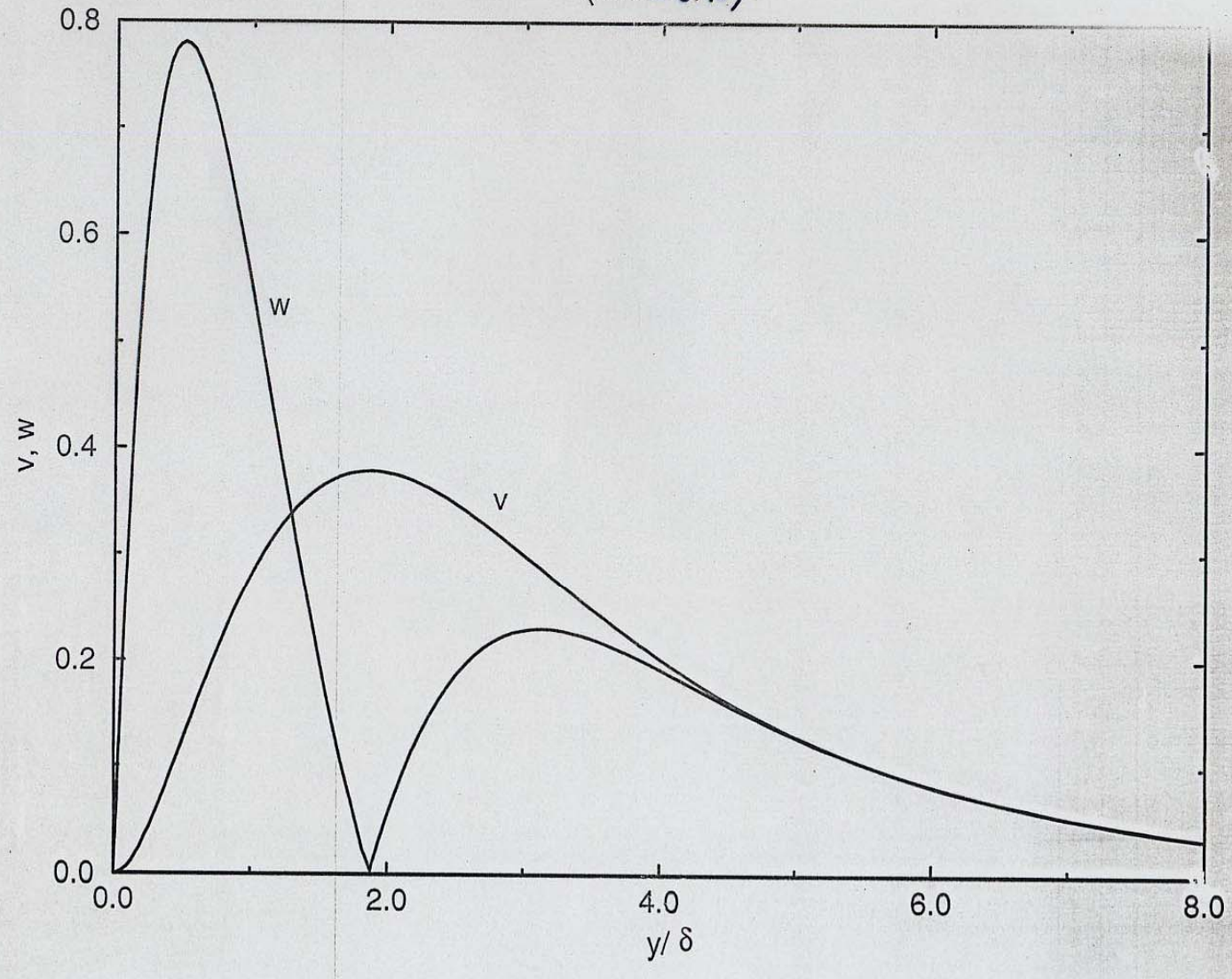
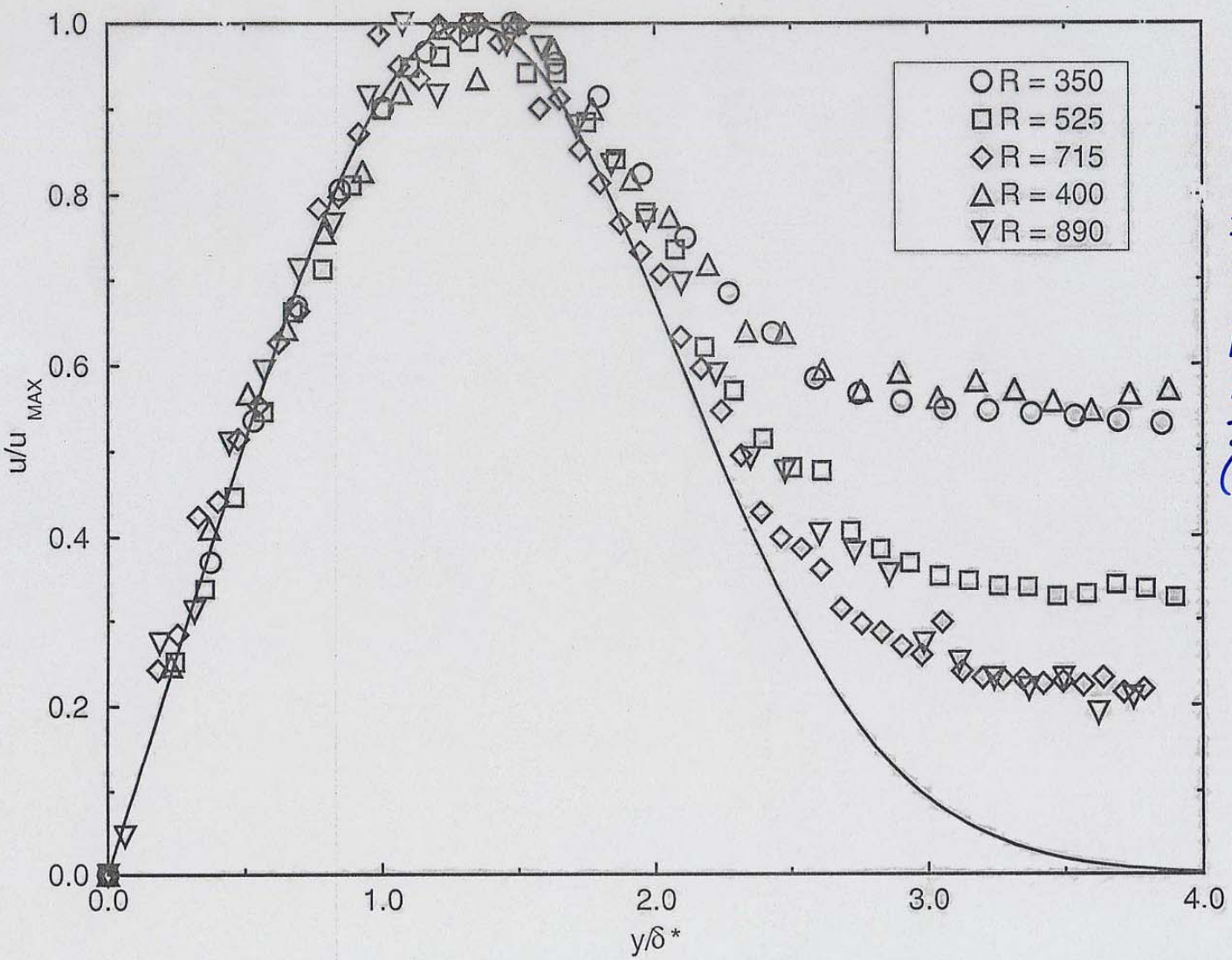
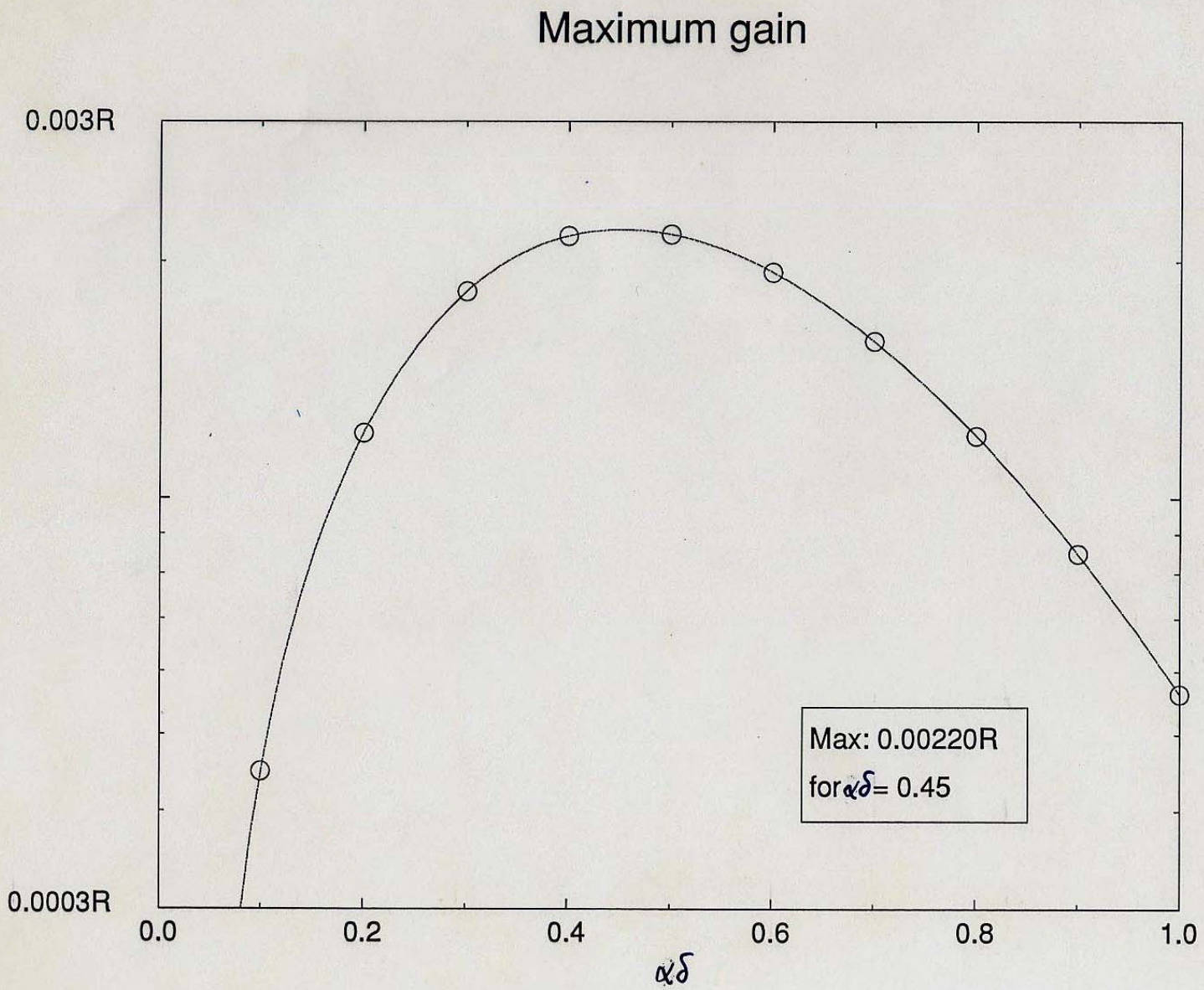


Fig. 5

Comparison with experimental data*



* From: Westin, Boiko, Klingmann, Kozlov & Alfredsson, J.F.M. 281 (1994) 193-218.

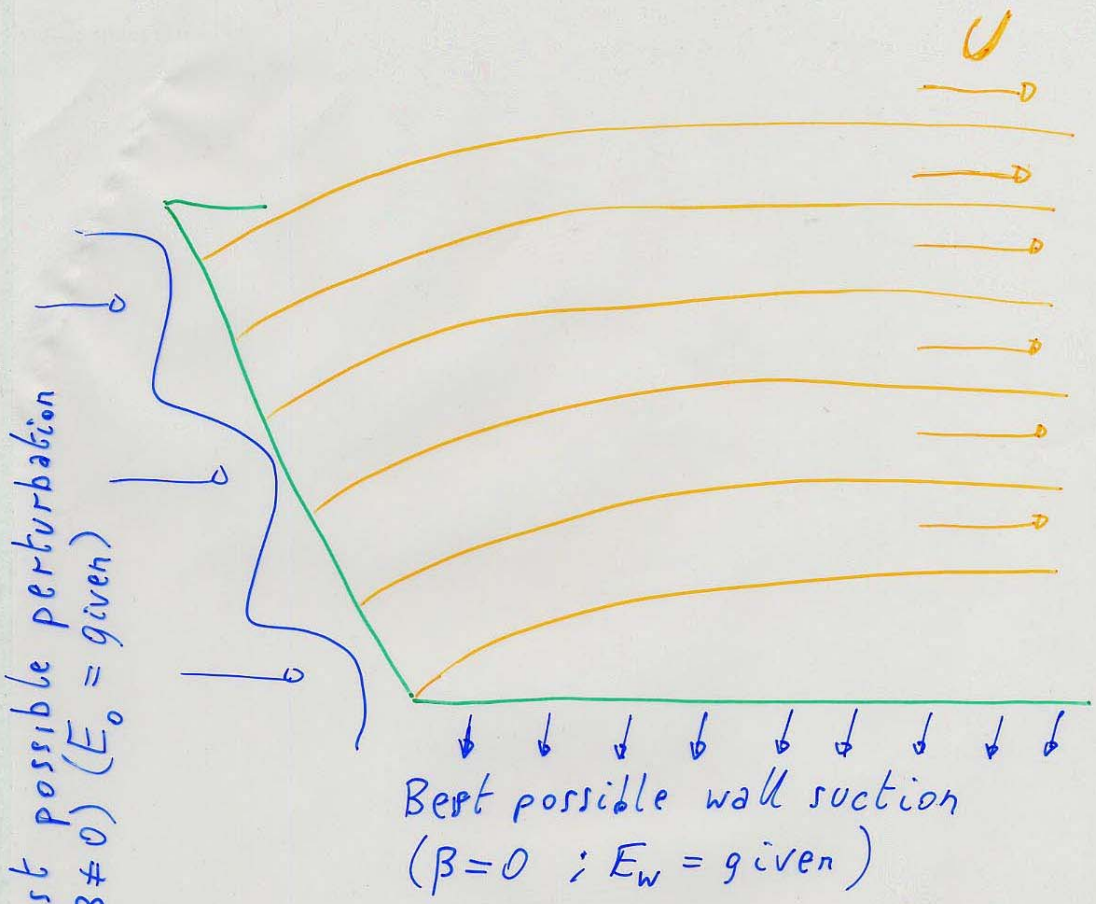

Maximum gain
0.003R
0.0003R
0.0
0.2
0.4
0.6
0.8
1.0
αδ
Max: 0.00220R
for αδ= 0.45

and now the nonlinear part...

S. Zuccher, A. Bottaro, P. Luchini: "Algebraic growth in a Blasius boundary layer: Nonlinear optimal disturbances", Eur. J. Mech. B/Fluids in press.

S. Zuccher, P. Luchini, A. Bottaro: "Algebraic growth in a Blasius boundary layer: Optimal and robust mean-suction control in the nonlinear regime", J. Fluid Mech. 513, 135-160 (2004).

Will suction counteract the transient growth ?



- $\omega = 0$
- 3 spatial variables

WHAT IS AN ADJOINT?
(nonlinear answer)

- complex conjugate transpose (matrix)
- dual space (operator)
- Lagrange multiplier (optimization problem)
- **Derivative of a massively compound function.**

Chain rule:

$$\underline{x}_1 = \underline{F}_1(\underline{x}_0)$$
$$\underline{x}_2 = \underline{F}_2(\underline{x}_1)$$
$$\underline{x}_3 = \underline{F}_3(\underline{x}_2)$$
$$\cdots\cdots$$
$$\underline{x}_n = \underline{F}_n(\underline{x}_{n-1})$$
$$g = \varphi(\underline{x}_0, \ldots \underline{x}_n)$$

$$\frac{\partial g}{\partial \underline{x}_0} = \frac{\partial g}{\partial \underline{x}_1} \cdot \frac{\partial \underline{F}_1}{\partial \underline{x}_0} + \frac{\partial \varphi}{\partial \underline{x}_0}$$
$$\frac{\partial g}{\partial \underline{x}_1} = \frac{\partial g}{\partial \underline{x}_2} \cdot \frac{\partial \underline{F}_2}{\partial \underline{x}_1} + \frac{\partial \varphi}{\partial \underline{x}_1}$$
$$\frac{\partial g}{\partial \underline{x}_2} = \frac{\partial g}{\partial \underline{x}_3} \cdot \frac{\partial \underline{F}_3}{\partial \underline{x}_2} + \frac{\partial \varphi}{\partial \underline{x}_2}$$
$$\cdots\cdots$$
$$\partial g / \partial \underline{x}_{n-1} = \frac{\partial g}{\partial \underline{x}_n} \cdot \frac{\partial \underline{F}_n}{\partial \underline{x}_{n-1}} + \frac{\partial \varphi}{\partial \underline{x}_{n-1}}$$
$$\partial g / \partial \underline{x}_n = \partial \varphi / \partial \underline{x}_n$$

How far can we go with gradient <u>vectors</u> (not matrices)?

Optimization

Objectives

⇒ Boundary layer, in the black–box fashion, receives boundary conditions and initial conditions as inputs: what is the most dangerous initial condition regarding transition (=which maximizes the perturbation energy)? **Optimal perturbation**

⇒ For the most dangerous initial condition, what is the best suction to be applied at the wall to minimize the perturbation energy? **Optimal control** and **robust control**

⇒ What happens if the initial energy is gradually increased and the **nonlinear** regime reached?

Methodology

☺ Use of an optimization technique based on the solution of the linear **adjoint equations** corresponding to the nonlinear direct ones.

Problem Formulation

3D incompressible and stationary boundary layer equations in conservative form:

$$\begin{aligned} u_x + v_y + w_z &= 0 \\ (uu)_x + (uv)_y + (uw)_z - u_{yy} - u_{zz} &= 0 \\ (uv)_x + (vv)_y + (vw)_z + p_y - v_{yy} - v_{zz} &= 0 \\ (uw)_x + (vw)_y + (ww)_z + p_x - w_{yy} - w_{zz} &= 0 \end{aligned}$$

u normalized with respect to U_∞, v and w normalized with respect to $Re^{-1/2}U_\infty$ ($Re = U_\infty L/\nu$)

initial conditions

$$\begin{aligned} u(0,y,z) &= 1 \\ v(0,y,z) &= v_0(y,z) \end{aligned}$$

boundary conditions

$$\begin{array}{llll} u = 0 & \text{at} \quad y = 0 & u = 1 & \text{for} \quad y \to \infty \\ v = v_W & \text{at} \quad y = 0 & w = 0 & \text{for} \quad y \to \infty \\ w = 0 & \text{at} \quad y = 0 & p = 0 & \text{for} \quad y \to \infty \end{array}$$

The flow field $\mathbf{V} = (u, v, w)$ is subdivided in two contributions $\mathbf{V}_0$ (independent of z) and $\bar{\mathbf{v}}$ (dependent on z)

$$\mathbf{V}(x,y,z) = \mathbf{V}_0(x,y) + \bar{\mathbf{v}}(x,y,z)$$

The kinetic energy of the contribution depending on z is taken as a measure of the level of perturbation:

$$E(x) = \int_{-Z}^{Z} \int_0^\infty [|\bar{u}|^2 + Re^{-1}(|\bar{v}|^2 + |\bar{w}|^2)] dy\, dz$$

Choice of the objective function

Gain at the end ($x = X$)

$$G_{\text{out}} = \frac{E_{\text{out}}}{E_{\text{in}}} = \frac{\left[\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{u}|^2 + Re^{-1}(|\bar{v}|^2 + |\bar{w}|^2)]dy\,dz\right]_{x=X}}{\left[\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{u}|^2 + Re^{-1}(|\bar{v}|^2 + |\bar{w}|^2)]dy\,dz\right]_{x=0}}$$

Gain of the mean energy (integral over whole the domain)

$$G_{\text{mean}} = \frac{E_{\text{mean}}}{E_{\text{in}}} = \frac{\int_0^X E(x)\,dx}{E_{\text{in}}} = \frac{\int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{X}[|\bar{u}|^2 + Re^{-1}(|\bar{v}|^2 + |\bar{w}|^2)]dx\,dy\,dz}{\left[\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{u}|^2 + Re^{-1}(|\bar{v}|^2 + |\bar{w}|^2)]dy\,dz\right]_{x=0}}$$

Under the hypotheses $Re \to \infty$ and $\bar{u}|_{x=0} = 0$:

$$G_{\text{out}} = \frac{E_{\text{out}}}{E_{\text{in}}} = Re\frac{\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{u}|^2]dy\,dz}{\left[\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{v}|^2 + |\bar{w}|^2]dy\,dz\right]_{x=0}}; \quad G_{\text{mean}} = \frac{E_{\text{mean}}}{E_{\text{in}}} = Re\frac{\int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{X}[|\bar{u}|^2]dx\,dy\,dz}{\left[\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{v}|^2 + |\bar{w}|^2]dy\,dz\right]_{x=0}}$$

Objective function:

$$\mathcal{J} = \alpha G_{\text{out}} + \beta G_{\text{mean}}$$

Constrained Optimization

Constraints on the initial energy E_{in} and control energy E_{w}

$$E_{\mathsf{in}} = \left[\int_{-Z}^{Z}\int_{0}^{\infty}[|\bar{v}|^2 + |\bar{w}|^2]dy\,dz\right]_{x=0} = E_0; \qquad E_{\mathsf{w}} = \left[\int_{x_{in}}^{X}|v_{\mathsf{w}}|^2\,dx\right]_{y=0} = E_{\mathsf{w}0}$$

Lagrange multipliers technique. Functional $\mathcal{L}(u, v, w, p, \bar{v}_0, v_{\mathsf{w}}, a, b, c, d, \lambda_0, \lambda_{\mathsf{w}})$:

$$\begin{aligned}
\mathcal{L} = \ \mathcal{J} \ &+ \int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{X} a[u_x + v_y + w_z]dx\,dy\,dz \\
&+ \int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{X} b[(uu)_x + (uv)_y + (uw)_z - u_{yy} - u_{zz}]dx\,dy\,dz \\
&+ \int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{X} c[(uv)_x + (vv)_y + (vw)_z + p_y - v_{yy} - v_{zz}]dx\,dy\,dz \\
&+ \int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{X} d[(uw)_x + (vw)_y + (ww)_z + p_x - w_{yy} - w_{zz}]dx\,dy\,dz \\
&+ \lambda_0[E_{\mathsf{in}}(\bar{v}_0) - E_0] + \lambda_{\mathsf{w}}[E_{\mathsf{w}}(v_{\mathsf{w}}) - E_{\mathsf{w}0}]
\end{aligned}$$

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta u}\delta u + \frac{\delta\mathcal{L}}{\delta v}\delta v + \frac{\delta\mathcal{L}}{\delta w}\delta w + \frac{\delta\mathcal{L}}{\delta p}\delta p + \frac{\delta\mathcal{L}}{\delta \bar{v}_0}\delta\bar{v}_0 + \frac{\delta\mathcal{L}}{\delta v_{\mathsf{w}}}\delta v_{\mathsf{w}} + \frac{\delta\mathcal{L}}{\delta a}\delta a + \frac{\delta\mathcal{L}}{\delta b}\delta b + \frac{\delta\mathcal{L}}{\delta c}\delta c + \frac{\delta\mathcal{L}}{\delta d}\delta d + \frac{\delta\mathcal{L}}{\delta\lambda_0}\delta\lambda_0 + \frac{\delta\mathcal{L}}{\delta\lambda_{\mathsf{w}}}\delta\lambda_{\mathsf{w}} = 0$$

$$\frac{\delta\mathcal{L}}{\delta u}\delta u = \lim_{\epsilon\to 0}\frac{\mathcal{L}(u + \epsilon\delta u, v, w, p, \bar{v}_0, v_{\mathsf{w}}, a, b, c, d, \lambda_0, \lambda_{\mathsf{w}}) - \mathcal{L}(u, v, w, p, \bar{v}_0, v_{\mathsf{w}}, a, b, c, d, \lambda_0, \lambda_{\mathsf{w}})}{\epsilon}$$

Adjoint problem

From integration by parts ($a^* = a + 2bu$):

$$
\begin{aligned}
c_y + d_z &= 0 \\
a^*_x - 2u_x b + b_y v + b_z w + c_z v + d_x v + d_x w + b_{yy} + b_{zz} &= \beta u \\
a^*_y - 2bu_y - b_y u + c_x u + 2c_y v + d_y w + c_z w + c_{yy} + c_{zz} &= 0 \\
a^*_z - 2bu_z - b_z u + c_z v + d_y v + d_x u + 2d_z w + d_{yy} + d_{zz} &= 0
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
b = 0 \quad &\text{at} \quad y = 0 & \qquad c = 0 \quad &\text{for} \quad y \to \infty \\
a^* - 2bu + c_y = 0 \quad &\text{at} \quad y = 0 & \qquad a^* - ub + c_y = 0 \quad &\text{for} \quad y \to \infty \\
d = 0 \quad &\text{at} \quad y = 0 & \qquad d = 0 \quad &\text{for} \quad y \to \infty
\end{aligned}
$$

and "initial conditions" at $x = X$

$$
\begin{aligned}
c = 0 \quad &\text{at} \quad x = X \\
d = 0 \quad &\text{at} \quad x = X \\
\int_{-Z}^{Z} \int_0^\infty a^* dy\, dz + \alpha \frac{\delta G_{\text{out}}}{\delta u} = 0 \quad &\text{at} \quad x = X
\end{aligned}
$$

From the integration by parts also "coupling conditions" between the adjoint and direct problem:

$$
\int_{-Z}^{Z} \int_0^\infty c\, dy\, dz + \lambda_0 \frac{\delta E_{\text{in}}}{\delta \overline{v}} = 0 \text{ per } x = 0; \qquad \int_0^X c\, dx - \lambda_{\text{w}} \frac{\delta E_{\text{w}}}{\delta v_{\text{w}}} = 0 \text{ per } y = 0
$$

Iterative optimization procedure

$x = 0$	$0 < x < X$	$x = X$
		$*$
$v_0^{(1)}, \quad v_w^{(1)}$	$u^{(1)}, v^{(1)}, w^{(1)}, p^{(1)} \Longrightarrow$	$\mathcal{J}^{(1)}, \quad \frac{\delta G_{\text{out}}}{\delta u}^{(1)}$ $\int_{-Z}^{Z}\int_{0}^{\infty} a^{*(1)}\, dy\, dz = -\alpha \frac{\delta G_{\text{out}}}{\delta u}^{(1)}$
		$\Downarrow$
$c^{(1)}$ $\int_{-Z}^{Z}\int_{0}^{\infty} c^{(1)}\, dy\, dz + \lambda_0 \frac{\delta E_{\text{in}}}{\delta v}^{(2)} = 0$ $\int_{0}^{X} c^{(1)}\, dx - \lambda_w \frac{\delta E_w}{\delta v_w}^{(2)} = 0$	$\Longleftarrow$ $a^{*(1)}, b^{(1)}, c^{(1)}, d^{(1)}$	$a^{*(1)}$ $c^{(1)} = 0$ $d^{(1)} = 0$
$\Downarrow$		
$v_0^{(2)}, \quad v_w^{(2)}$	$u^{(2)}, v^{(2)}, w^{(2)}, p^{(2)} \Longrightarrow$	$\mathcal{J}^{(2)}$ $\Downarrow$ $\underbrace{\underbrace{\left\lvert \mathcal{J}^{(n)} - \mathcal{J}^{(n-1)} \right\rvert < \epsilon}_{\text{no}}}_{\text{yes}}$ no $\downarrow$ repeat from $*$ yes $\downarrow$ done

Discretization and implementation

Fourier modes in z

$$u(x,y,z) = \sum_{n=-N}^{N} U_n(x,y)e^{\mathrm{i}n\beta z}; \quad v(x,y,z) = \sum_{n=-N}^{N} V_n(x,y)e^{\mathrm{i}n\beta z}; \quad w(x,y,z) = \sum_{n=-N}^{N} W_n(x,y)e^{\mathrm{i}n\beta z}$$

Nonlinear terms like uu, uv,uw, ... introduce coupling coefficients C^{UU}, C^{UV},C^{UW}, ...

$$u(x,y,z)v(x,y,z) = \sum_{n=-N}^{N} e^{\mathrm{i}n\beta z}\left[\sum_{k=a}^{b} U_k(x,y)V_{n-k}(x,y)\right] = \sum_{n=-N}^{N} e^{\mathrm{i}n\beta z}C_n^{UV}(x,y); \qquad \begin{array}{rcl} a & = & \max(-N, n+N) \\ b & = & \min(N, n-N) \end{array}$$

$$\begin{aligned}
(U_n)_x + (V_n)_y + in\beta W_n &= 0 \\
(C_n^{UU})_x + (C_n^{UV})_y + in\beta C_n^{UW} - (U_n)_{yy} + n^2\beta^2 U_n &= 0 \\
(C_n^{UV})_x + (C_n^{VV})_y + in\beta C_n^{VW} - (V_n)_y + n^2\beta^2 V_n + (P_n)_y &= 0 \\
(C_n^{UW})_x + (C_n^{VW})_y + in\beta C_n^{WW} - (W_n)_{yy} + n^2\beta^2 W_n + in\beta P_n &= 0
\end{aligned}$$

Fourier modes in z, finite differences in x and y (second order, non uniform).

Non linear discretized problem with **all modes coupled with each other**.
Possible approaches:
1. Newton–like linearization for the coupled and large system
2. Newton–like linearization and decoupling of the modes in order to solve a small linear system for each mode

Discretization and implementation

"Initial condition" for the adjoint problem at $x = X$:

$$\int_{-Z}^{Z}\int_{0}^{\infty} a^* \, dy \, dz = -\alpha \frac{\delta G_{\text{out}}}{\delta u} \quad \Rightarrow \quad a^*(y) = -\boldsymbol{L}_u(y) u^*(y)$$

Coupling conditions between the direct and adjoint problem

$$\int_{-Z}^{Z}\int_{0}^{\infty} c \, dy \, dz + \lambda_0 \frac{\delta E_{\text{in}}}{\delta v} = 0; \quad c + \lambda_0 \boldsymbol{L}_v v^* = 0 \quad \Rightarrow \quad v_0^{t+1} = -\left[\frac{1}{\lambda^t}\boldsymbol{L}_v^{-1} c_0^t\right]^*$$

$$\int_{0}^{X} c_{\text{w}} \, dx - \lambda \frac{\delta E_{\text{w}}}{\delta v_{\text{w}}} = 0; \quad c_{\text{w}} - \lambda_{\text{w}} \boldsymbol{L}_{\text{w}} v_{\text{w}}^* = 0 \quad \Rightarrow \quad v_{\text{w}}^{t+1} = \left[\frac{1}{\lambda^t}\boldsymbol{L}_{\text{w}}^{-1} c_{\text{w}}^t\right]^*$$

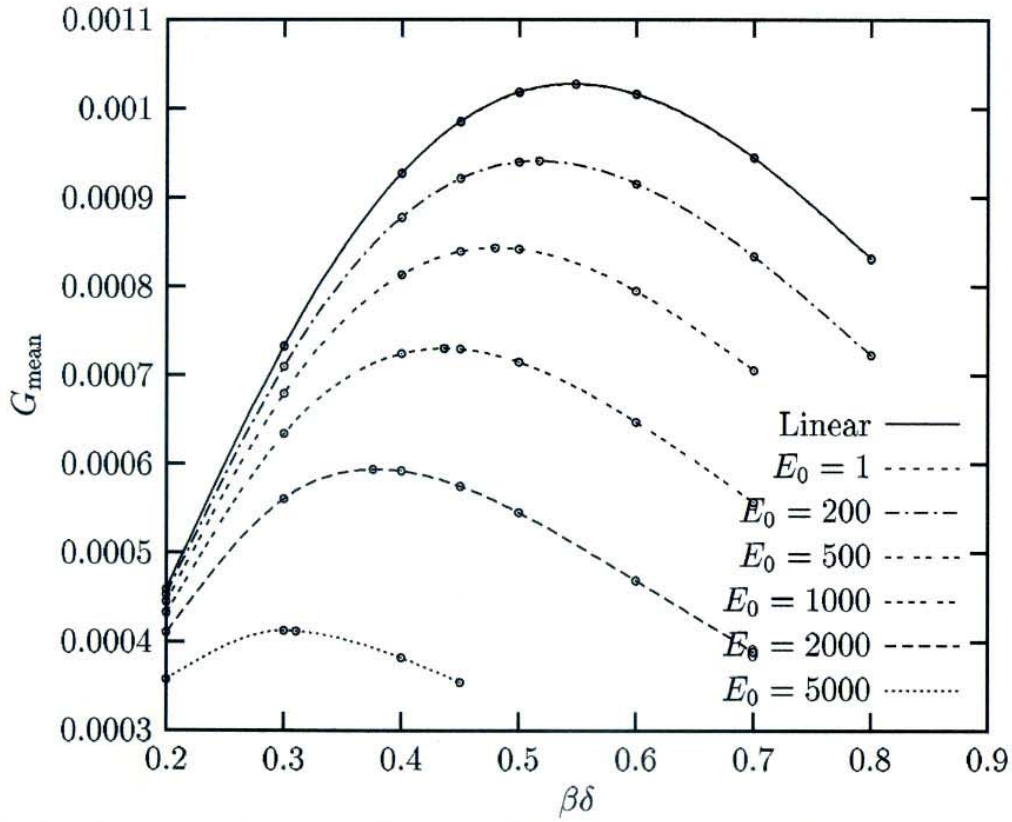
With relaxation ($0 < k < 1$)

Initial condition: $$v_0^{t+1} = v_0^t(1-k) - k\left[\frac{1}{\lambda^t}\boldsymbol{L}_v^{-1} c_0^t\right]^*$$

Boundary condition at the wall: $$v_{\text{w}}^{t+1} = v_{\text{w}}^t(1-k) + k\left[\frac{1}{\lambda^t}\boldsymbol{L}_{\text{w}}^{-1} c_{\text{w}}^t\right]^*$$

Optimal perturbation		Optimal control
mode $n = 0$	mode $n = 1$	mode $n = 0$
$U_0 = 1$	$U_1 = 0$	$U_0 = 1$
$V_0 = 0$	$V_1 = V_1^0$	$V_0 = v_{\text{w}}$
$W_0 = 0$	$W_1 = W_1^0(V_1^0)$	$W_0 = 0$

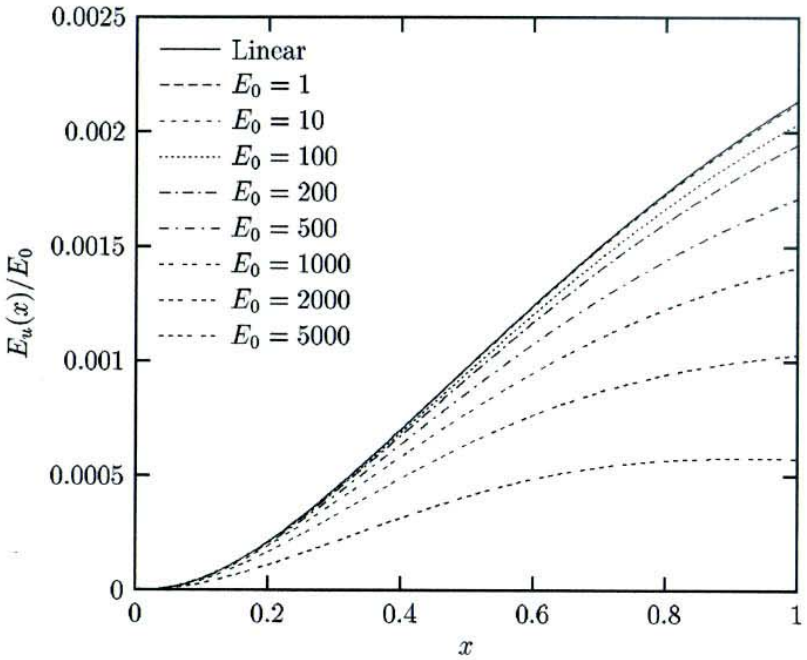
Optimal perturbation



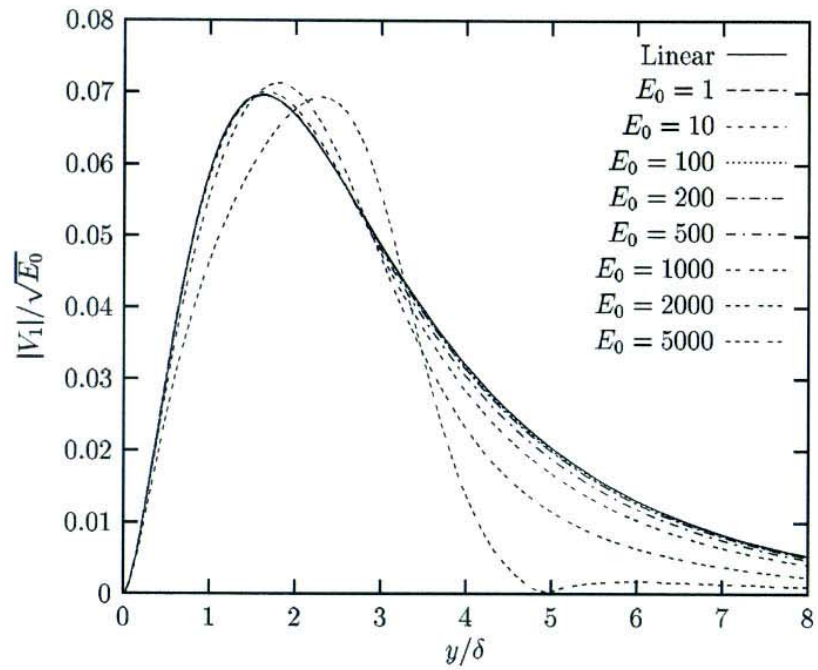
Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ for different values of the initial energy E_0 and different wavenumbers $\beta\delta$

Optimal perturbation – $\beta\delta = 0.45$

Optimal perturbation for varying E_0 and at $\beta\delta$ fixed



Energy behavior $E_u(x)/E_0$

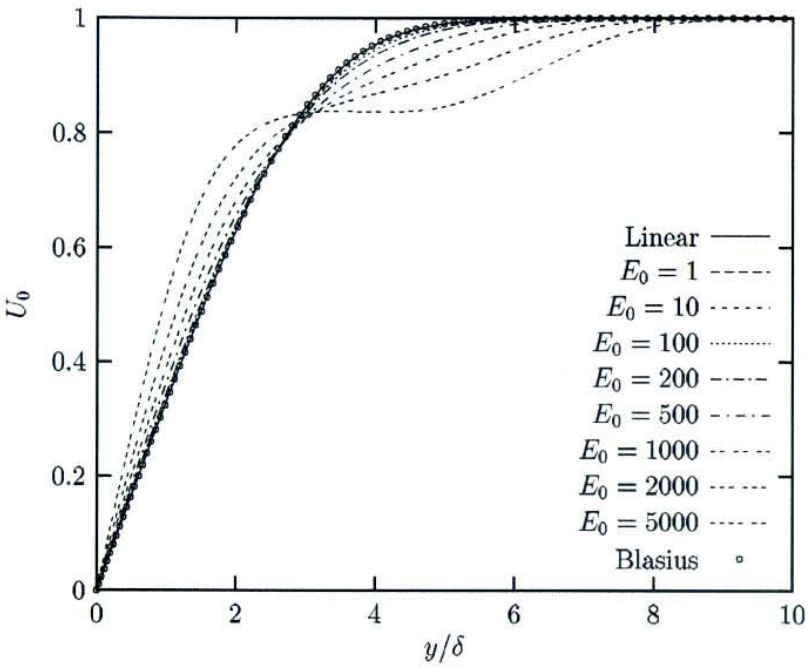


Initial perturbation $|V_1|/\sqrt{E_0}$

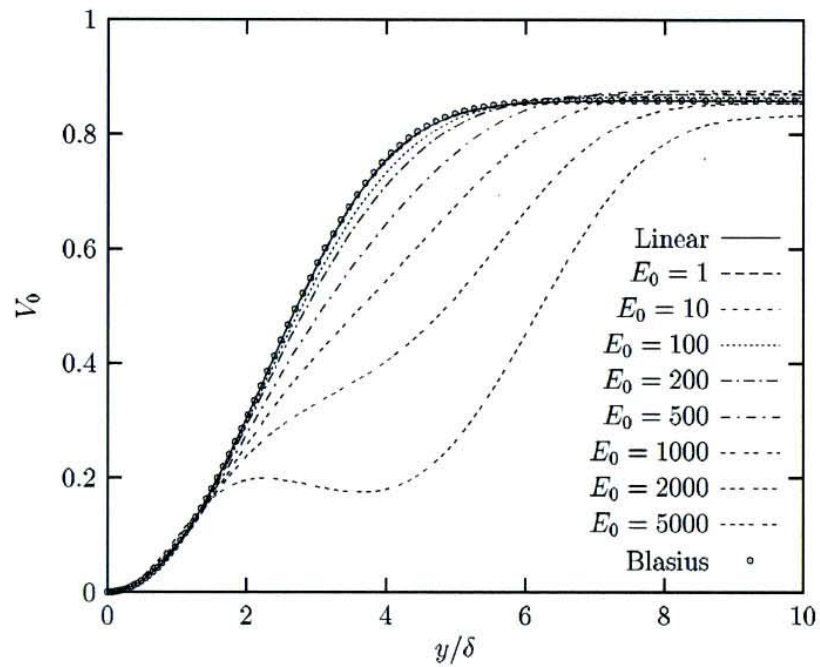
$\Rightarrow$ Saturation for high initial energy?

Optimal perturbation – $\beta\delta = 0.45$

Mean flow contribution (independent of z). Profiles at the final station $x = 1$



$|U_0|$ with varying E_0

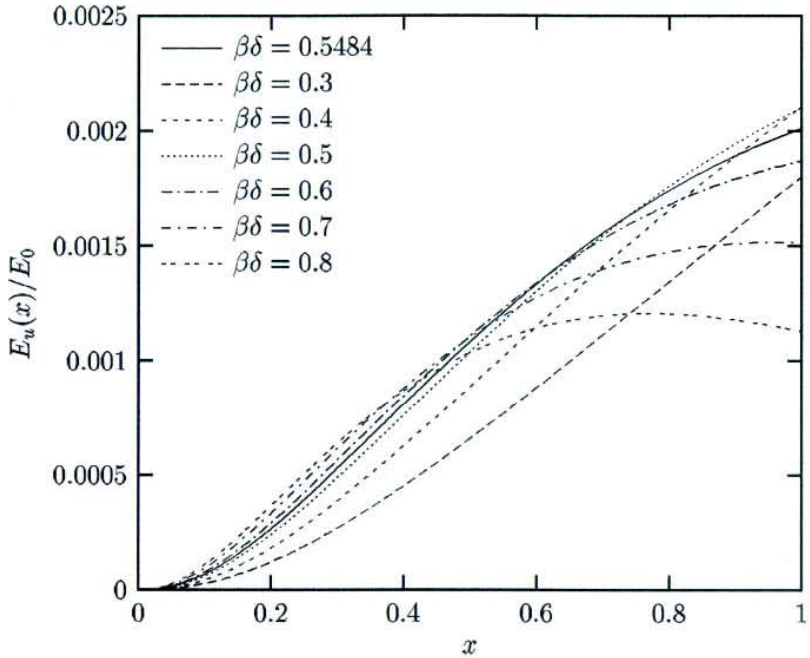


$|V_0|$ with varying E_0

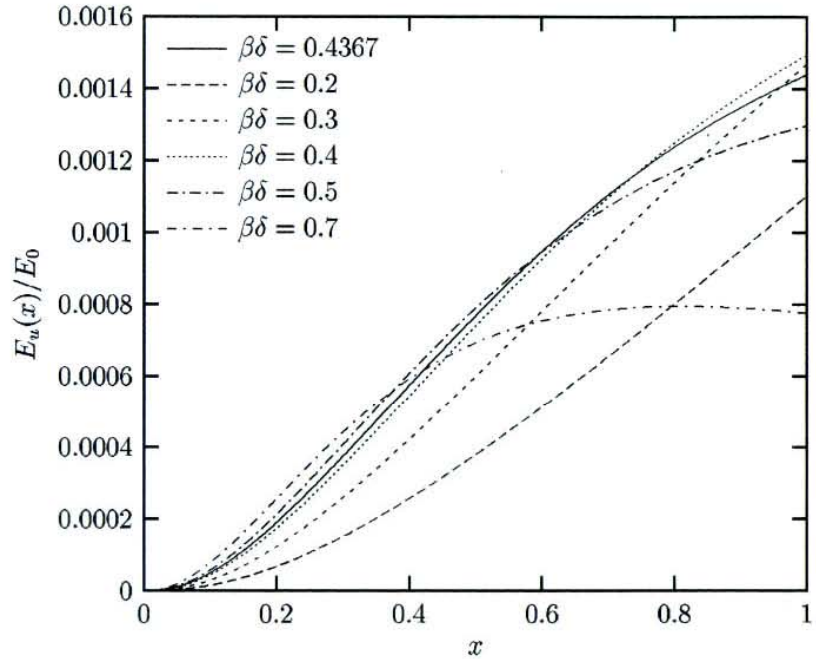
$\Rightarrow$ Mean flow distortion with respect to Blasius

Optimal perturbation – fixed E_0

Energy behavior $E_u(x)/E_0$ for varying $\beta\delta$ at fixed E_0



$E_0 = 1$

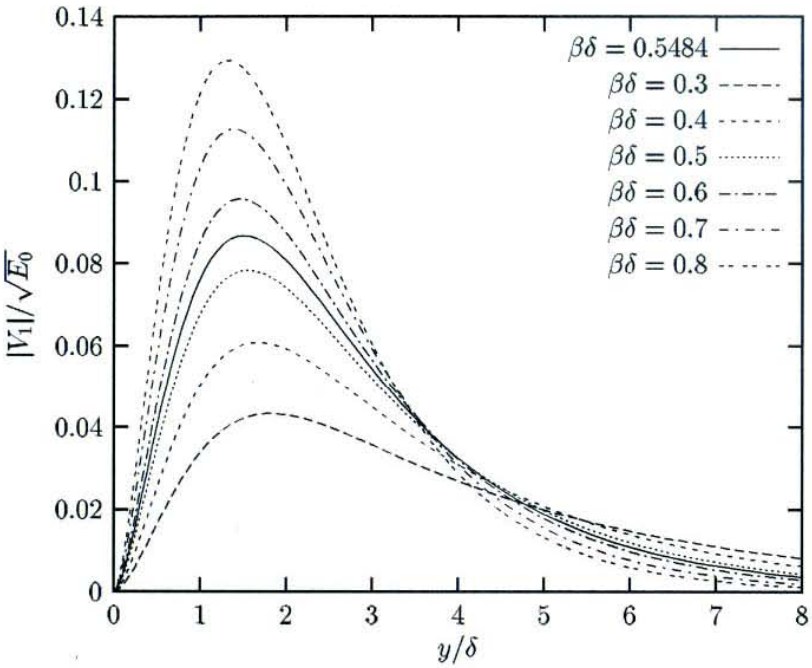


$E_0 = 1000$

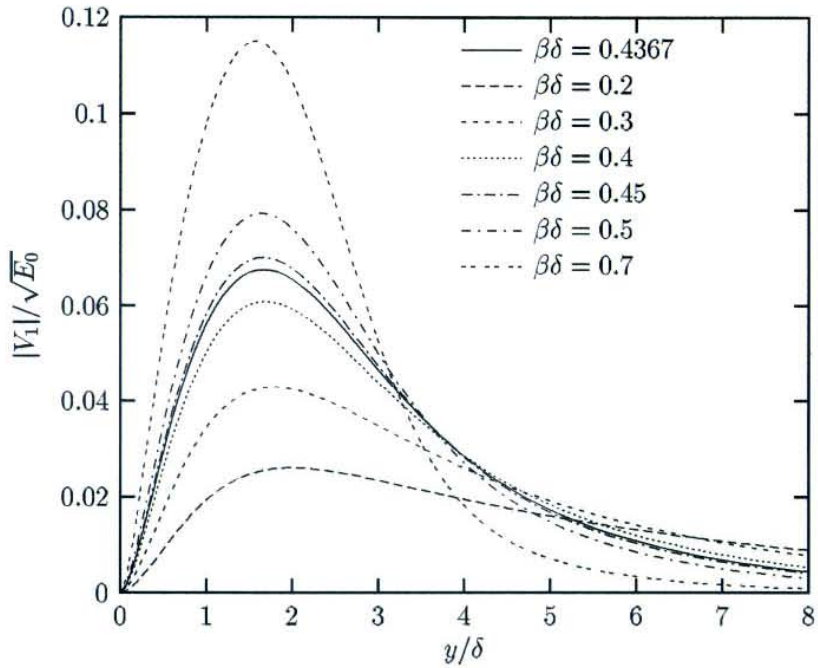
$\Rightarrow$ Saturation or effect of high $\beta\delta$?

Optimal perturbation – fixed E_0

Optimal perturbation $|V_1|/\sqrt{E_0}$ for varying $\beta\delta$ at fixed E_0



$E_0 = 1$

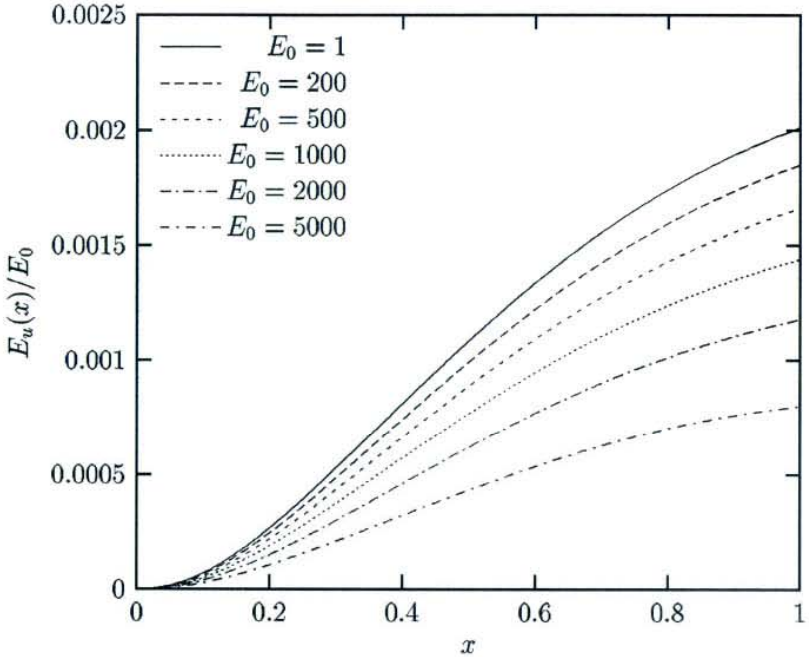


$E_0 = 1000$

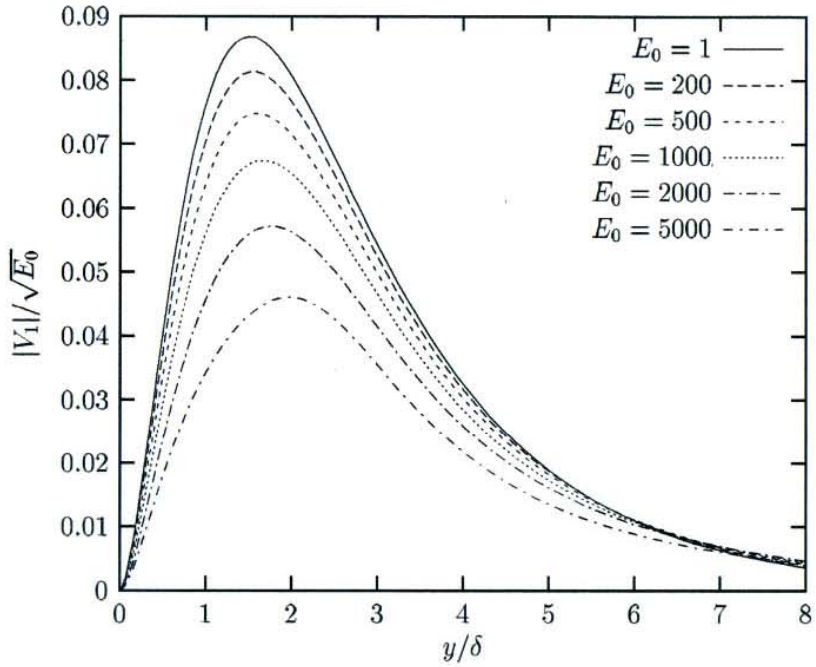
$\Rightarrow$ Effect of the wavenumber $\beta\delta$, weak dependence on E_0

Optimal perturbation – optimal $\beta\delta$

Comparisons at optimal $\beta\delta$ for different values of initial energy



Energy behavior $E_u(x)/E_0$

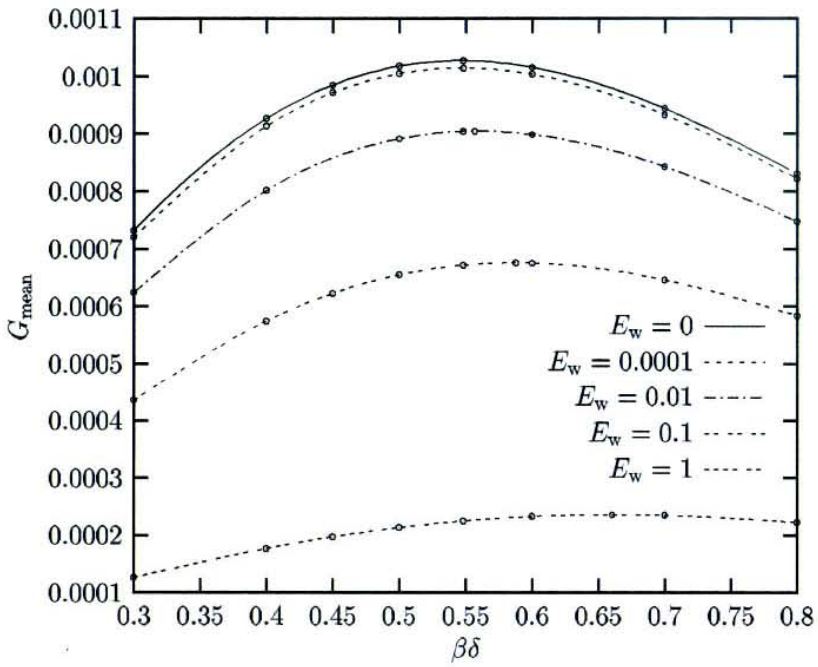


Optimal perturbation $|V_1|/\sqrt{E_0}$

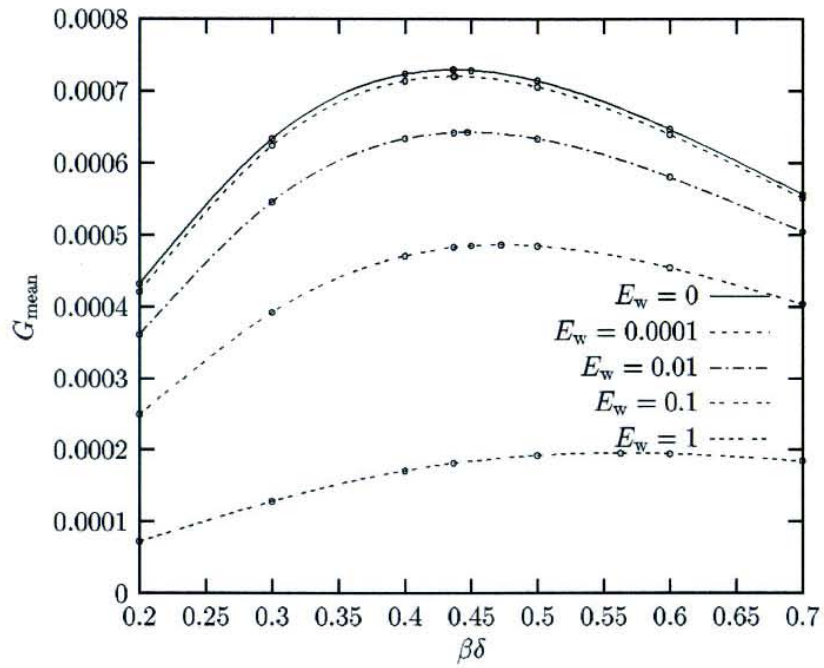
$\Rightarrow$ Much more regular behavior with varying E_0

Optimal control

Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ with varying control energy E_{w} and wavenumber $\beta\delta$



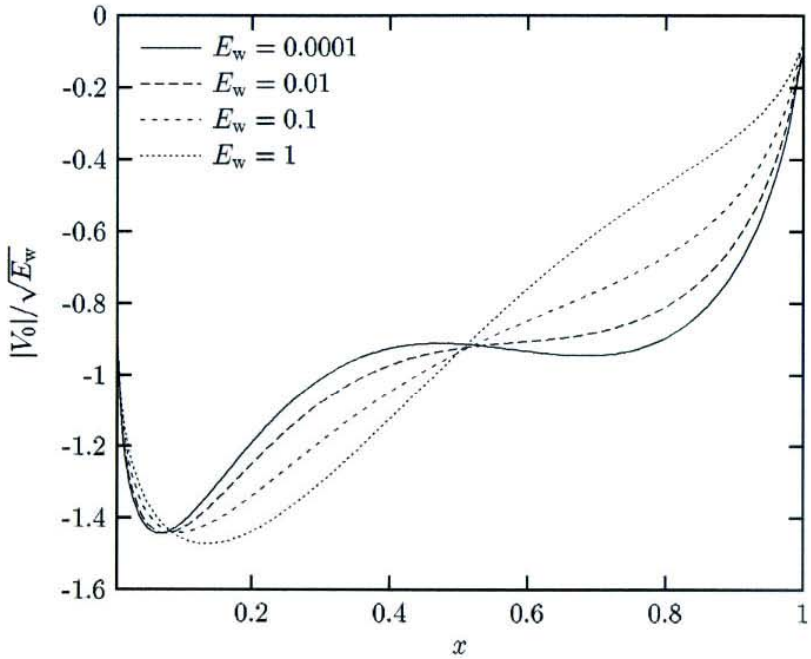
$E_0 = 1$



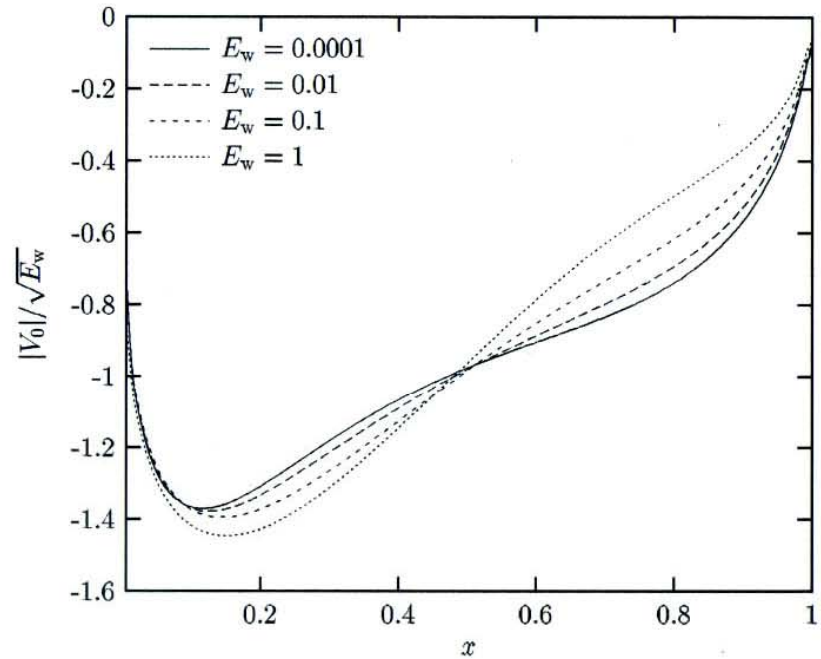
$E_0 = 1000$

Optimal control – optimal $\beta\delta$

Optimal control at the wall $|V_0|/\sqrt{E_w}$ for varying E_w at fixed $\beta\delta$



$E_0 = 1$

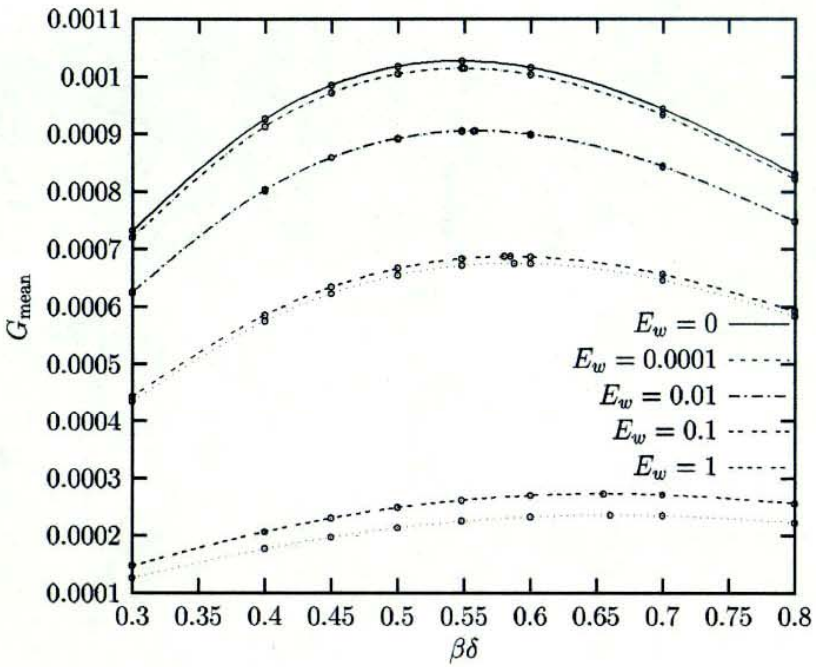


$E_0 = 1000$

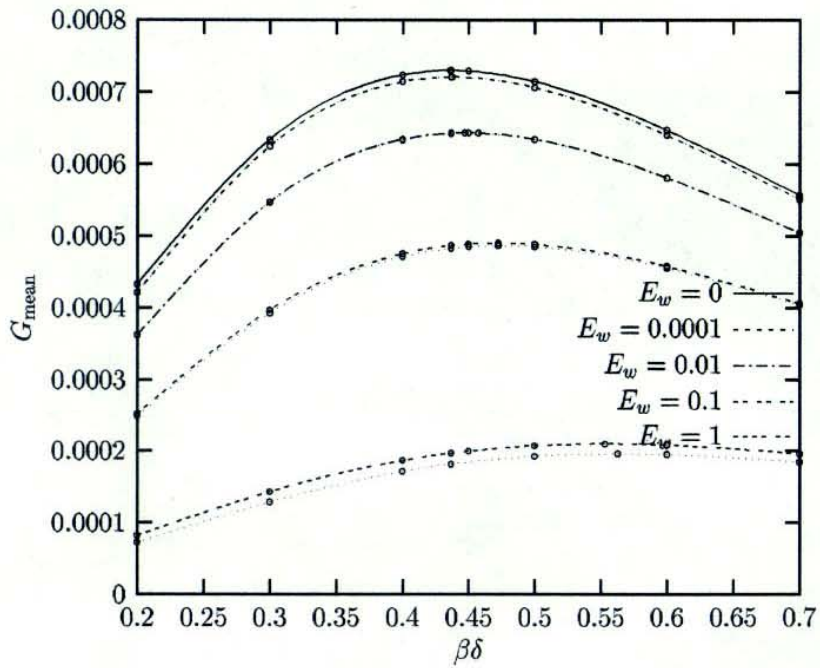
$\Rightarrow$ Effect of the initial energy E_0 on the regularity of the profile

Robust control

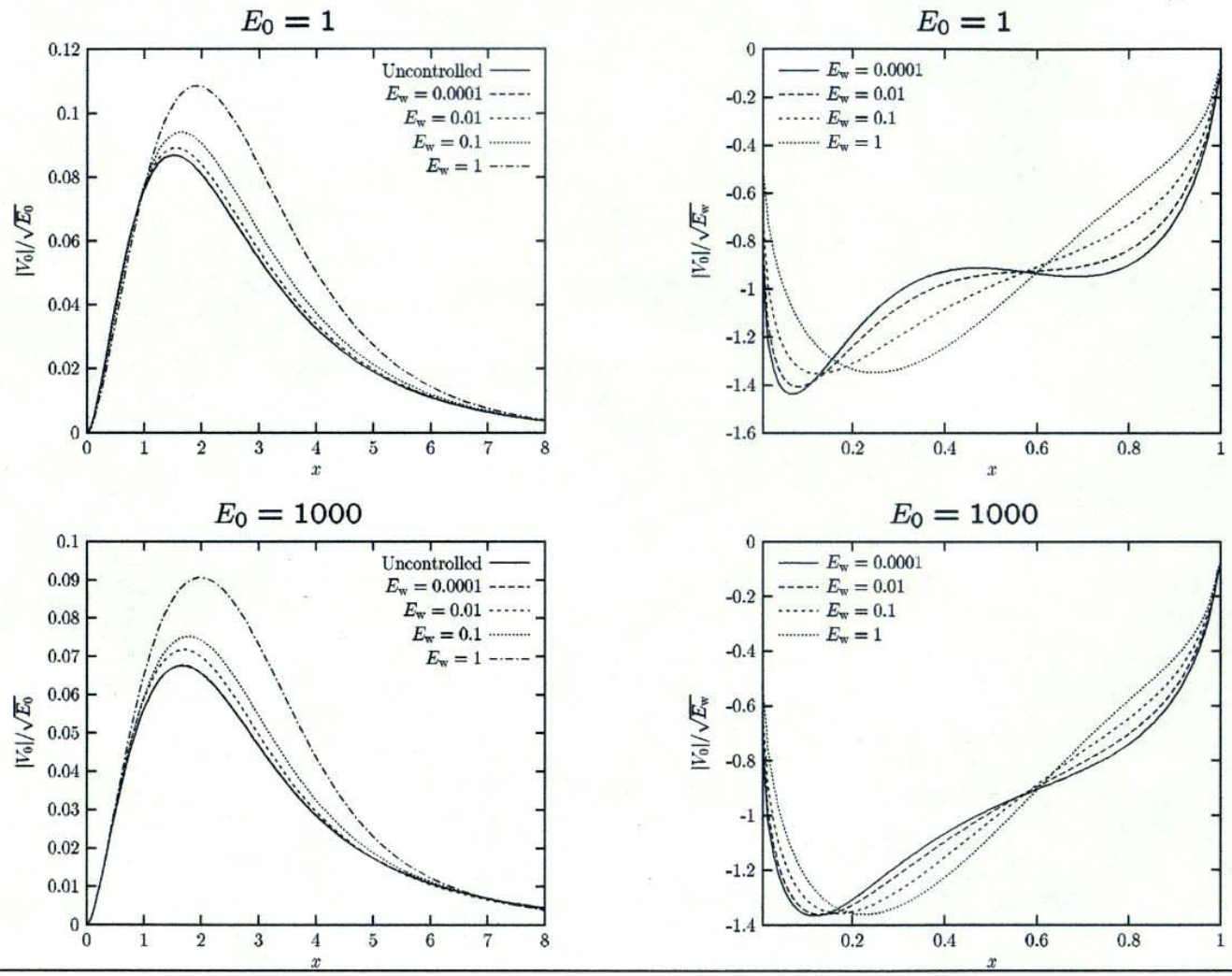
Gain $G_{\text{mean}} = E_{\text{mean}}/E_0$ for varying E_{w} and $\beta\delta$



$E_0 = 1$



$E_0 = 1000$

$\Rightarrow$ Robust control curves are always above optimal control curves.

Robust control – optimal $\beta\delta$