Models for Large-Scale Robust Optimization

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Agenda

- Tractable Robust Optimization Models
- Robust Linear Optimization
- Approximation of Multiperiod Stochastic Linear Optimization
- Conclusions

Tractable Robust Optimization Models

- Consider a family of optimization models with uncertain parameters:

$$\left\{ \begin{array}{rl} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & f_i(\boldsymbol{x}, \boldsymbol{z}) \geq 0, \quad i \in I \\ & \boldsymbol{x} \in X, \end{array} \right\}_{\boldsymbol{z} \in \mathcal{U}}$$

Tractable Robust Optimization Models

- Large Scale Robust Optimization
  - Computationally tractable in practice. Tractable in theory may not necessarily practical in large scale problems.
  - Mild distributional assumption, such as known support, mean and deviation measures
  - Scalable multiperiod models

Tractable Robust Optimization Models

- Classical Chance Constraint (Charnes and Cooper (1959))

$$\begin{array}{rl} \text{min} & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \mathsf{P}(f_i(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \geq 0) \geq 1 - \epsilon_i, \quad \boldsymbol{i} \in I \\ & \boldsymbol{x} \in X, \end{array}$$

Tractable Robust Optimization Models

- Destroy convexity of original model
- Hard to compute probability
  - Require multidimensional integration
- Exact distribution is unknown. Impossible to collect such information

Tractable Robust Optimization Models

- Robust Counterpart Approach
  - Soyster (1973), Ben-Tal and Nemirovski (1997), El-Ghaoui et al (1997), Bertsimas and Sim (2003)

$$
\begin{array}{rl}
\min & \boldsymbol{c}'\boldsymbol{x} \\
\text{s.t.} & f_i(\boldsymbol{x}, \boldsymbol{z}) \geq 0 \quad \forall \boldsymbol{z} \in \mathcal{U}_i \quad i \in I \\
 & \boldsymbol{x} \in X,
\end{array}
$$

Tractable Robust Optimization Models

- Preserve convexity but lead to explosion in constraints (possibly infinite)
- Tractability depends on function and uncertainty set
  - Most promising when f(**x**,**z**) is biaffine (e.g in a linear constraint) where robust counterpart is concisely formulated and efficiently solved.

$$f(\boldsymbol{x}, \boldsymbol{z}) = f_0(\boldsymbol{x}) + \sum_{j=1}^{N} f_j(\boldsymbol{x}) z_j$$

Tractable Robust Optimization Models

- Robust Linear Optimization Models
  - Second Order Cone Programming (SOCP)
    - Ben-Tal and Nemirovski (1997), El-Ghaoui et al (1997), Chen, Sim and Sun (2005)
    - SOCP Solvers are getting better and robust
      - MOSEK, Frontline, SiDuMe, SDPT3
  - Linear Optimization – Suited for MIP
    - Price of Robustness - Bertsimas and Sim (2003)
    - Robust Discrete Optimization and Network Flows - Bertsimas and Sim (2004)
    - Robust Discrete Optimization and Downside Risk Measures - Bertsimas and Sim (2004)

Tractable Robust Optimization Models

- Robust Conic Optimization
  - Generally intractable
  - Not directly linked to chance constraint

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{z}) \in \boldsymbol{K} \quad \forall \boldsymbol{z} \in \mathcal{U}$$

$$\boldsymbol{F}(\boldsymbol{x}, \tilde{\boldsymbol{z}}) = \boldsymbol{F}_0(\boldsymbol{x}) + \sum_{j=1}^{N} \boldsymbol{F}_j(\boldsymbol{x}) \tilde{z}_j$$

Tractable Robust Optimization Models

- Tractable Approximations to Robust Conic Optimization Problems - Bertsimas and Sim (2004)

$$
\begin{aligned}
&\boldsymbol{F}_0(\boldsymbol{x}) - \Omega y \boldsymbol{V} \in \boldsymbol{K} \\
&t_j \boldsymbol{V} + \boldsymbol{F}_j(\boldsymbol{x}) \in \boldsymbol{K} \ \forall j \in N \\
&t_j \boldsymbol{V} - \boldsymbol{F}_j(\boldsymbol{x}) \in \boldsymbol{K} \ \forall j \in N \\
&\|\boldsymbol{t}\|_2 \leq y. \\
&y \in \Re, \boldsymbol{t} \in \Re^{|N|} \qquad \boldsymbol{V} \in \mathrm{int}(\boldsymbol{K})
\end{aligned}
$$

Tractable Robust Optimization Models

- Large deviation results of Nemirovski (2004)

Assume $\tilde{z}_j$ independently distrbuted and $\ln \mathsf{E}(\exp(\tilde{z}_j^2)) \leq 1$

$$\mathsf{P}(\boldsymbol{F}(\boldsymbol{x}, \tilde{\boldsymbol{z}}) \notin \boldsymbol{K}) \leq \exp(1 - c\Omega^2/\alpha^2)$$

α depends on the cone, $\boldsymbol{K}$.
For example, $\alpha = \sqrt{2}$ for second order cone and $\alpha = \sqrt{\ln(m)}$ for positive semidefinite cone.

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Robust Linear Optimization

- Uncertain Linear Constraint

$$\{\tilde{\boldsymbol{a}}'\boldsymbol{x} \geq \tilde{b}\}$$

$$\tilde{\boldsymbol{a}} = \boldsymbol{a}(\tilde{\boldsymbol{z}})$$
$$\tilde{b} = b(\tilde{\boldsymbol{z}})$$

Robust Linear Optimization

- Affine Uncertainty

$$\tilde{a} = a(\tilde{z}) = a^0 + \sum_{j=1}^{N} \Delta a^j \tilde{z}_j$$

$$\tilde{b} = b(\tilde{z}) = b^0 + \sum_{j=1}^{N} \Delta b^j \tilde{z}_j$$

$\tilde{z}_j \in [\underline{z}_j, \bar{z}_j]$: zero mean, independent but not necessarily identically distributed

Robust Linear Optimization

- Goal of Robust Optimization:
  - Easy to obtain feasible solutions that satisfy chance constraint
  - Not as conservative as worst case

$$
\begin{gathered}
\boldsymbol{a}(\boldsymbol{z})'\boldsymbol{x} \leq b(\boldsymbol{z}) \ \ \forall \boldsymbol{z} \in \mathcal{U}_{\Omega} \\
\Downarrow \\
\mathsf{P}(\tilde{\boldsymbol{a}}'\boldsymbol{x} > \tilde{b}) \leq \epsilon(\Omega)
\end{gathered}
$$

Robust Linear Optimization

- Worst case
  - Relies only on the distribution support
  - Easy to solve (Soyster 1973)
  - Extremely conservative

$$\mathrm{P}(\tilde{a}'\boldsymbol{x} > \tilde{b}) = 0$$

$$\Updownarrow$$

$$\boldsymbol{a}(\boldsymbol{z})'\boldsymbol{x} \le b(\boldsymbol{z}) \ \forall \boldsymbol{z} \in \underbrace{[-\underline{\boldsymbol{z}}, \bar{\boldsymbol{z}}]}_{=\mathcal{W}}$$

Robust Linear Optimization

- Asymmetrical Uncertainty Set
  - Chen, Sim and Sun (2005)

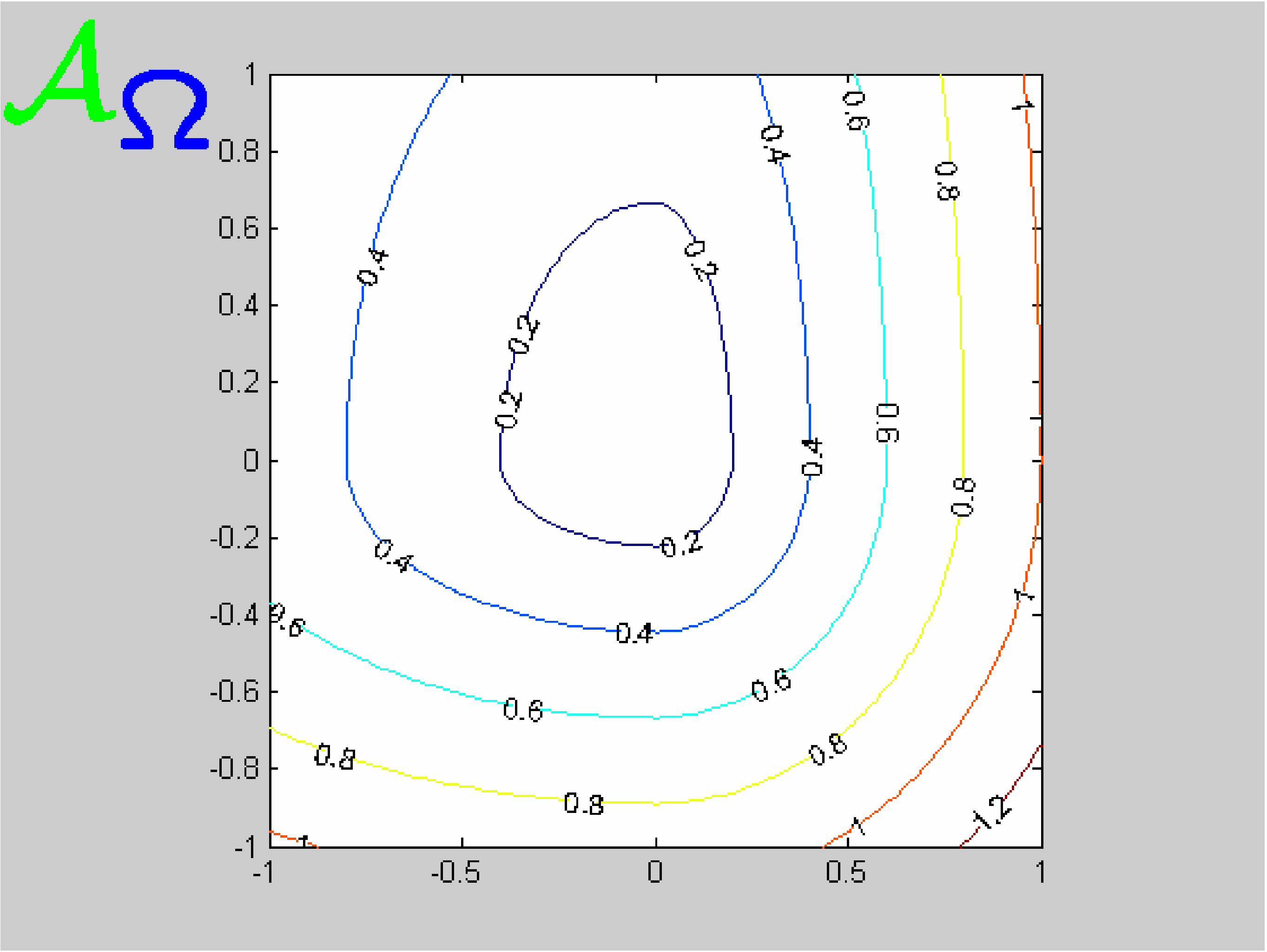
$$\mathcal{U}_{\Omega} = \mathcal{A}_{\Omega} \cap \mathcal{W}$$

$$\mathcal{A}_{\Omega} = \left\{ \boldsymbol{z} | \exists \boldsymbol{w}, \boldsymbol{v} \geq \boldsymbol{0} : \boldsymbol{z} = \boldsymbol{w} - \boldsymbol{v}, \left\| \boldsymbol{P}^{-1}\boldsymbol{w} + \boldsymbol{Q}^{-1}\boldsymbol{v} \right\| \leq \Omega \right\}$$

$$\boldsymbol{P} = \mathrm{diag}(p_1, \ldots, p_N)$$

$$\boldsymbol{Q} = \mathrm{diag}(q_1, \ldots, q_N)$$

Uncertainty Sets and Probability Bounds



Robust Linear Optimization

- *Forward* deviation

$$p_j = \sup_{\phi>0} \left\{ \sqrt{2\mathrm{E}\left(\exp\left(\phi \tilde{z}_j\right)\right)/\phi^2} \right\}$$

- *Backward* deviation

$$q_j = \sup_{\phi>0} \left\{ \sqrt{2\mathrm{E}\left(\exp\left(-\phi \tilde{z}_j\right)\right)/\phi^2} \right\}$$

Robust Linear Optimization

- Forward and Backward deviation measures
  - $p = q$ if distribution is symmetrical
  - p, q ¸ σ (standard deviation)
  - $p = q = \sigma$ if distribution is Normal
- Can be estimated from past samples

$$\mathsf{P}(\tilde{z}_j > \Omega p_j) \leq \exp(-\Omega^2/2)$$
$$\mathsf{P}(\tilde{z}_j < -\Omega q_j) \leq \exp(-\Omega^2/2)$$

Robust Linear Optimization

- Deviation measures exist for all bounded deviations
  - Suppose distribution is bounded in [-a,b]
  - Not practically restrictive

$$p, q \leq \frac{a+b}{2}$$

Robust Linear Optimization

- Robust Counterpart

$$a(z)'x \leq b(z) \ \forall z \in \mathcal{A}_\Omega \cap \mathcal{W}$$

$$\Updownarrow$$

$$\left\{ x \ : \ \begin{array}{l} \exists u, r, s \in \Re^N, h \in \Re \\ a^{0\prime}x + \Omega h + r'\bar{z} + s'\underline{z} \leq b^0 \\ \|u\|_2 \leq h \\ u_j \geq p_j(\Delta a^{j\prime}x - \Delta b^j - r_j) \quad \forall j = \{1, \ldots, N\}, \\ u_j \geq -q_j(\Delta a^{j\prime}x - \Delta b^j + s_j) \quad \forall j = \{1, \ldots, N\}, \\ u, r, s \geq 0. \end{array} \right\}$$

Robust Linear Optimization

- Probability Bound

$$\boldsymbol{a}(\boldsymbol{z})'\boldsymbol{x} \le b(\boldsymbol{z}) \;\; \forall \boldsymbol{z} \in \mathcal{A}_{\Omega} \cap \mathcal{W}$$

$$\Downarrow$$

$$\mathsf{P}\left(\tilde{\boldsymbol{a}}'\boldsymbol{x} > \tilde{b}\right) \le \exp\left(-\frac{\Omega^2}{2}\right)$$

Robust Linear Optimization

- Portfolio optimization computations studies
  - Natarajan, Pachamanova and Sim (2005)
  - Minimize (1-ε)-Value at Risk subjected to target return
  - Value at Risk Approximation
    - Conditional Value at Risk (CVaR)
      - Make use of past returns in the optimization models
    - Asymmetric Value at Risk
      - Asymmetic uncertainty set mapped from past returns

Robust Linear Optimization

Target Return	Asymmetric VaR	CVaR-Based VaR
0.600	3.814	4.412
0.800	4.026	4.936
1.000	4.149	5.758
1.200	4.456	6.617
1.400	4.683	7.279
1.600	4.886	7.405
1.800	5.130	8.397
2.000	5.322	8.909
2.200	5.501	9.249
2.400	5.682	9.547
2.600	5.862	10.545

Out-of-sample experiments for $\varepsilon = 1\%$

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Stochastic linear optimization

- Multiperiod Stochastic Optimization is generally hard to solve
  - Samples required for multiperiod can be very large
    - Shapiro (2004), Shapiro and Nemirovski (2004)
  - Actual distributions may not be known

Stochastic linear optimization

- Two-stage stochastic model with fixed recurse and chance constraints

$$
\begin{array}{rl}
\min & \boldsymbol{c}'\boldsymbol{x} + \mathsf{E}(\boldsymbol{d}'\boldsymbol{y}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & \boldsymbol{D}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{T}\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{f}(\tilde{\boldsymbol{z}}) \\
 & \boldsymbol{x} \geq \boldsymbol{0} \\
 & \mathsf{P}(y_i(\tilde{\boldsymbol{z}}) \geq 0) \geq 1 - \epsilon_i \ \forall i
\end{array}
$$

Stochastic linear optimization

- Linear Decision Rule
  - Appeared in early Stochastic Optimization
    - Garstka and Wets (1974)
  - Resurface recently as *affinely adjustable robust counterpart*
    - Ben-Tal et al. (2004)
  - "First order estimation" of future costs
  - Easily extendable to multiperiod models to capture non-anticipative affine decision rules

$$\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{y}_0 + \sum_{j=1}^{N} \boldsymbol{y}_j \tilde{z}_j$$

Stochastic linear optimization

- Linear Decision Rule

$$y(\tilde{z}) = y_0 + \sum_{j=1}^{N} y_j \tilde{z}_j$$

$$\mathrm{E}(d' y(\tilde{z})) = d' y_0$$

Stochastic linear optimization

- Linear Decision Rule

$$\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{y}_0 + \sum_{j=1}^{N} \boldsymbol{y}_j \tilde{z}_j$$

$$\boldsymbol{D}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{T}\boldsymbol{y}(\tilde{\boldsymbol{z}}) = \boldsymbol{f}(\tilde{\boldsymbol{z}})$$

$$\Updownarrow$$

$$\boldsymbol{D}_j\boldsymbol{x} + \boldsymbol{T}\boldsymbol{y}_j = \boldsymbol{f}_j \quad \forall j = 0, \ldots, N$$

Stochastic linear optimization

- Linear Decision Rule

$$y(\tilde{z}) = y_0 + \sum_{j=1}^{N} y_j \tilde{z}_j$$

$$y(z) \geq 0 \quad \forall z \in \mathcal{U}_\Omega$$
$$\Downarrow$$
$$\mathrm{P}(y(\tilde{z}) \geq 0) \geq 1 - \exp(\Omega^2/2)$$

Stochastic linear optimization

- Linear Decision Rule
  - Can perform poorly in hard constraints
    - Becomes "zeroth" decison rule

$$y(\tilde{z}) = y_0 + \sum_{j=1}^{N} y_j \tilde{z}_j$$

$$\mathrm{P}(y_i(\tilde{z}) \geq 0) = 1$$
$$\Updownarrow$$
$$y(z) \geq 0 \quad \forall z \in \Re^n$$
$$\Downarrow$$
$$y(z) = y_0 \geq 0, \text{ i.e. } y_j = 0 \ \forall j = 1, \ldots, N$$

Stochastic linear optimization

- Linear Decision Rule
  - Resonable performance in soft constraints
  - E.g. Forward and backward deviations equal one
    - Ω is small (· 6) even for high reliability 1-10^{-7}
    - As opposed to Ω = 1 for reliability one.

$$\mathrm{P}(y_i(\tilde{z}) \geq 0) \geq 1 - \exp(-\Omega^2/2)$$

$$\Uparrow$$

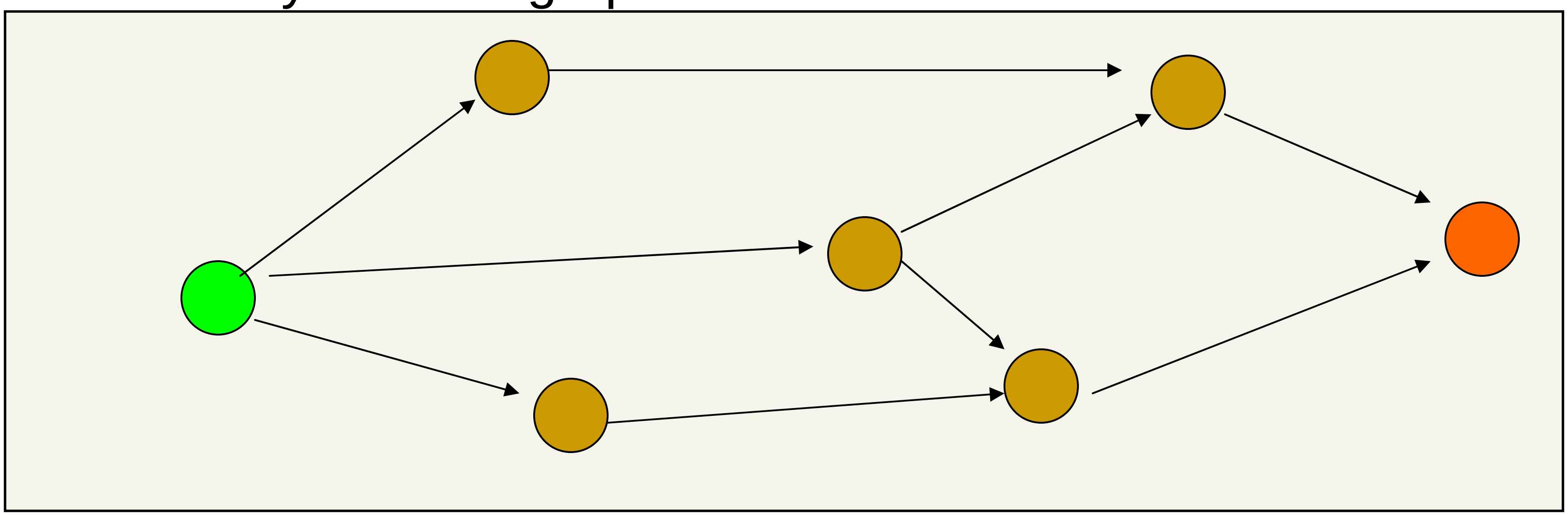
$$y(z) \geq 0 \quad \forall z \in \mathcal{U}_\Omega$$

$$\Updownarrow$$

$$y_0 \geq \Omega \sqrt{\textstyle\sum_{j=1}^N y_j^2}$$

Stochastic Activity Network

- Classical Project Management Problem
  - Each activity has deterministic completion time
  - Activities must satisfy precedence constraints
  - Determine project completion time LP
  - Activity network/graph



Stochastic Activity Network

- Project Crashing
  - Find the minimum cost for allocating additional resources so that project can be completed on time, T

$$
\begin{aligned}
\min \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & y_n \leq T \\
& y_j - y_i \geq b_{ij} - a_{ij}x_{ij} \forall (i,j) \in \mathcal{E} \\
& y_1 = 0 \\
& \mathbf{0} \leq \mathbf{x} \leq \bar{\mathbf{x}} \\
& \mathbf{y} \in \Re^n
\end{aligned}
$$

Stochastic Activity Network

- Robust Project Crashing
  - Stochastic Activity Time
  - Find the minimum cost for allocating additional resources so that project can be completed on time, T with high probability, say 99%

Stochastic Activity Network

- Robust Project Crashing

$$
\begin{aligned}
Z_r^* = \min \quad & \mathbf{c}'\mathbf{x} \\
\text{s.t.} \quad & y_n^0 + \Omega_0 h_0 + \mathbf{r}^{0\prime}\bar{\mathbf{z}} + \mathbf{s}^{0\prime}\underline{\mathbf{z}} \le T \\
& \|\mathbf{u}^0\|_2 \le h_0 \\
& u_{ij}^0 \ge p_{ij}(y_n^{ij} - r_{ij}^0) \quad \forall (i,j) \in \mathcal{E} \\
& u_{ij}^0 \ge -q_{ij}(y_n^{ij} + s_{ij}^0) \quad \forall (i,j) \in \mathcal{E} \\
& y_j^0 - y_i^0 \ge b_{ij} - a_{ij}x_{ij} + \Omega_{ij} h_{ij} + \mathbf{r}^{ij\prime}\bar{\mathbf{z}} + \mathbf{s}^{ij\prime}\underline{\mathbf{z}} \quad \forall (i,j) \in \mathcal{E} \\
& \|\mathbf{u}^{ij}\|_2 \le h_{ij} \quad \forall (i,j) \in \mathcal{E} \\
& u_{ij}^{ij} \ge p_{ij}(b_{ij} + y_i^{ij} - y_j^{ij} - r_{ij}^{ij}) \quad \forall (i,j) \in \mathcal{E} \\
& u_{ij}^{kl} \ge p_{ij}(y_k^{ij} - y_l^{ij} - r_{ij}^{kl}) \quad \forall (i,j),(k,l) \in \mathcal{E}, (i,j) \ne (k,l) \\
& u_{ij}^{ij} \ge -q_{ij}(b_{ij} + y_i^{ij} - y_j^{ij} + s_{ij}^{ij}) \quad \forall (i,j) \in \mathcal{E} \\
& u_{ij}^{kl} \ge -q_{ij}(y_k^{ij} - y_l^{ij} + s_{ij}^{kl}) \quad \forall (i,j),(k,l) \in \mathcal{E}, (i,j) \ne (k,l) \\
& y_1^0 = 0, y_1^{ij} = 0 \quad \forall (i,j) \in \mathcal{E} \\
& 0 \le \mathbf{x} \le \bar{\mathbf{x}} \\
& \mathbf{u}^0, \mathbf{u}^{ij}, \mathbf{r}^0, \mathbf{r}^{ij}, \mathbf{s}^0, \mathbf{s}^{ij} \in \Re_+^{|\mathcal{E}|} \quad \forall (i,j) \in \mathcal{E} \\
& h_0, h_{ij} \in \Re \quad \forall (i,j) \in \mathcal{E} \\
& \mathbf{x} \in \Re^{|\mathcal{E}|} \\
& \mathbf{y}^0, \mathbf{y}^{ij} \in \Re^n \quad \forall (i,j) \in \mathcal{E}.
\end{aligned}
$$

Stochastic Activity Network

- Stochastic Activity Time

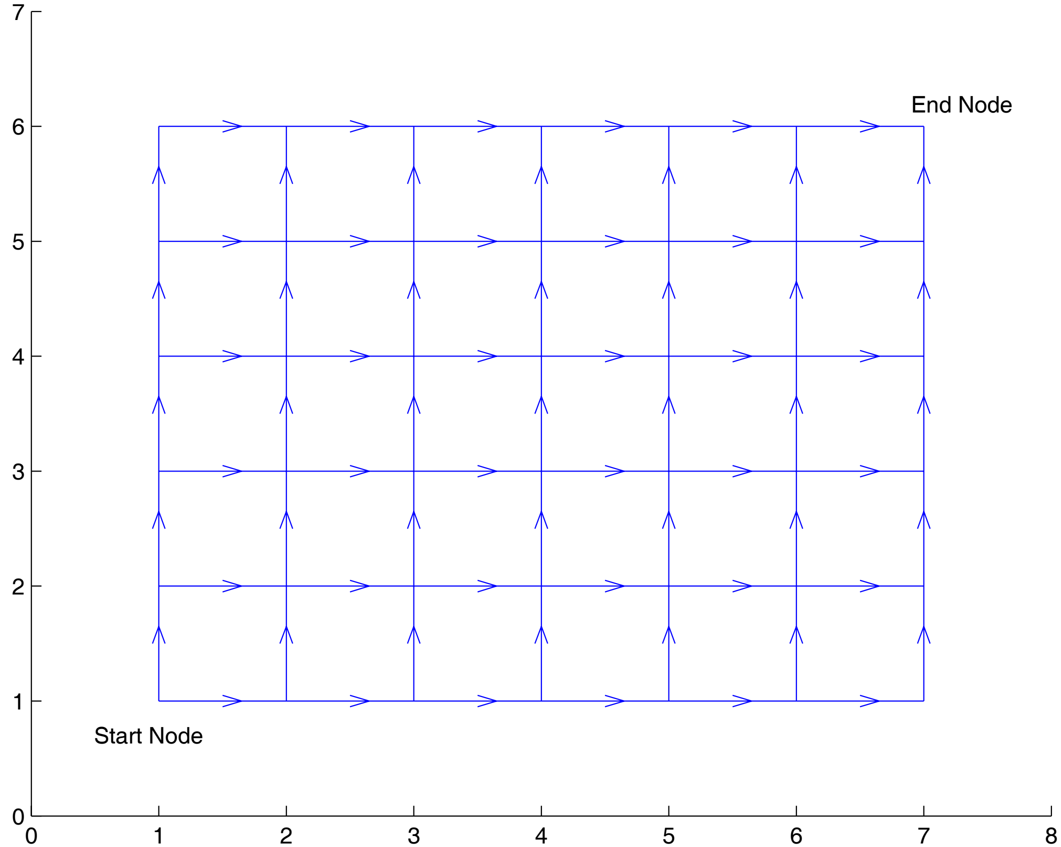
$$c_{ij} = a_{ij} = 1$$
$$\tilde{b}_{ij} = 100(1 + \tilde{z}_{ij})$$
$$T = 100(H + W - 2)$$

$$\mathrm{P}(\tilde{z}_{ij} = z) = \begin{cases} 0.6 & \text{if } z = -0.06 \\ 0.3 & \text{if } z = 0.04 \\ 0.1 & \text{if } z = 0.24 \end{cases}$$

$$\underline{z}_{ij} = 0.06, \bar{z}_{ij} = 0.24$$
$$p_{ij} = 0.1154, q_{ij} = 0.0917$$

Stochastic Activity Network

- Example: Grid Network
  - H=6 by W=7



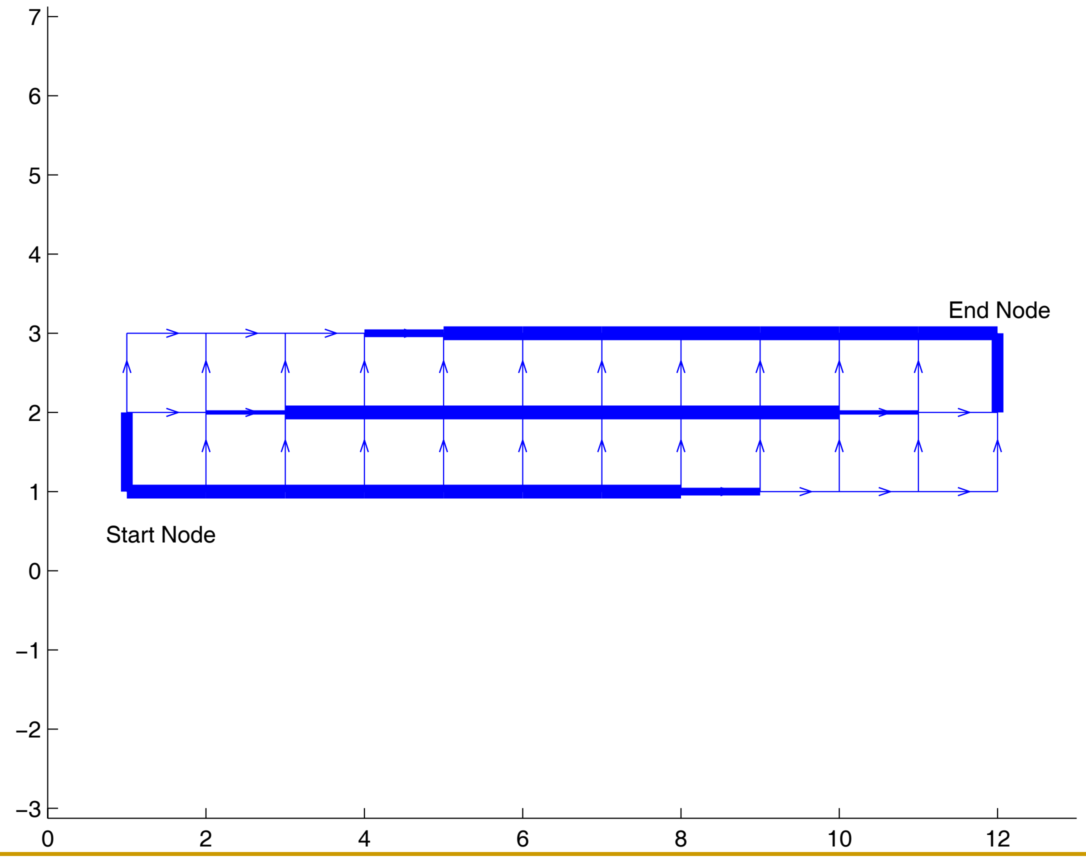
Stochastic Activity Network

- For 99% confidence of completion
  - Solved SOCP using SDPT3

H	W	m	n	Ω	Z_r^*	Z_w^*	$\frac{Z_r^*}{Z_w^*}$
3	3	12	9	3.77	269.83	288	0.94
3	4	17	12	3.86	367.06	408	0.90
3	8	37	24	4.05	519.69	888.	0.59
3	12	57	36	4.16	587.09	1368	0.43

Stochastic Activity Network

- Grid Network Solution
  - H=3 by W=12



Stochastic linear optimization

Do soft constriants always make sense in the stochastic optimization model?

Stochastic linear optimization

- E.g. Newsboy Model
  - What is the meaning of having probabilistic constraints?

$$
\begin{aligned}
\text{max} \quad & -cx + p\mathsf{E}(y(\tilde{d})) \\
\text{s.t.} \quad & y(\tilde{d}) \le \tilde{d} \\
& y(\tilde{d}) \le x \\
& x \ge 0
\end{aligned}
$$

Stochastic linear optimization

- Complete recourse problems
  - Second stage is always feasible.
  - Would chance constraint make any sense?

$$
\begin{array}{ll}
\min & \boldsymbol{c}'\boldsymbol{x} + \mathrm{E}(\boldsymbol{f}'\boldsymbol{w}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{U}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{W}\boldsymbol{w}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\
& \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq 0 \\
& \boldsymbol{x} \geq 0,
\end{array}
$$

$$\forall \boldsymbol{t}, \exists \boldsymbol{w} \geq 0 \text{ such that } \boldsymbol{W}\boldsymbol{w} = \boldsymbol{t}$$

Stochastic linear optimization

- Chen, Sim, Sun and Zhang (2005)

$$
\begin{aligned}
\min \quad & \boldsymbol{c}'\boldsymbol{x} + \mathsf{E}(\boldsymbol{d}'\boldsymbol{v}(\tilde{\boldsymbol{z}})) + \mathsf{E}(\boldsymbol{f}'\boldsymbol{w}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{U}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{V}\boldsymbol{v}(\tilde{\boldsymbol{z}}) + \boldsymbol{W}\boldsymbol{w}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\
& \mathsf{P}(v_i(\tilde{\boldsymbol{z}}) \geq 0) \geq 1 - \epsilon_i \\
& \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq 0 \\
& \boldsymbol{x} \geq 0,
\end{aligned}
$$

$\boldsymbol{U}$ and $\boldsymbol{h}$ are affine in $\tilde{\boldsymbol{z}}$

Stochastic linear optimization

- Semi-complete Recourse

$$\begin{array}{ll} \min & c'x + \mathrm{E}(d'v(\tilde{z})) + \mathrm{E}(f'w(\tilde{z})) \\ \text{s.t.} & Ax = b \\ & U(\tilde{z})x + Vv(\tilde{z}) + Ww(\tilde{z}) = h(\tilde{z}) \\ & \mathrm{P}(v_i(\tilde{z}) \geq 0) \geq 1 - \epsilon_i \\ & w(\tilde{z}) \geq 0 \\ & x \geq 0, \end{array}$$

$$\exists w > 0 \text{ such that } Ww = 0$$

Stochastic linear optimization

- Complete recourse implies semi-complete

$$
\begin{array}{ll}
\min & \boldsymbol{c}'\boldsymbol{x} + \mathsf{E}(\boldsymbol{d}'\boldsymbol{v}(\tilde{\boldsymbol{z}})) + \mathsf{E}(\boldsymbol{f}'\boldsymbol{w}(\tilde{\boldsymbol{z}})) \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{U}(\tilde{\boldsymbol{z}})\boldsymbol{x} + \boldsymbol{V}\boldsymbol{v}(\tilde{\boldsymbol{z}}) + \boldsymbol{W}\boldsymbol{w}(\tilde{\boldsymbol{z}}) = \boldsymbol{h}(\tilde{\boldsymbol{z}}) \\
& \mathsf{P}(v_i(\tilde{\boldsymbol{z}}) \geq 0) \geq 1 - \epsilon_i \\
& \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq 0 \\
& \boldsymbol{x} \geq 0,
\end{array}
$$

$$\forall \boldsymbol{t}, \exists \boldsymbol{w} \geq 0 \text{ such that } \boldsymbol{W}\boldsymbol{w} = \boldsymbol{t}$$

$$\Downarrow$$

$$\exists \boldsymbol{w} > 0 \text{ such that } \boldsymbol{W}\boldsymbol{w} = 0$$

Stochastic linear optimization

- Modified Linear Decision Rule

$$\boldsymbol{w}(\tilde{\boldsymbol{z}}) = \boldsymbol{r}^0 + \sum_{k=1}^{N} \boldsymbol{r}^k \tilde{z}_k + \sum_{i=1}^{n_3} ((\boldsymbol{r}^0 + \sum_{k=1}^{N} \boldsymbol{r}^k \tilde{z}_k)_i)^- \boldsymbol{p}^i$$

where $\boldsymbol{p}^i$ is optimal for

$$\begin{array}{ll} \min & \boldsymbol{f}'\boldsymbol{w} \\ \text{s.t.} & \boldsymbol{W}\boldsymbol{w} = 0 \\ & w_i = 1 \\ & \boldsymbol{w} \geq 0 \end{array}$$

$\boldsymbol{W}\boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0}$

Define $\bar{f}_i = \boldsymbol{f}'\boldsymbol{p}^i$

Stochastic linear optimization

- Modified Linear Decision Rule

$$
\begin{array}{ll}
\min & \boldsymbol{c}'\boldsymbol{x} + \boldsymbol{d}'\boldsymbol{v}^0 + \boldsymbol{f}'\boldsymbol{r}^0 + \mathsf{E}\left[\bar{\boldsymbol{f}}'(\boldsymbol{r}(\tilde{\boldsymbol{z}}))^-\right] \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{U}_k\boldsymbol{x} + \boldsymbol{V}\boldsymbol{v}_k + \boldsymbol{W}\boldsymbol{r}_k = \boldsymbol{h}_k \quad k = 0, \ldots, N \\
& \mathsf{P}(v_i(\tilde{\boldsymbol{z}}) \geq 0) \geq 1 - \epsilon_i \quad \forall i \\
& \boldsymbol{x} \geq 0
\end{array}
$$

Stochastic linear optimization

- Approximating Objective Function
  - Can be approximated very well as SOCP!!

$$
\mathrm{E}((r(\tilde{z}))^-) \le \inf_{\mu>0} \left\{ \frac{\mu}{e} \exp\left(-\frac{r_0 - \boldsymbol{t}'\bar{\boldsymbol{z}} - \boldsymbol{s}'\underline{\boldsymbol{z}}}{\mu} + \frac{u_0^2}{2\mu^2} \right) \right\}
$$

for all $u_0 \in \Re, \boldsymbol{u} \in \Re^N$, $\boldsymbol{t}, \boldsymbol{s} \in \Re^N$, satisfying

$$
\begin{array}{rcll}
u_k & \ge & q_k(r_k + t_k - s_k) & \forall k \in \{1, \ldots, N\} \\
u_k & \ge & -p_k(r_k + t_k - s_k) & \forall k \in \{1, \ldots, N\} \\
\|\boldsymbol{u}\|_2 & \le & u_0 & \\
\boldsymbol{t}, \boldsymbol{s} & \ge & \boldsymbol{0} &
\end{array}
$$

Stochastic linear optimization

- Distributional System with Transshipment (Chou, Sim and So (2005))
  - N retailers to stock up inventory
  - Uncertain demand
    - Service level: Satisfy all demands with high probability, as opposed to shortage costs
  - Transshipment to other retailers in shortage
  - Minimize long term replenishment costs and transshipment while satisfying service constraints.

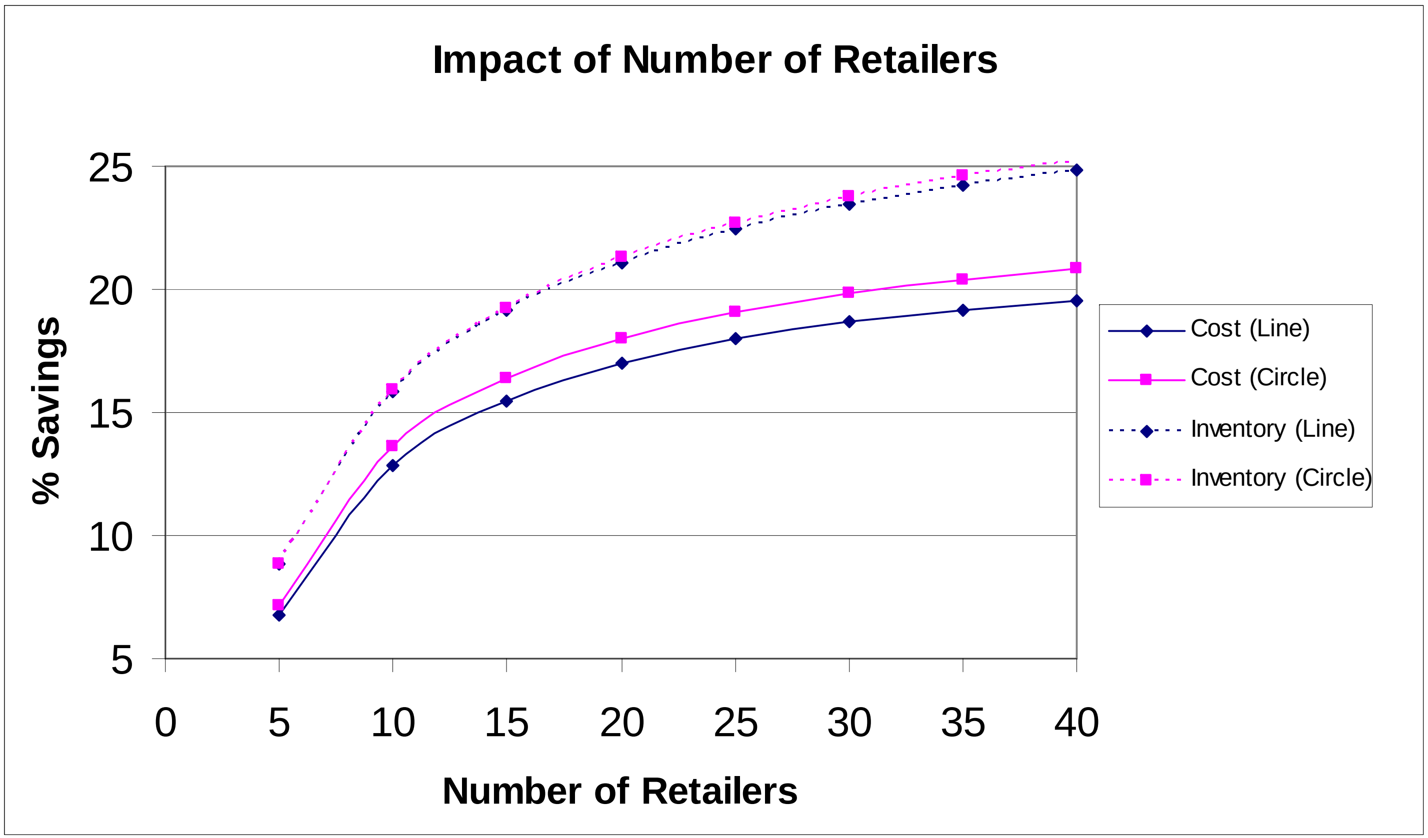
Stochastic linear optimization

- Distributional System with Transshipment
  - Case of Semi-complete Recourse

$$
\begin{array}{lll}
\min & \boldsymbol{h}'\boldsymbol{x} + \sum_{(i,j)\in\mathcal{E}} c_{ij}\mathsf{E}\left(w_{ij}(\tilde{\boldsymbol{z}})\right) & \\
\text{s.t.} & v_i(\tilde{\boldsymbol{z}}) = x_i - d_i(\tilde{\boldsymbol{z}}) - \sum_{j:(i,j)\in\mathcal{E}} w_{ij}(\tilde{\boldsymbol{z}}) + \sum_{j:(j,i)\in\mathcal{E}} w_{ji}(\tilde{\boldsymbol{z}}) & \forall i \\
& \mathsf{P}(v_i(\tilde{\boldsymbol{z}}) \geq 0) \geq 1 - \epsilon_i & \forall i \in I \\
& \boldsymbol{w}(\tilde{\boldsymbol{z}}) \geq \boldsymbol{0} & \\
& \boldsymbol{x} \geq 0 &
\end{array}
$$

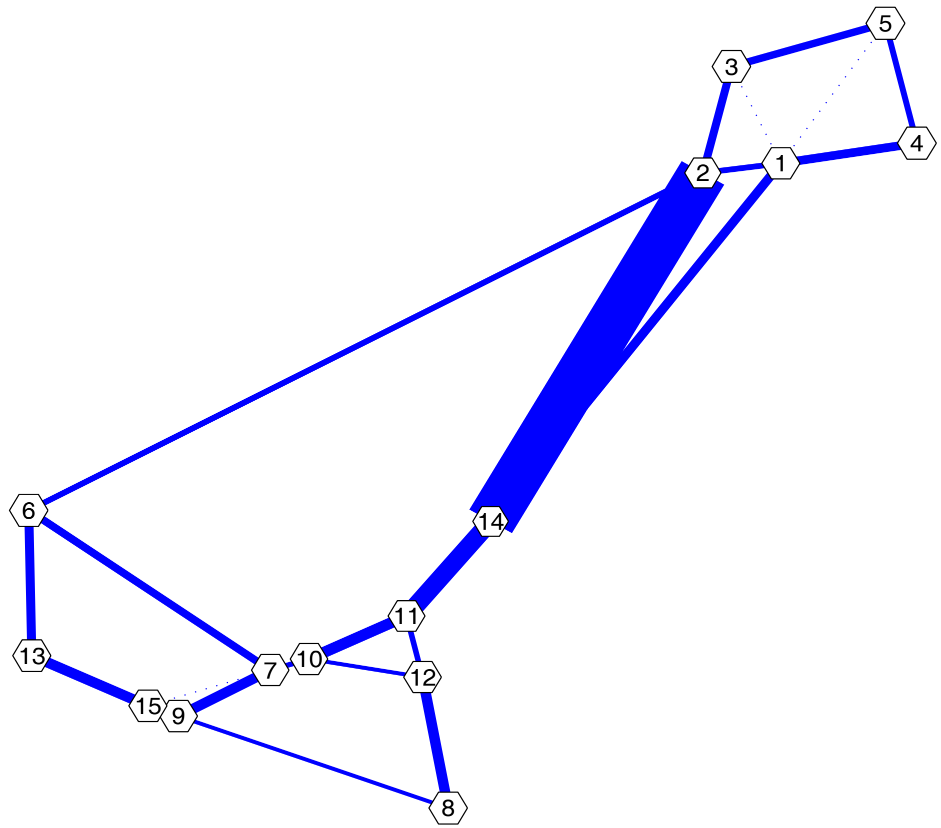
Stochastic linear optimization

- Distributional System with Transshipment



Stochastic linear optimization

- Distributional System with Transshipment



Conclusions

- Current Work
  - Joint chance constraints
- Future work
  - How to obtain a lower bound for the framework?
  - How to incorporate discrete conditions?
- Papers:
  - http://www.bschool.nus.edu/STAFF/dscsimm/research.htm