A Robust optimization formulation for general nonlinear programming

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Abstract

For optimal design or control problems involving safety constraints and parameter uncertainty, a central task is to select safety margins. Robust optimization provides a natural mechanism for this task. We propose a general robust formulation for nonlinear programming, and discuss the advantages and limitations of this formulation.

An ODE Control Example

$$\begin{aligned} \min_{y,u\,\in\mathcal{U}} \quad & \left\| y(T) - y^{end} \right\|_2^2 \\ \text{s.t.} \quad & \dot{y} = f(y,u,s), \qquad y(0) = y^{ini}, \\ & y_i(t) \le \gamma, \qquad t \in (0,T) \end{aligned}$$

— $y(t)$ is state, $u(t)$ is control and s is a parameter vector.

— Parameter s contains uncertainty.

— There are important (inequality) “safety constraints”.

– The system and constraints are generally nonlinear.

Task: Optimize while maintaining safety for all “reasonable” s values.

Robust Optimization: Existing Results and Need for Extension

- Robust optimization has been actively studied for linear conic programming: LP, SOCP, SDP (Ben-Tal & Nemirovski, El Ghaoui, ...).

- Data uncertainty are restricted to cases where

  – every uncertain constraint is a linear inequality

  – every uncertain parameter appears linearly

  – e.g. LP: $Ax \leq b$ where (A, b) is uncertain (no $A(s)x \leq b(s)$)

- The applicability of the existing formulations needs to be extended.

- Our extension uses simple ideas: implicit function theorem, linearization

General Nonlinear Programming

$$\begin{aligned} \min_{y,u\,\in\mathcal{U}} \quad & \phi(y,u,s) \\ \text{s.t.} \quad & F(y,u,s) = 0 \\ & G(y,u,s) \le 0. \end{aligned}$$

- $y =$ state variable; $u =$ design/control variable; $s =$ system parameter.
- Assume that state equation $F = 0$ implicitly defines $y = y(u,s)$.
- Assume that a good parameter estimation $\hat{s}$ is available.
- Assume that all inequalities are “safety constraints”.

A Look at A Robust Model

$$
\begin{array}{ll}
\min\limits_{y,u\,\in\mathcal{U}} & \phi(y,u,\hat{s}) \\
\text{s.t.} & F(y,u,\hat{s}) = 0 \\
 & (F_y y_s + F_s)(y,u,\hat{s}) = 0 \\
 & G(y,u,s)e^T \pm \tau(G_y y_s + G_s) \le 0.
\end{array}
$$

A nonlinear program involving only $\hat{s}$, new unknown y_s, $e^T = (1 \cdots 1)$.

$$
\begin{array}{lcl}
\min & \left\| y(T) - y^{end} \right\|_2^2 & \\
\text{s.t.} & \dot{y} = f(y,u,\hat{s}), & y(0) = y^{ini}, \\
 & \dot{y}_s = f_y(y,u,\hat{s})y_s + f_s(y,u,\hat{s}), & y_s(0) = 0, \\
 & y(t)e^T \pm \tau y_s(t) \le 0, & t \in (0,T)
\end{array}
$$

Robust Model Derivation: First Step

We start from the Inequality-only case:

$$\begin{array}{lll} \min & \phi(u) & \\ \text{s.t.} & h_i(u,s) \le 0, \quad \forall s \in \mathcal{S} & \forall i \end{array}$$

$$\Updownarrow$$

$$\max\{h_i(u,s) : s \in \mathcal{S}\} \le 0, \quad \forall i$$

- Can we handle this comfortably?

$$\max_{s \in \mathcal{S}} h_i(u,s)$$

- Maybe too hard for general $\mathcal{S}$ and $H = (h_1 \cdots, h_m)^T$.

Restrict $\mathcal{S}$ and Linearize H

- For $\tau > 0$, define
$$\mathcal{S}_\tau = \{\hat{s} + \tau D\delta : \|\delta\|_p \le 1\}$$
$\hat{s}$ is the nominal parameter value, δ the unit variation (in p-norm) and τ the magnitude of the variation. D is a scale or basis matrix for the variations.

- By Taylor approximation,
$$h_i(u, \hat{s} + \tau D\delta) \approx h_i(u, \hat{s}) + \tau \langle \nabla_s h_i(u, \hat{s}), D\delta \rangle.$$

- Hence,
$$\max_{s \in S_\tau} h_i(u, s) \approx h_i(u, \hat{s}) + \tau \max_{\|\delta\|_p = 1} \langle D^T \nabla_s h_i(u, \hat{s}), \delta \rangle$$

Solution of Linearized Maximization

Recall

$$|\langle c, x\rangle| \le \|x\|_p \|c\|_q \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \le p, q \le +\infty,$$

and the equality can be achieved, i.e.,

$$\max_{\|x\|_p = 1} \langle c, x\rangle = \|c\|_q.$$

- Therefore,

$$\begin{aligned} \max_{s \in S_\tau} h_i(u, s) \quad &\approx \quad h_i(u, \hat{s}) + \tau \max_{\|\delta\|_p = 1} \langle D^T \nabla_s h_i(u, \hat{s}), \delta\rangle \\ &= \quad h_i(u, \hat{s}) + \tau \, \|D^T \nabla_s h_i(u, \hat{s})\|_q. \end{aligned}$$

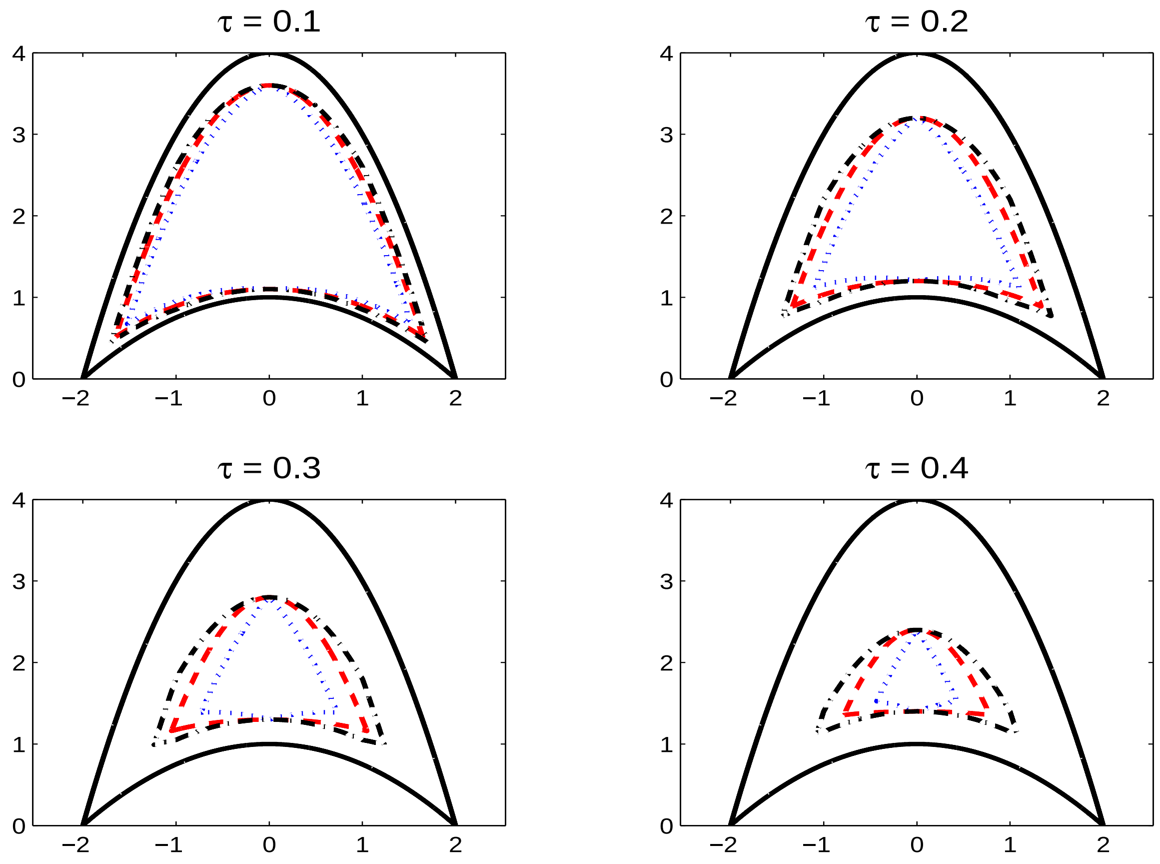
- Now replace $\max h_i(u, s) \le 0$ by $h_i(u, \hat{s}) + \tau \, \|D^T \nabla_s h_i(u, \hat{s})\|_q \le 0$.

Robust Model: Inequality-only

$$\begin{array}{ll} \min & \phi(u) \\ \text{s.t.} & h_i(u,\hat{s}) + \tau \, \|D^T \nabla_s h_i(u,\hat{s})\|_q \le 0, \quad \forall i \end{array}$$

- Only need to optimize once at the nominal value $s = \hat{s}$.
- The added nonnegative terms are "safety margins".
- The margins are proportional to the magnitude τ of the variations.
- Also proportional to the sensitivity of h_i to parameter s at $\hat{s}$.

Robust Feasible Set Illustrations



Feasible set is $\{(x, y) : |x| \leq 2, (ax^2 + b) \leq y \leq 4(ax^2 + b)\}$ with $\hat{a} = 1/4$ and $\hat{b} = 1$. Robust feasible sets are plotted for $q = 1, 2, \infty$ and various τ values.

General Nonlinear Programming

$$\begin{aligned} \min_{y,u\in\mathcal{U}} \quad & \phi(y,u,s) \\ \text{s.t.} \quad & F(y,u,s) = 0 \\ & G(y,u,s) \leq 0 \end{aligned}$$

Approach: **Implicit Function, elimination, and Chain Rule**.

Let $y(u,s)$ satisfy $F(y,u,s) = 0$. Then

$$H(u,s) := G(y(u,s),u,s) \leq 0$$

Even if $G = G(y,u)$, H can still be sensitive to s through the state y.

Be explicit with Implicit Functions

After elimination and robustification:

$$F(y,u,\hat{s}) = 0,\ g_i(y,u,\hat{s}) \le 0 \quad \Longrightarrow \quad h_i(u,\hat{s}) + \tau \| D^T \nabla_s h_i(u,\hat{s}) \|_q \le 0.$$

Recall $H(u,s) = G(y(u,s),u,s)$,

$$\nabla_s h_i = (H_s)^T e_i, \qquad H_s = G_s + G_y y_s.$$

Implicit functions are hard to handle. We put definitions into constraints.

$$\begin{aligned} F(y,u,s) &= 0, \quad \text{for } y(u,s) \\ F_y y_s + F_s &= 0, \quad \text{for } y_s(u,s) \end{aligned}$$

With the above equality imposed,

$$\Longleftrightarrow \quad g_i(y,u,\hat{s}) + \tau \| D^T (G_s + G_y\, y_s)^T e_i \|_q \le 0$$

General Robust Model

$$\begin{array}{ll} \displaystyle\min_{y,u\,\in\mathcal{U}} & \phi(y,u,\hat{s}) \\ \text{s.t.} & F(y,u,\hat{s}) = 0 \\ & \tau(F_y y_s + F_s)(y,u,\hat{s}) = 0 \\ & g_i(y,u,\hat{s}) + \tau\|D^T(G_s + G_y\, y_s)^T e_i\|_q \le 0 \end{array}$$

$$\tau = 0 \implies \text{The original model}$$

- The robust model remains a one-level, nonlinear program involving only the nominal values of uncertain parameters.
- The robust model produces a robust design/control variable u^*. For every parameter value s near $\hat{s}$, we can find a corresponding **safe state** $y(u^*, s)$.

A Robust Model for $p = 1 (q = \infty)$

$$\begin{aligned} \min_{y,u \in \mathcal{U}} \quad & \phi(y, u, \hat{s}) \\ \text{s.t.} \quad & F(y, u, \hat{s}) = 0 \\ & \tau(F_y y_s + F_s)(y, u, \hat{s}) = 0 \\ & G(y, u, \hat{s}) e^T \pm \tau(G_y y_s + G_s) D \leq 0 \end{aligned}$$

If everything is sensitive to $s \in \Re^k$, then:

$n \rightarrow n + n * k$ state variables (y and y_s)

$n \rightarrow n + n * k$ equations (for y and y_s)

$m \rightarrow 2 * m * k$ inequalities

Computational costs remain in the same order if $k = O(1)$
(not so if $k = O(n)$ or $O(m)$).

ODE Example Revisited

$$
\begin{array}{lll}
\min & \left\| y(T) - y^{end} \right\|_2^2 & \\
\text{s.t.} & \dot{y} = f(y, u, \hat{s}), & y(0) = y^{ini}, \\
& \dot{y}_s = f_y(y, u, \hat{s}) y_s + f_s(y, u, \hat{s}), & y_s(0) = 0, \\
& y(t) e^T \pm \tau y_s(t) \leq 0, & t \in (0, T)
\end{array}
$$

Let $y \in \Re^n$ and $s \in \Re^k$.

- A new ODE system of size $n * k$ is added for the sensitivity y_s.
- Safety constraints become a set of $2n * k$ constraints.

1st-Order Robustness

Theorem

Let $(\hat{y}, \hat{u})$ *be strictly feasible to corresponding to* $\hat{s}$ *and* $\tau > 0$*.*

Assume that in the set $\mathcal{S}_\tau$*,*

1. $y(\hat{u}, s)$ *is implicitly defined as a differentiable function of* s *via the equation* $F(y, \hat{u}, s) = 0$*;*
2. *every row of* $[G_y\, y_s + G_s](y(s), \hat{u}, s)$ *is Lipschitz continuous modulo to* L*.*

Then

$$G(y(\hat{u}, s), \hat{u}, s) \leq \frac{L}{2}\tau^2, \ \forall s \in \mathcal{S}_\tau.$$

Parameterized Linear Programs:

$$
\begin{aligned}
\min_{y,u} \quad & \langle c_0(s), y\rangle + \langle d_0(s), u\rangle + \gamma_0(s) \\
\text{s.t.} \quad & A(s)y + B(s)u - h(s) = 0 \\
& \langle c_i(s), y\rangle + \langle d_i(s), u\rangle + \gamma_i(s) \leq 0 \\
& i = 1, 2, \cdots, m
\end{aligned}
$$

Proposition

If all the functions of s are in C^1 (not necessarily linear) and $A(s)$ is invertible, then the robust model of the above parameterized LP is again an LP for $p = 1$ or ∞, and an SOCP for $p = 2$.

— Nonlinear parameter dependency can be handled.

— Equality constraints can be handled (at least partially).

Applicability, Advantages and Limitations

- Provide intelligent safety margins to maintain safety under parameter uncertainty.
- Can handle a wide range of general nonlinear programs.
- Solve a one-level nonlinear program at the nominal value; relatively inexpensive when the number of uncertain parameters is moderate.
- Require good parameter estimates. Parameter variations need to be relatively small around their nominal values.
- Robust only to the 1st-order around the nominal value.
- A technical report is available at: http://www.caam.rice.edu/caam/trs/2004/TR04-13.pdf