Inverse problems, model reduction and uncertainties

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1 – Practical setting, an example

Process tomography

- Measurement model: electrical impedance tomography
- Measurements carried out in 2 – 40 ms: frame rate
- Evolution model: Stochastic convection-diffusion model
- Estimation scheme: Linear or extended Kalman filter
- The models are only approximate and include several auxiliary unknowns
- Compute the estimates (and the associated covariances) within a frame

- Large scale problems – relative to the computational resources
- The approximation problem dual:
  - Given computational resources, can we exploit increasing accuracy of measurements?
  - Given the accuracy of the measurements: how much can we reduce the model?
- The uncertainty problem: How to deal with auxiliary unknowns: unknown boundary data, uninteresting (distributed) parameters, slightly unknown geometry, grossly approximative physical models etc.
- In many applications the modelling errors dominate the measurement errors
- Examples
  - Optical tomography
  - Electrical impedance tomography, stationary and nonstationary
  - Estimation of the coefficients of the heat equation

2 – The idea behind approximation/modelling error model

- Let $\bar{A}$ and $\bar{x}$ represent physical reality

$$\begin{aligned} y &= \bar{A}\bar{x} + e = Ax + (\bar{A}\bar{x} - Ax) + e \\ &= Ax + \varepsilon(\bar{x}) + e \end{aligned}$$

- The idea: treat ε as a random variable (additional noise) and compute its (second order joint) statistics.
- Clearly, ε cannot be computed since $\bar{A}$ and $\bar{x}$ cannot be.
- Construct a sequence of increasingly accurate approximations $A < A_1 < A_2, \ldots$ and $x < x_1 < x_2, \ldots$
- Can we capture the statistical structure by approximating $\varepsilon \approx A_k x_k - Ax$ for some small k over the prior model for x_k? The answer is often yes.
- Joint statistics of (y, x).

3 – Modelling errors

- Typical sources:
  - Unknown boundary conditions
  - Inaccurate geometry
  - Physical models: convection-diffusion model with Navier-Stokes flow
- Feasible models for the uncertainties can often be constructed
- Simulations (forward problems) may be carried out readily: model for $\bar{A}$?
- Treatment of model errors: fix the prior model and uncertain parameters for A, compute the approximative second order statistics for ε over the prior

4 – Optical tomography

- Single modulation frequency data

$$\begin{aligned} -\nabla\cdot\kappa\nabla\Phi(\omega)+\mu_a\Phi(\omega)+\frac{i\omega}{c} &= q_0, \quad \text{in } \Omega \\ \Phi(\omega)+2\kappa\vartheta\frac{\partial\Phi(\omega)}{\partial\nu} &= g_s, \quad \text{on } \partial\Omega \end{aligned}$$

- Prior model: smooth Gaussian MRF's in 2D, simulations
- Typically required number of nodes $\sim 4000 - -15000$, use a mesh with 700 nodes
- Diminishing noise level $\longrightarrow$ more accurate estimates?

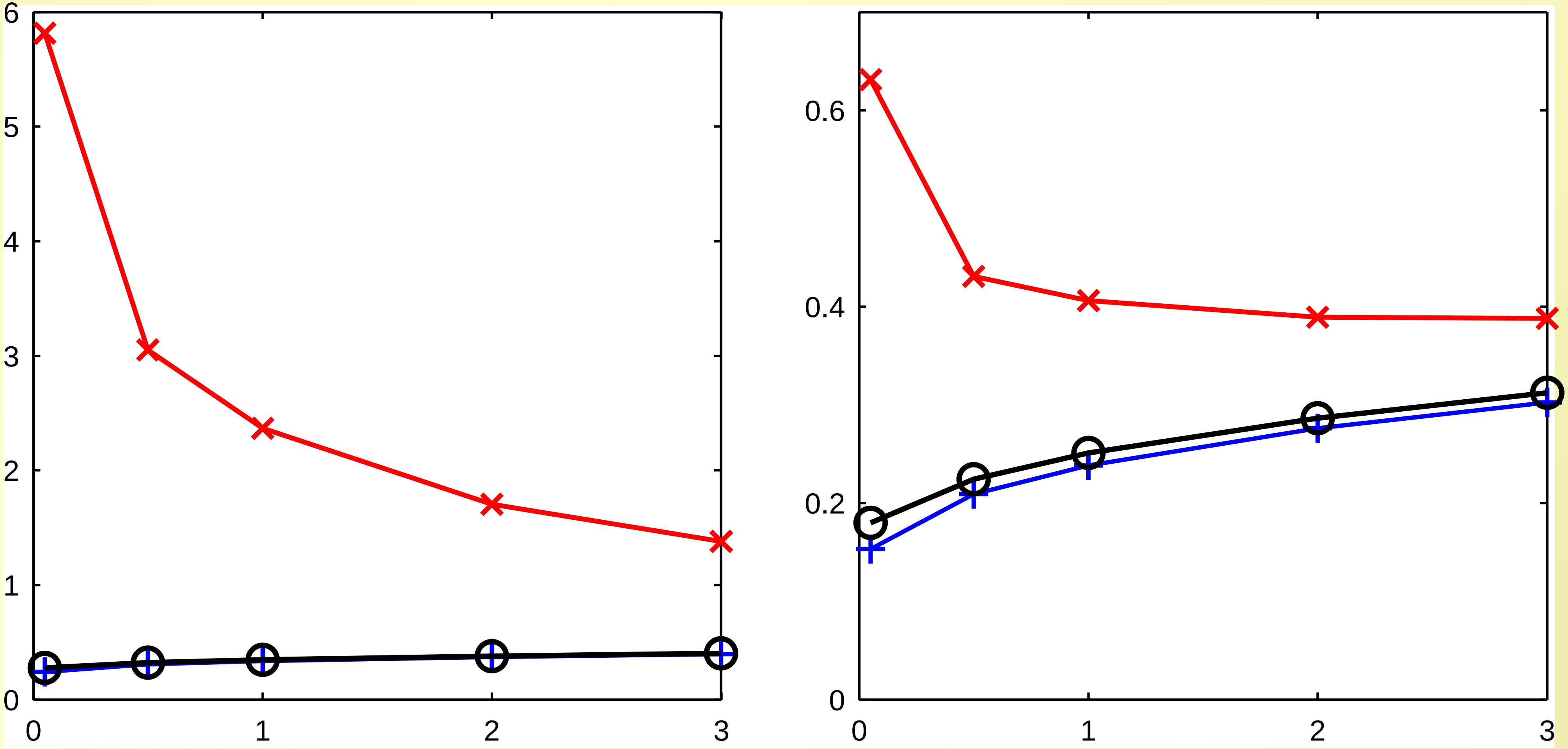


Figure 1: Effect of noise level, left: scattering coefficient, right: absorbtion coefficient. Red: conventional error model @ 700 nodes, Black: approximation error model @ 700 nodes, Blue: conventional error model @ 4200 nodes.

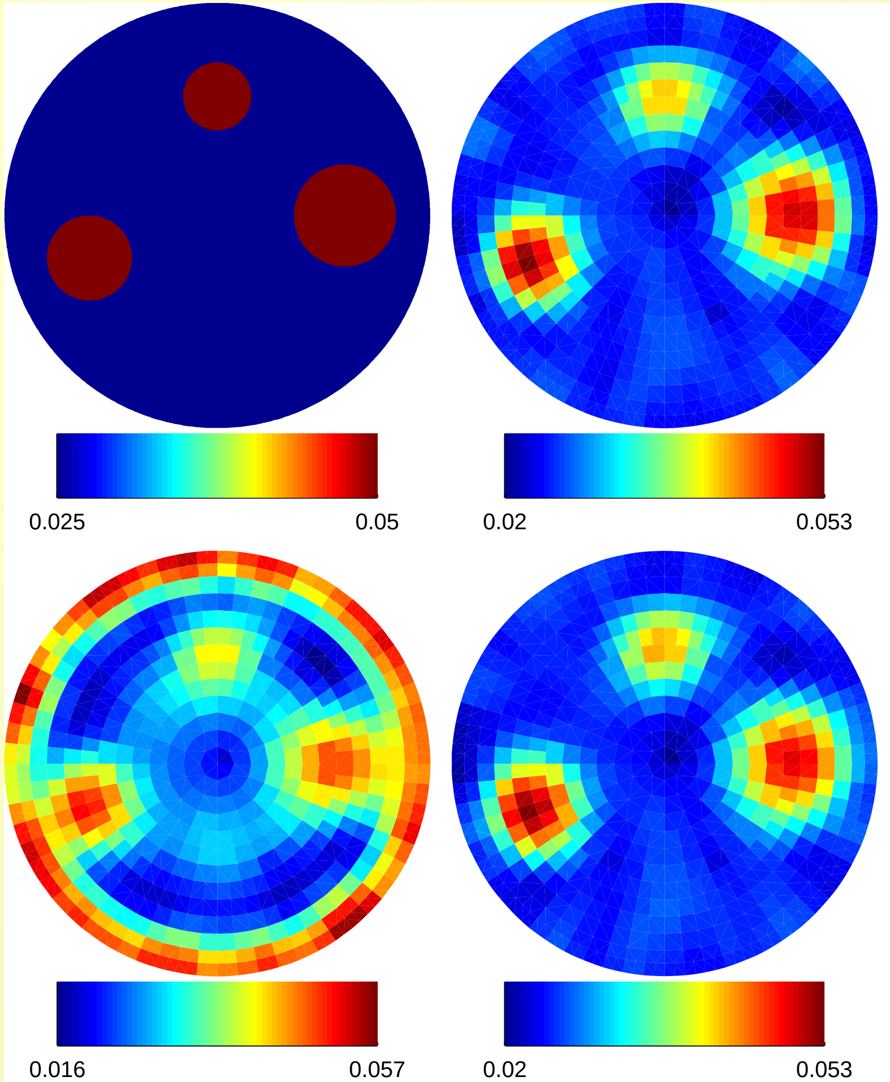


Figure 2: Blocky object, low prior density: absorbtion coefficient

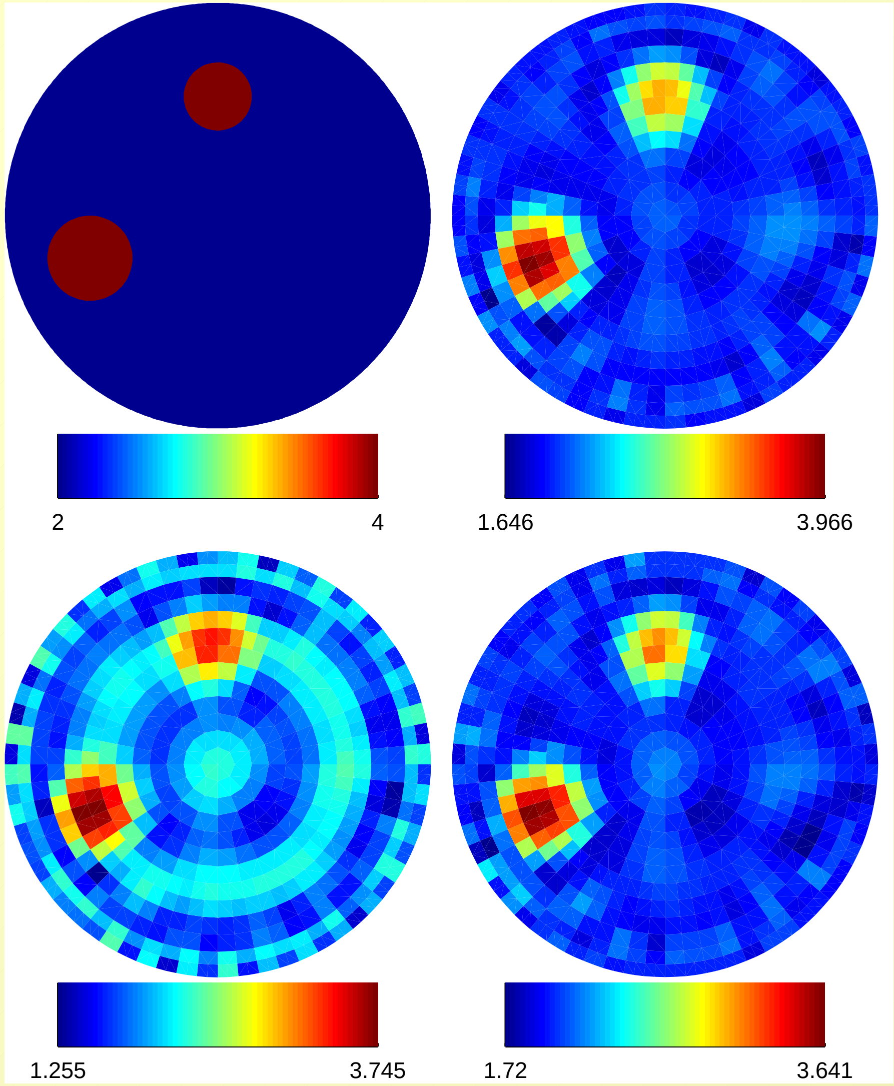


Figure 3: Blocky object, low prior density: scattering coefficient

5 – Electrical impedance tomography

- The governing equations for the *complete electrode model* are

$$\nabla \cdot (\sigma \nabla u) = 0, \quad x \in \Omega$$
$$u + z_\ell \sigma \frac{\partial u}{\partial \nu} = U^{(\ell)}, \quad x \in e_\ell,\ \ell = 1, 2, \ldots, L$$
$$\int_{e_\ell} \sigma \frac{\partial u}{\partial \nu}\, \mathrm{d}S = I^{(\ell)}, \quad x \in e_\ell,\ \ell = 1, 2, \ldots, L$$
$$\sigma \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega \backslash \cup_{\ell=1}^{L} e_\ell$$

- Simple test with real data: a 3D tank filled with tap water, a single conductivity to estimate, $y \in \mathbb{R}^{1500}$
  - Forward problem mesh density with 800 nodes: error 40% $\longrightarrow$ 0.5% when employing the approximation error model

6 – Structure of the approximation errors

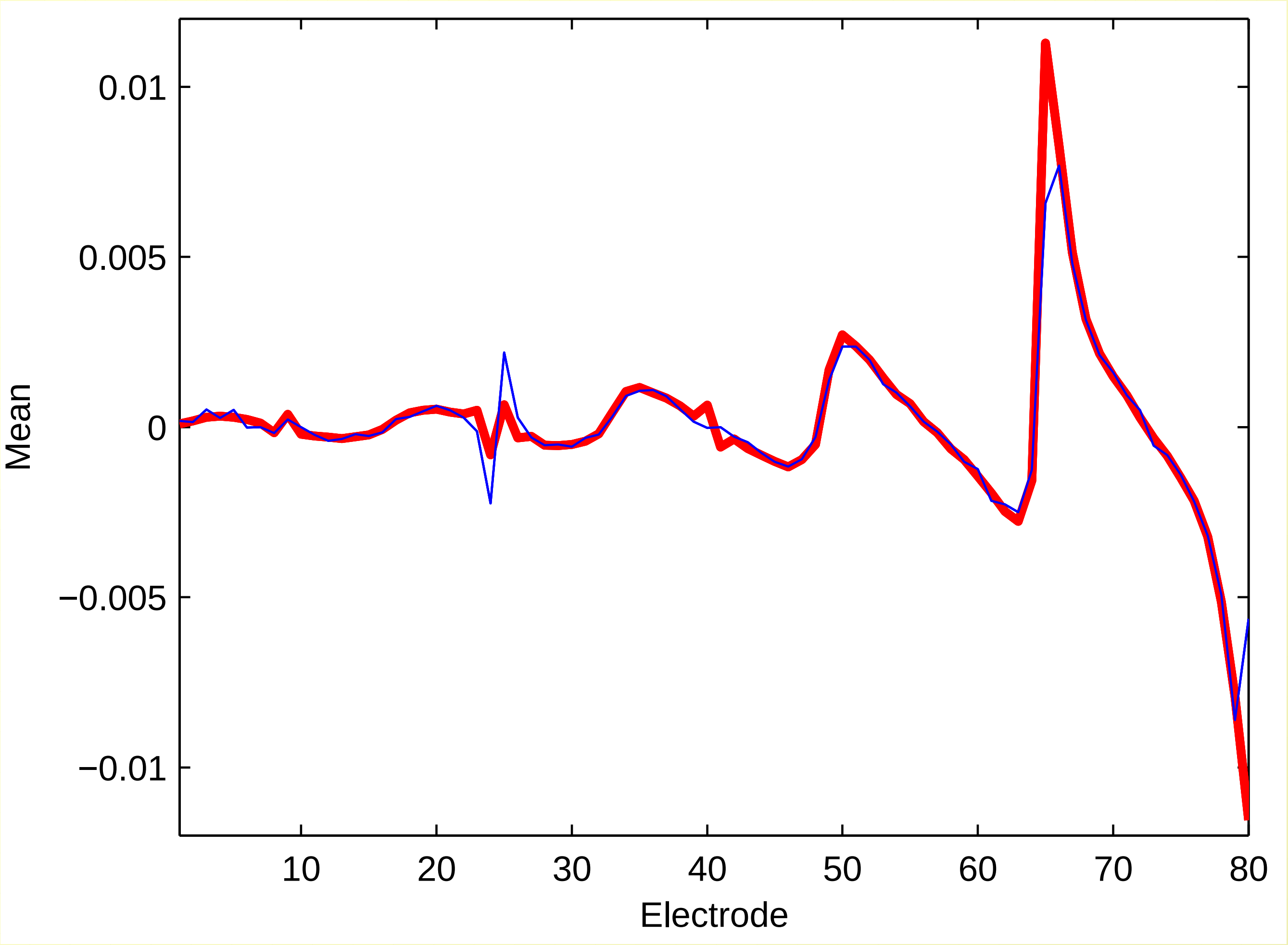
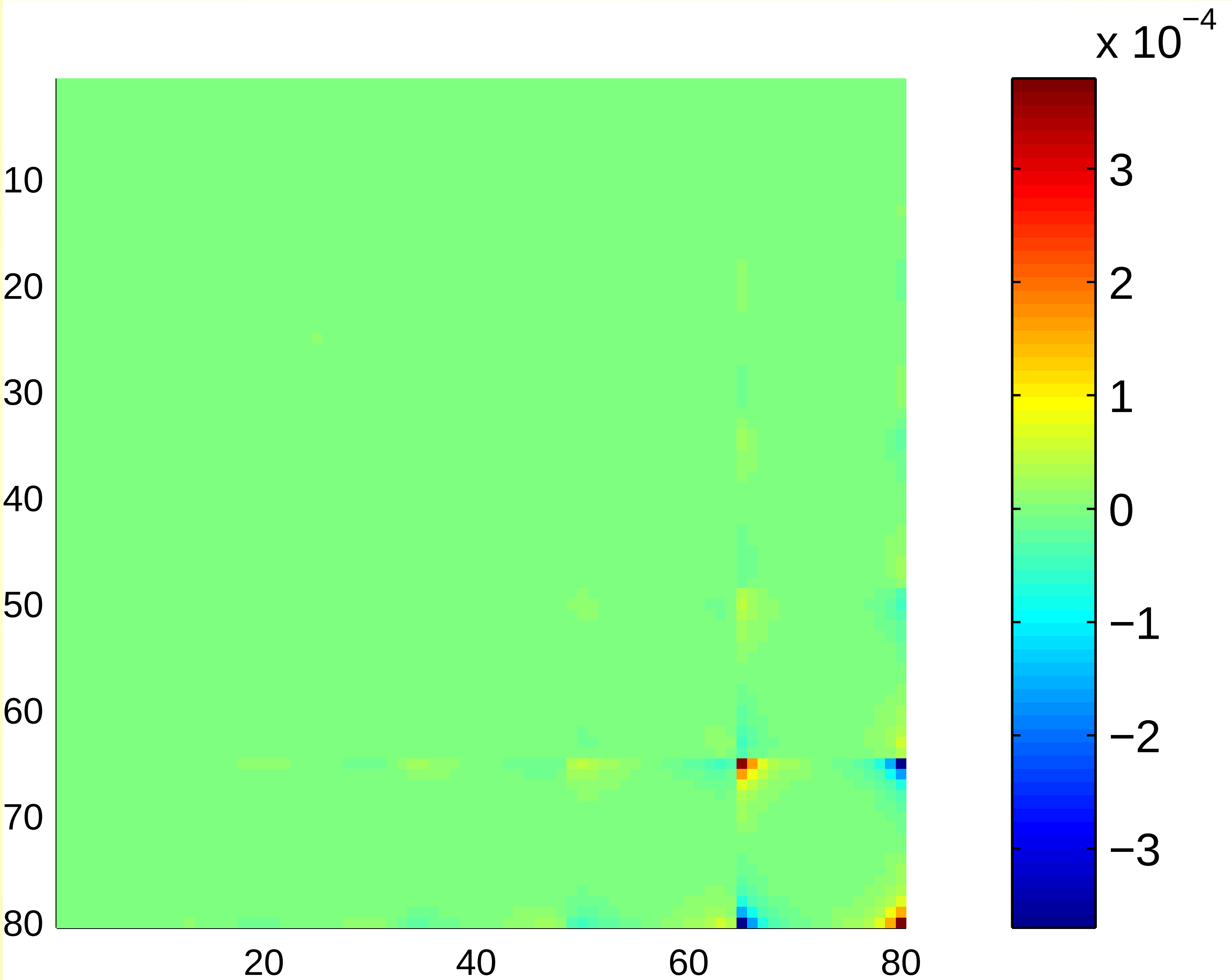


Figure 4: The measurements and predictions for a homogeneous tank(part).


x 10^−4
3
2
1
0
−1
−2
−3
10
20
30
40
50
60
70
80
20
40
60
80

7 – 3D tank EIT with two insulating rods, smooth MRF prior model

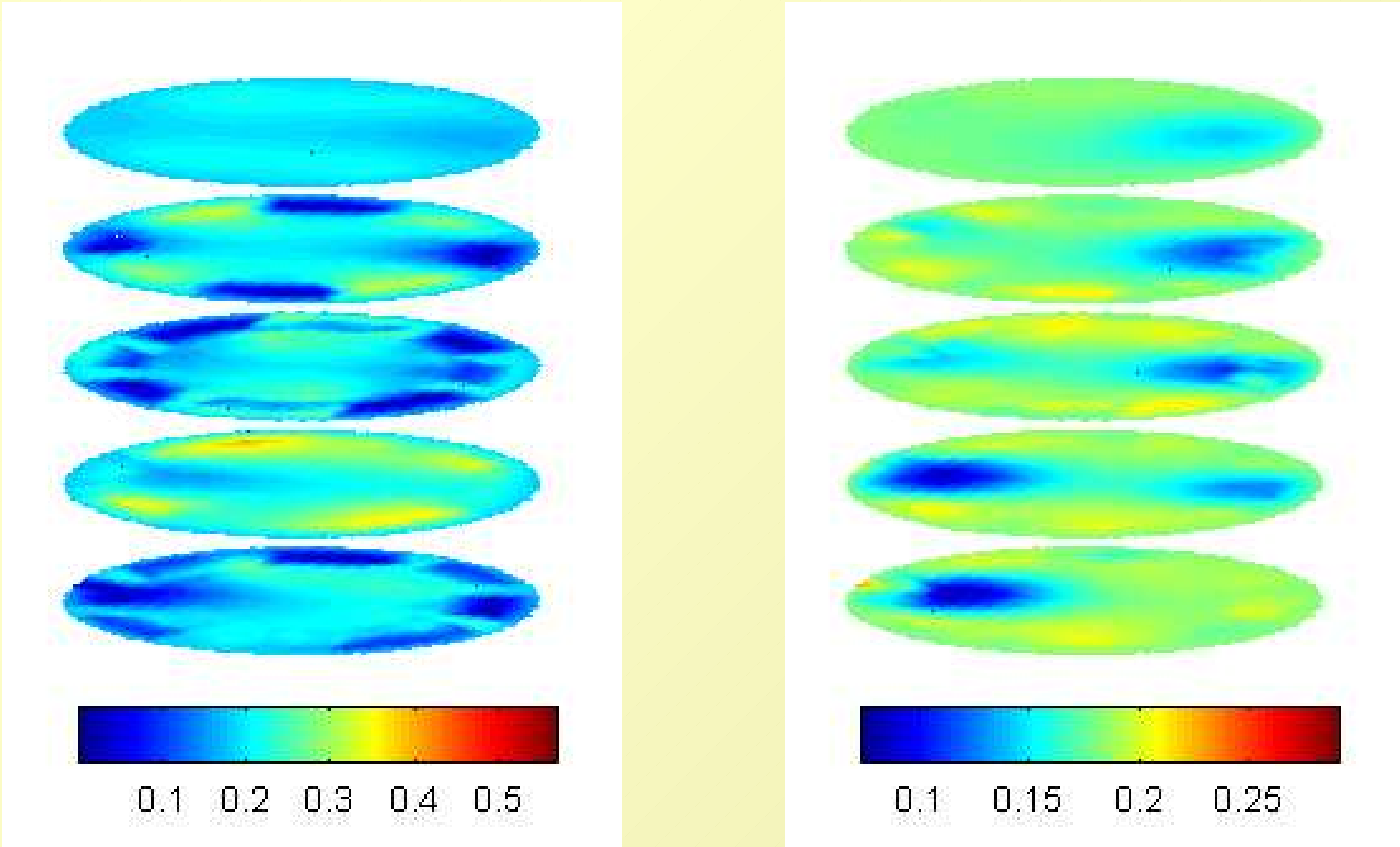


Figure 5: Left: Using ordinary likelihood model. Right: enhanced error model.

8 – State estimation and Kalman filters

- The state space representation

$$\begin{aligned} x_{t+1} &= F_t x_t + B_t u_t + w_t && \text{state evolution equation} \\ y_t &= G_t x_t + v_t && \text{observation equation} \end{aligned}$$

- Representation for the state, models for F_t, G_t and the covariances Γ_{w_t} and Γ_{v_t}
- Trivial models for the covariances?
- Approximation/modelling error model for the (extended) Kalman filter (linear Gaussian case: closed form and nonlinear cases: Monte Carlo simulations)

9 – The stochastic CD equation

- The stochastic CD equation

$$dx_t = \nabla \cdot \kappa \nabla x_t \, dt - \vec{v}_t \cdot \nabla x_t \, dt + dW_t$$

  where dW_t is Brownian motion.

- Need: initial and boundary conditions - these are usually partially unknown
- Model the uncertainty in parameters, discretize space, time and compute the statistics for $\int dW_t$ over the measurement intervals.
  - Build a statistical model for the missing boundary conditions
  - Use approximation error theory for a) discretization and b) parameters of the CD model etc.

10 – Practical example

- Is the statistical model good enough in practice?
- A practical example
  - Model: Stationary 2D single phase rotating flow in a circular tank
  - Reality: Nonstationary 3D turbulent two phase flow in a tank with a rotating blade

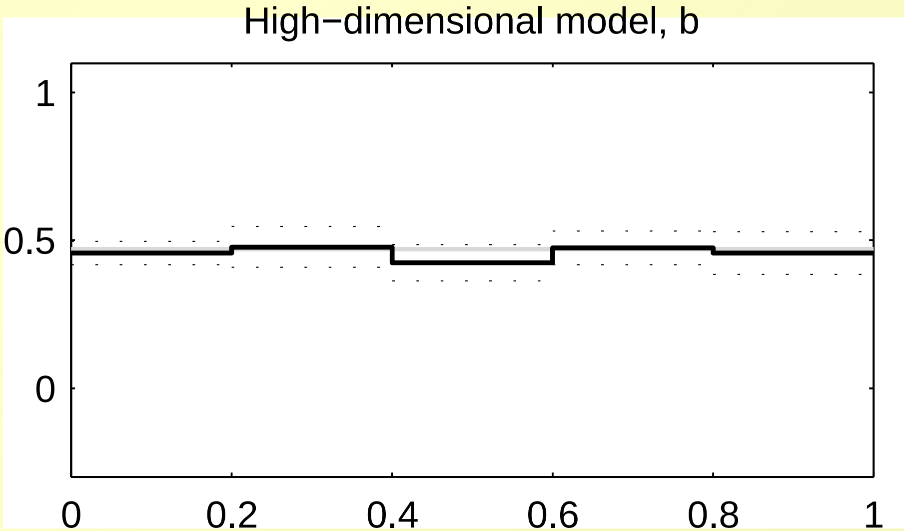
Figure 6: The water-filled ping-pong ball.

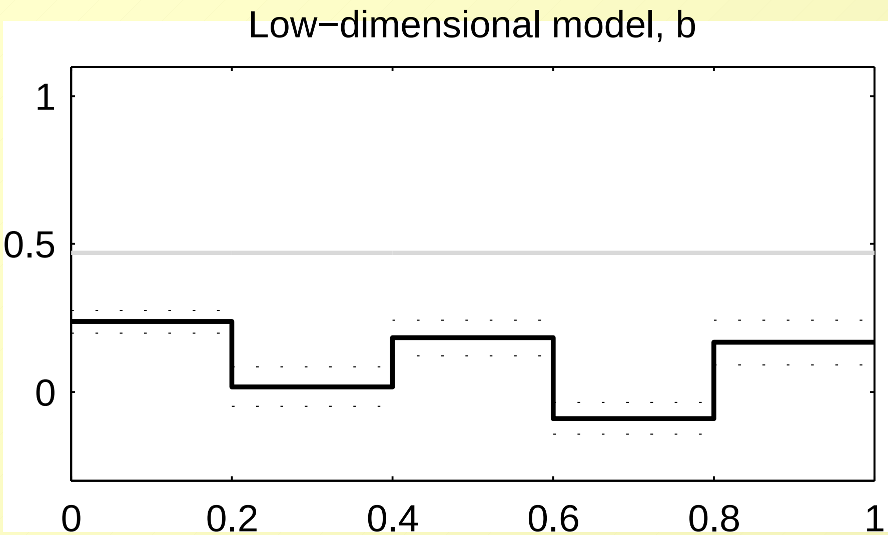
11 – State space identification and error estimates

- The Bayesian framework in principle allows for feasible error estimates
- Inverse problems typically exhibit small measurement errors
- Infeasible and misleading prior models can easily be constructed
- The key point in modelling

Feasible modelling of the uncertainty!

- Reliability and quality of the estimates and the error estimates.
- An example: 1D heat equation, the other end insulated, the other end can be heated (Dirichlet), estimate the segmentwise constant diffusion and perfusion coefficients: effect of model reduction


High−dimensional model, b
1
0.5
0
0
0.2
0.4
0.6
0.8
1


Low−dimensional model, b
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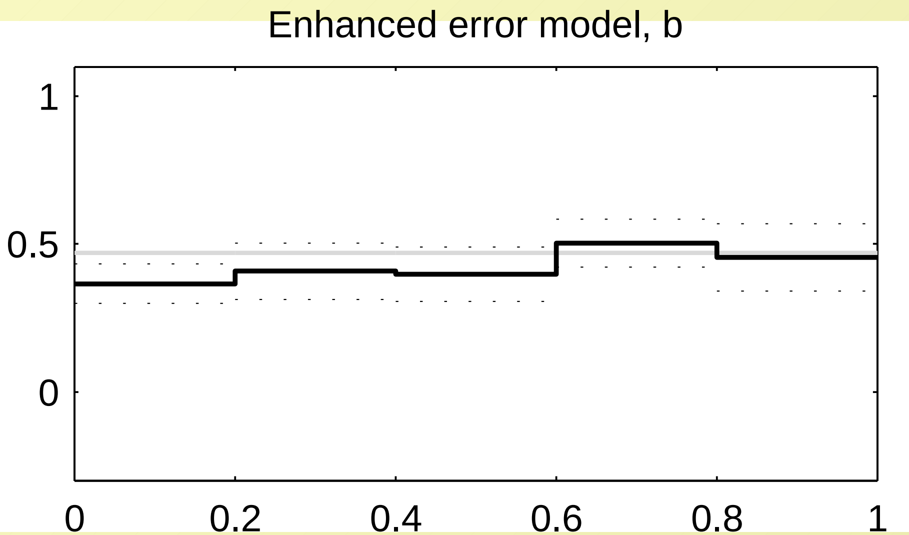
Enhanced error model, b
1
0.5
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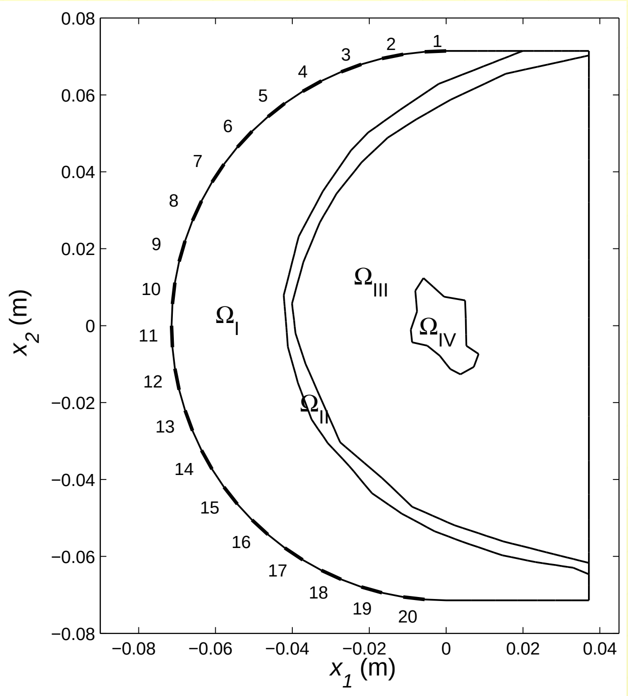
0
0.2
0.4
0.6
0.8
1

12 – State space identification: Ultrasound probing

- Use ultrasound therapy setting to estimate the thermal characteristics of the target
- Scan the target with an ultrasound focus, raise the temperature locally a few degrees
- Use MRI to measure the temperature evolution, errors 1°C.



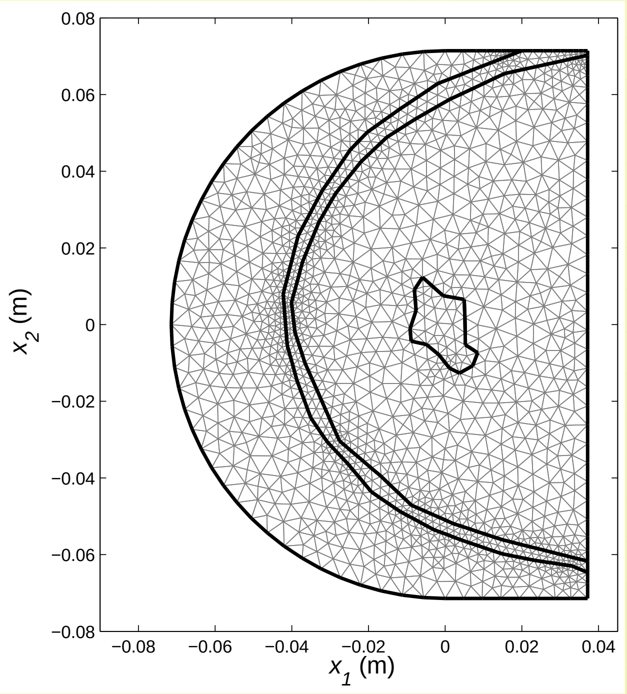


Figure 7: Mesh for estimation: 2965 elements and 1533 nodes. Essential convergence with 65000 elements.

	Approximation error model				Conventional model		
	True	$\hat{X}$	$\hat{\sigma}$	True Error	$\hat{X}$	$\hat{\sigma}$	True Error
$\kappa^{(\mathrm{II})}$	0.650	0.597	0.025	0.053 ($2.1 \times \hat{\sigma}$)	0.567	0.020	0.083 ($4.2 \times \hat{\sigma}$)
$\kappa^{(\mathrm{III})}$	0.400	0.421	0.014	0.021 ($1.5 \times \hat{\sigma}$)	0.523	0.024	0.123 ($5.2 \times \hat{\sigma}$)
$\kappa^{(\mathrm{IV})}$	0.800	0.732	0.093	0.069 ($0.7 \times \hat{\sigma}$)	1.367	0.142	0.567 ($4.0 \times \hat{\sigma}$)
$\beta^{(\mathrm{II})}$	4524	6777	634	2253 ($3.6 \times \hat{\sigma}$)	2104	485	2420 ($5.0 \times \hat{\sigma}$)
$\beta^{(\mathrm{III})}$	3393	3283	82	110 ($1.3 \times \hat{\sigma}$)	4695	85	1302 ($15.3 \times \hat{\sigma}$)
$\beta^{(\mathrm{IV})}$	6409	6241	318	168 ($0.5 \times \hat{\sigma}$)	8951	360	2542 ($7.1 \times \hat{\sigma}$)

13 – Comments

- Simple and less simple approaches for how to model the model uncertainties and the effect of (too much) reduced computational models
- The converse to model reduction: how to exploit increased accuracy of data without practically too complex computational models
- In practice there are three models: a) the real one (physics), b) a relatively accurate approximation and c) the one to be used in the inversion.
- Setting up the model b) is usually a tedious job but once the joint statistics of (y, x) has been computed, applying the enhanced error model leads to approximately the same computational complexity as the conventional error model.