Uncertainty Quantification using Probability

- What does probability mean?
  - General agreement on probability axioms (e.g. as stated by Kolmogorov in 1933) but they are devoid of any meaning
  - Common interpretation is that P(A) denotes the relative frequency of an "inherently random" event A "in the long run"
  - "Inherently random" is poorly defined, cannot be proved and limits domain of applicability, e.g. is "probability of a model" meaningful?

$a \Rightarrow b$

Probability Logic

- Alternative interpretation of probability (e.g. P.S. Laplace, H. Jeffreys, R.T. Cox, E.T. Jaynes)

  $P(b|a)$ = measure of how **plausible** proposition b is, conditional on information in proposition a

  - It extends Boolean propositional logic to allow quantification of plausible reasoning when information is missing
  - No division into aleatoric vs epistemic uncertainty – it's all epistemic!
  - Probability models are viewed as (lack of) knowledge models to cover missing information, not something approximating "true probabilities" for real phenomena

Probability Logic (Cont.)

- By enforcing consistency with Boolean logic, Cox *derived* axioms for probability logic (Cox, 1946, 1961; Jaynes 1983, 2003):

  For any propositions a, b and c,

  - P1: $P(b|a) \geq 0$ [By convention]
  - P2: $P(\sim b|a) = 1 - P(b|a)$ [Negation Function]
  - P3: $P(c\&b|a) = P(c|b\&a)P(b|a)$ [Conjunction Function]
  - These axioms and De Morgan's Law imply Disjunction Function:

$$P(c \text{ or } b|a) = P(c|a) + P(b|a) - P(c\&b|a)$$

Probability Logic (Cont.)

- Kolmogorov's axioms for finite-set probability measure can be derived from those of probability logic

Model Validation

- Since any deterministic model only approximates real system behavior, cannot strictly validate a model
- Must allow for *uncertain prediction error* and use system data to update probability distribution on this
- Soft validation: e.g. is updated prediction-error variance acceptable?
- Must also allow for uncertain parameters in model
  - leads to concept of *stochastic model class* which includes parametric uncertainty and prediction-error uncertainty

Robust System Analysis

- A stochastic model class of a system allows prior and posterior *robust* predictive analysis

  - predictions of all models in the model class are considered, each weighted by its prior or posterior probability (Total Probability Thm)

  - automatically treats sensitivity of predictions to the set of parameter values

- *Stochastic simulation* methods make required calcs feasible:

  Prior case - MCS (or Subset Simulation for very low probability events)

  Posterior case – MCMC methods with tempering (i.e. evolution to high probability-content region of parameter space)

Posterior Comparison of Competing Model Classes for a System

- Posterior probability of model class via Bayes' Theorem is controlled by *evidence* given by the data
- Recently shown (Muto & Beck, 2007):

Log evidence

= Average (posterior mean) data fit of model class

– Expected information gain (Shannon's relative entropy) about the model parameters from the data

= Measure of consistency of the model class with the data

– Penalty for more complex models that extract more information from the data

Bayes' Theorem for model classes automatically gives a quantitative Principle of Parsimony in model building

Final Remark

- We have applied UQ based on probability logic using stochastic simulation tools to:
  - modeling, analysis, identification and control of uncertain dynamic systems subjected to uncertain excitation
  - Bayesian classification and regression with an automatic relevance determination prior (e.g. relevance vector machine)