Reduced-Order Models Over Parameter Ranges

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ROM in V& V

- Reduced-order models (ROMs) are an enabling tool in control and optimization.

- *A priori* and *a posteriori* error estimates need more development.

- Need ROMs that represent solutions over a wide range of parameter space.
- Not just the parameter samples used to produce the basis.

Proper Orthogonal Decomposition

Standard ROM for PDEs

- Find a *good* low-dimensional basis $\{\phi_j\}_{j=1}^r$
- Represent the solution as

$$w(x,t) \approx w^r(x,t) \equiv \sum_{j=1}^{r} \phi_j(x) a_j(t)$$

- Find a dynamical system for coefficients $\{a_j\}_{j=1}^r$

Proper Orthogonal Decomposition

Proper Orthogonal Decomposition

Given an collection of simulations $\{w(x,t;q_k)\}$, find ϕ that solves

$$\min_{\bar{a}(\cdot),\|\phi\|=1} \frac{1}{K|\mathcal{T}|} \int_{\mathcal{T}} \sum_{k=1}^{K} \left\| w(\cdot,t;q_k) - \phi(\cdot)\bar{a}(t;q_k) \right\|^2 dt$$

ϕ minimizes the error between the projection and the data.

Limitations

The reduced-basis $\{\phi_j\}$ needs to be suitable over

- all time
- all relevant parameters

PID: Principal Interval Decomposition

Simultaneously find

- Bases and relevant time intervals
- Bases and relevant parameter ranges
- Combination

Idea: We can impose extra conditions

$$\int_{\mathcal{T}} \|w(\cdot,t) - \phi(\cdot)\bar{a}(t)\|^2 \, dt \leq \epsilon \int_{\mathcal{T}} \|w(x,t)\|^2 \, dt$$

and in practice

$$\int_{t_i}^{t_{i+1}} \|w(\cdot,t) - \phi(\cdot)\bar{a}(t)\|^2 \, dt \leq \epsilon \int_{t_i}^{t_{i+1}} \|w(x,t)\|^2 \, dt$$

PID: Features/Limitations

- Can realize prescribed approximation properties of the basis.
- Replacing $\bar{a}$ by a

$$\dot{a} = f(a, t; q)$$

  can realize prescribed error tolerances of the model *at* the snapshot parameters.
- The number of intervals is determined by the tolerance ϵ.
- Requires a large number of simulations to obtain good resolution of the intervals (that would allow fewer intervals to achieve the same tolerance)

PID References

W. IJzerman and E. van Groesen. Low-dimensional model for vortex merging in the two-dimensional temporal mixing layer. *European Journal of Mechanics B - Fluids*, 20:821840, 2001.

J. Borggaard, A. Hay, and D. Pelletier. Interval-based reduced-order models for unsteady fluid flow. *International Journal of Numerical Analysis and Modeling*, 4(34):353367, 2007.

Uncertainty Estimates

Given

$$w(x, t; q) \qquad \text{and} \qquad \frac{\partial w}{\partial q} w(x, t; q)$$

we can estimate

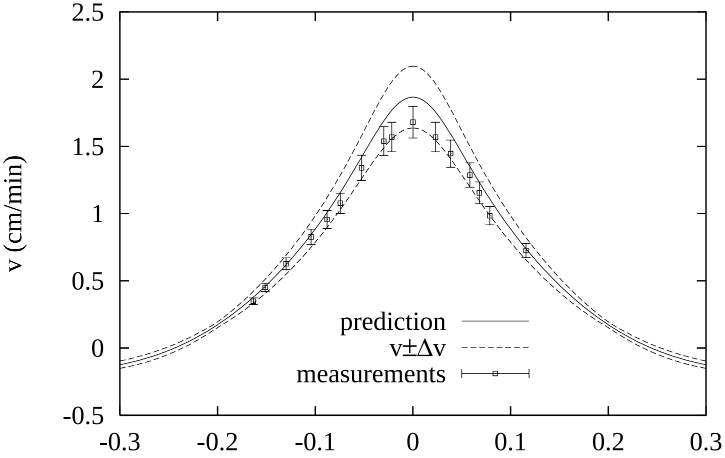
$$|w(x, t; \bar{q}) - w(x, t; q)| \ \tilde{\leq}\ \Delta q \left| \frac{\partial w}{\partial q}(x, t; q) \right|$$

for $\bar{q} \in (q - \Delta q, q + \Delta q)$.

Example

w/ É. Turgeon and D. Pelletier

Experiments and Computation, both with Error Bars



Parameter Dependence in ROMs

$$w^r(x,t;q) = \sum_{j=1}^{r} \phi_j(x;q) a_j(t;q)$$

Sensitivity of the POD basis may lead to a number of applications

- Similar parametric interval analysis

$$|w^r(x,t;\bar{q}) - w^r(x,t;q)| \leq \Delta q \left| \sum_{j=1}^{r} \frac{\partial \phi_j}{\partial q}(x;q) a_j(t;q) + \phi_j(x;q) \frac{\partial a_j}{\partial q}\right.$$

- Augment the reduced-basis for better representation away from snapshot parameters

$$\text{span} \left\{ \{\phi_j\}_{j=1}^{r}, \left\{ \frac{\partial \phi_j}{\partial q} \right\}_{j=1}^{r} \right\}.$$