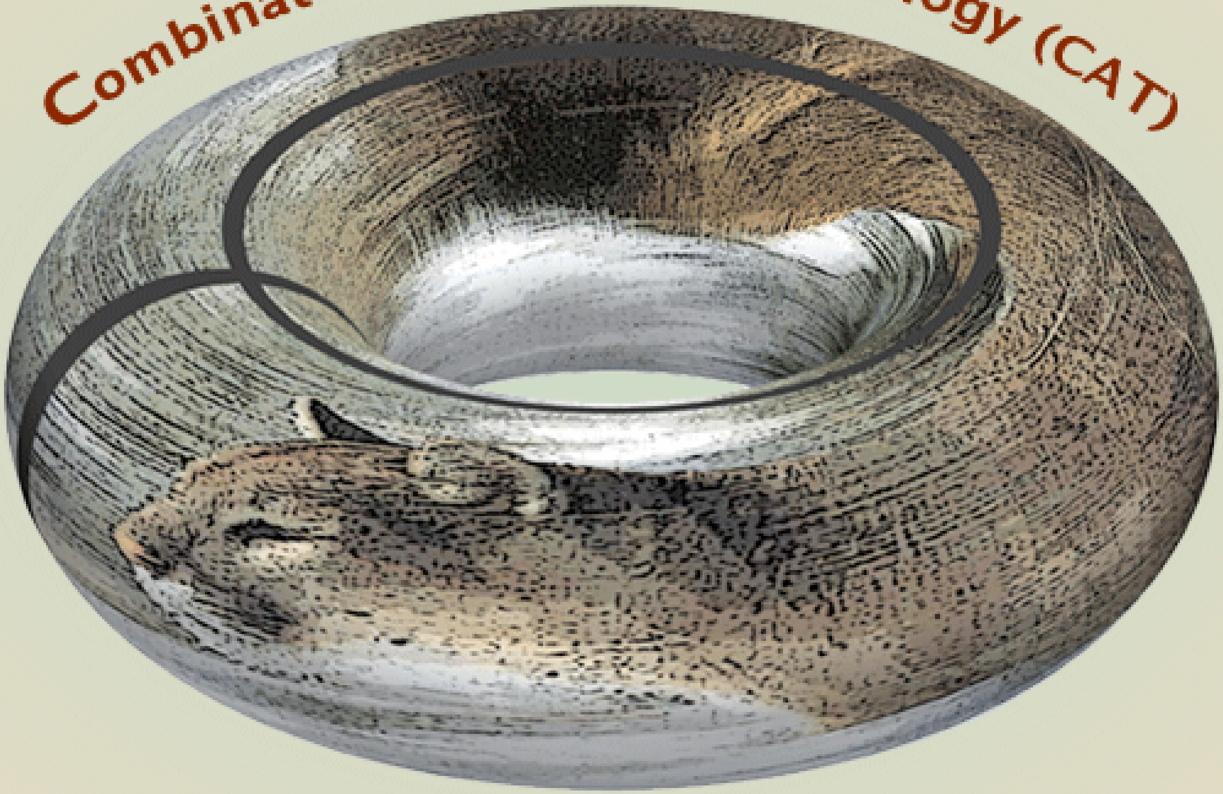
Combinatorial Algebraic Topology (CAT)



Sandia National Laboratories - CSRI Workshop on

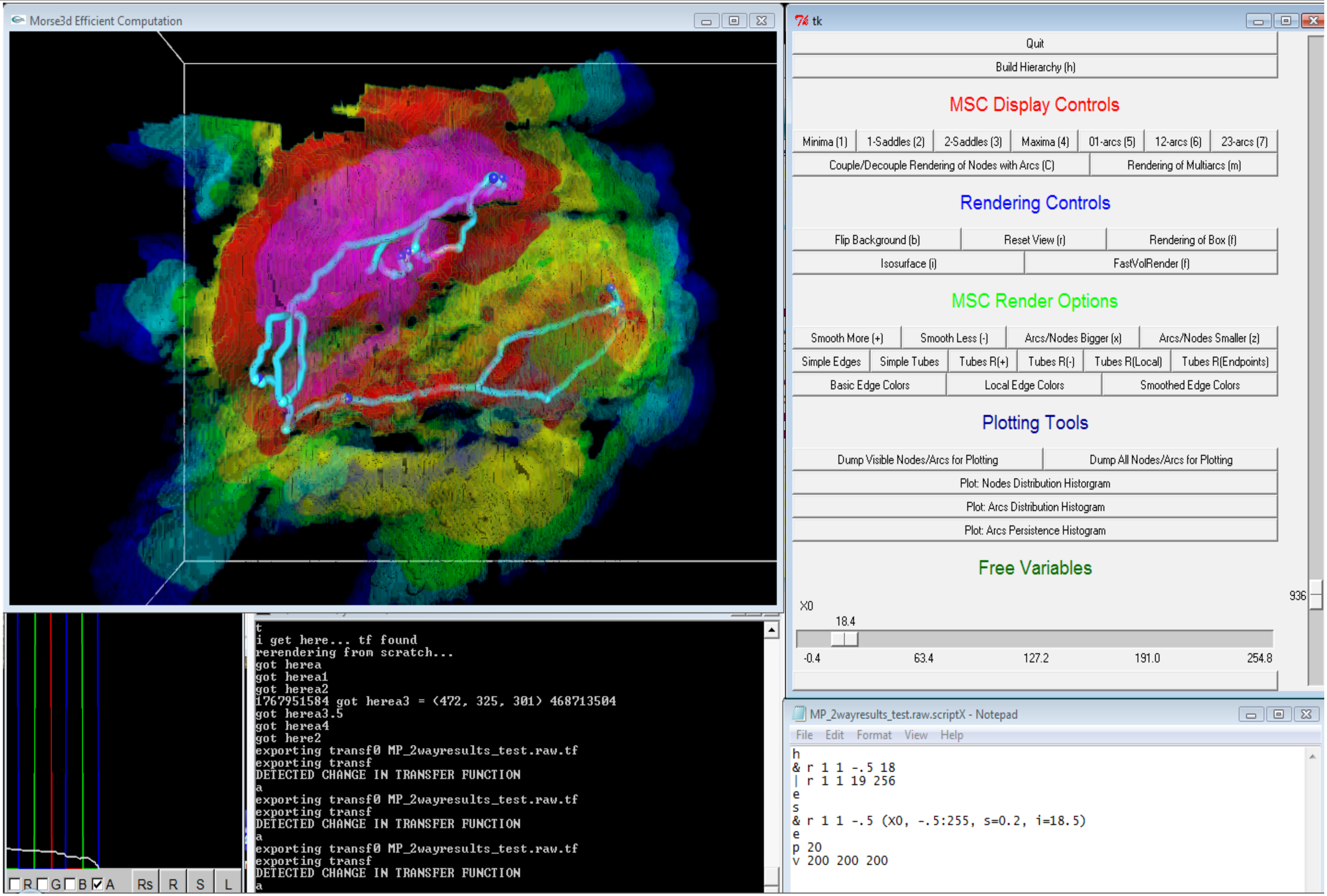
Combinatorial Algebraic Topology (CAT): software, applications & algorithms

(by invitation only)

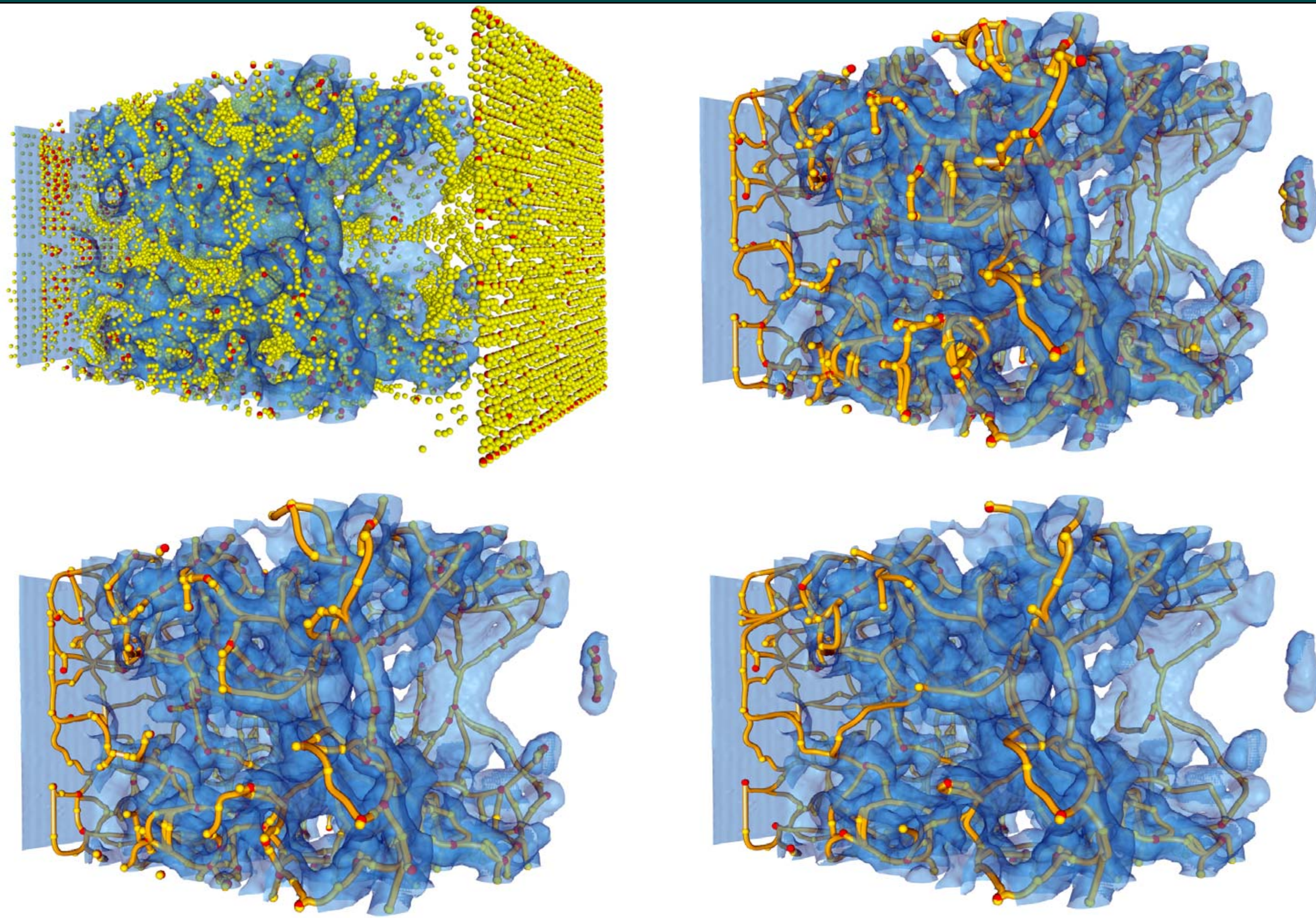
29–30 August 2009, Reception Evening 28 August

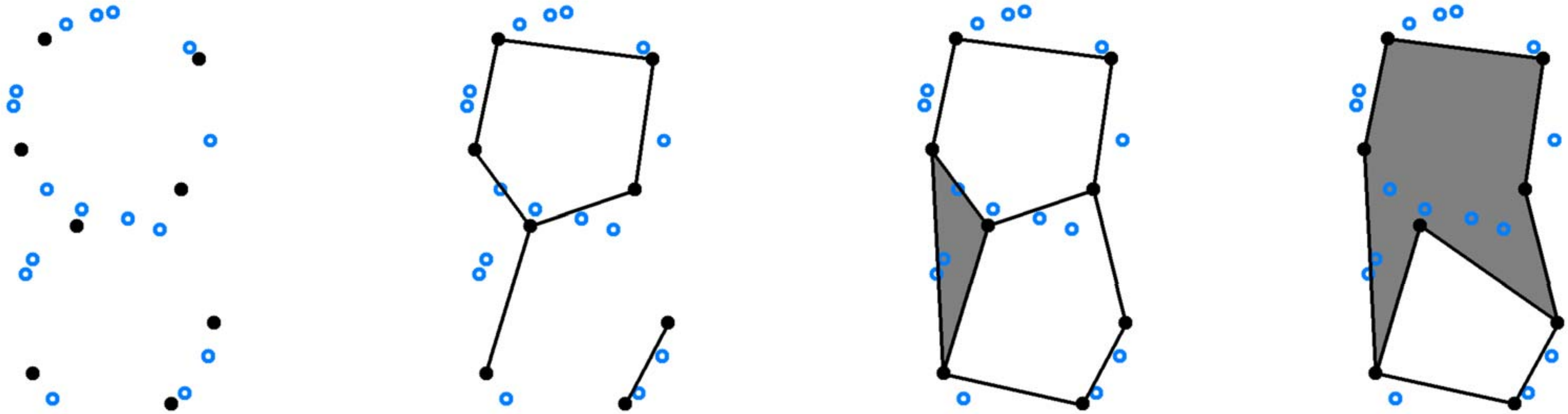
Hilton Santa Fe, NM

Interactive Data Analysis Session Using Morse-Smale Complexes

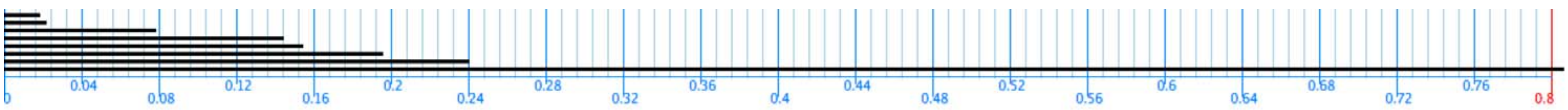


Multiple Topological Scales for Analysis of Porous Media

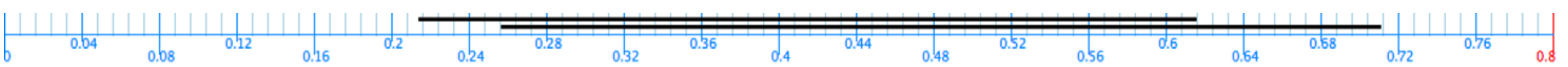




Betti barcodes for a figure-eight

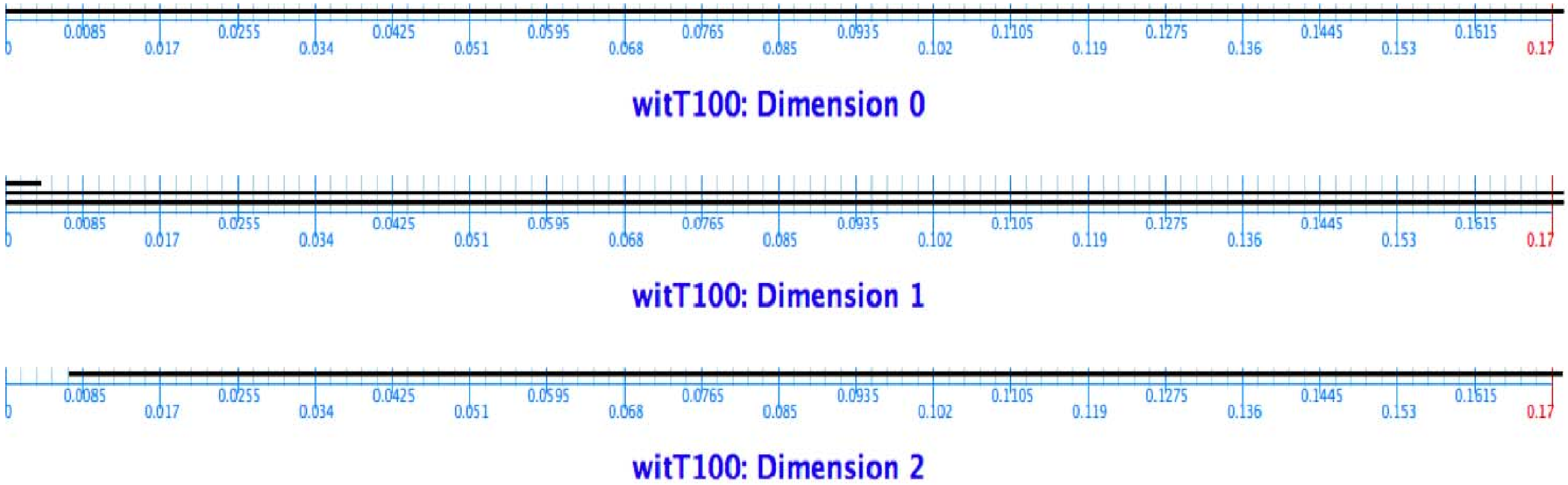


Betti plot: Dimension 0

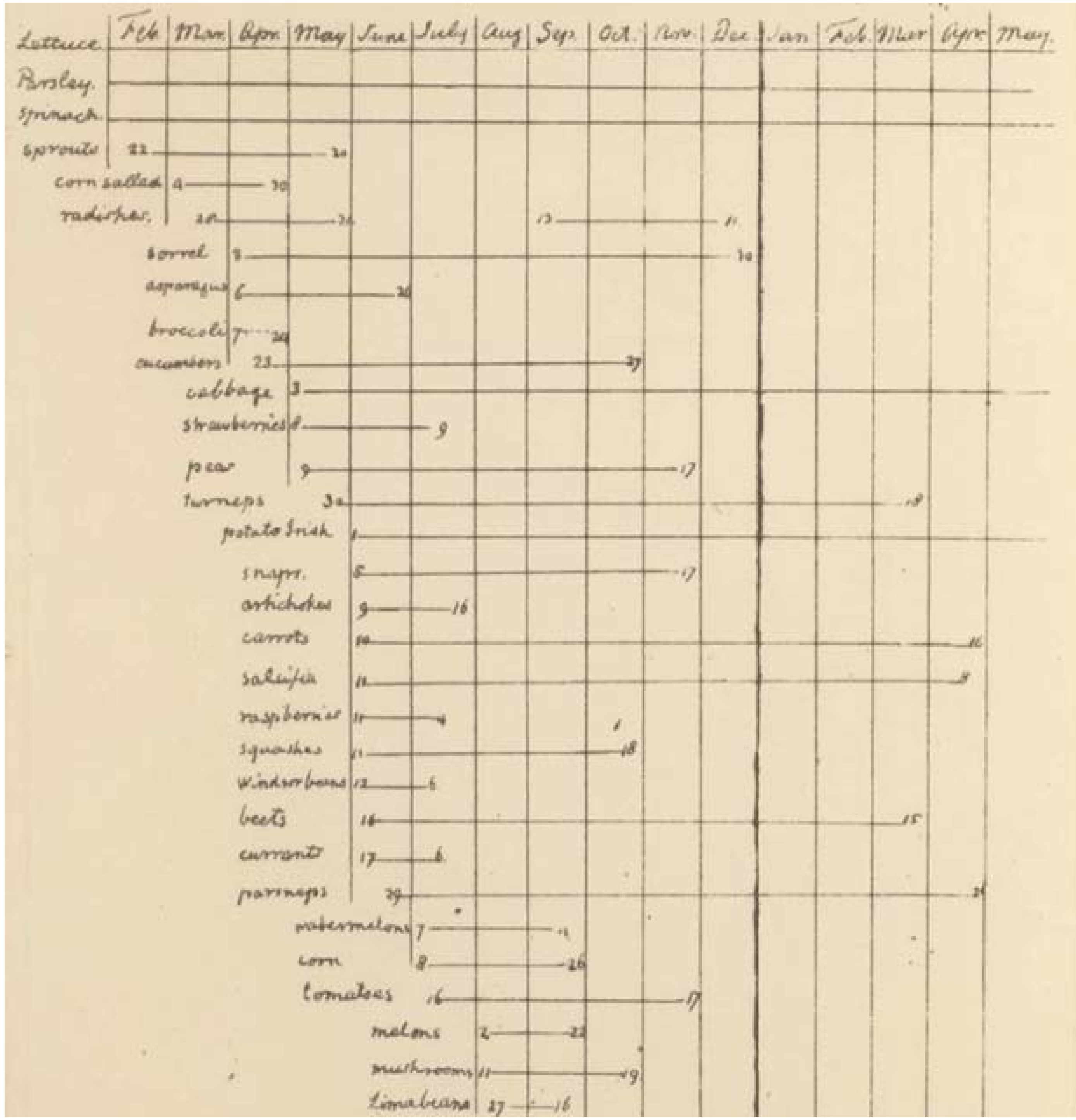


Betti plot: Dimension 1

Betti barcodes for a torus

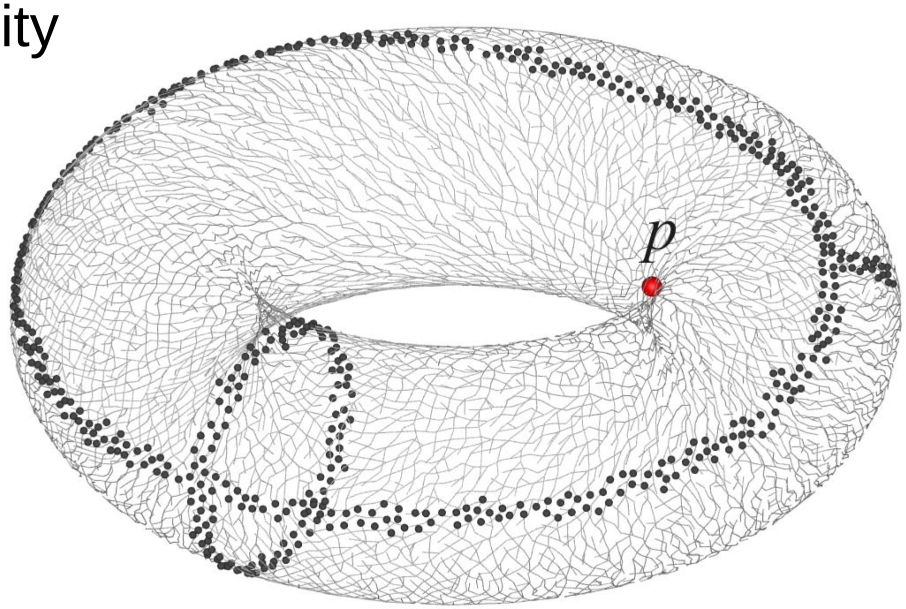


Thomas Jefferson made barcodes too.



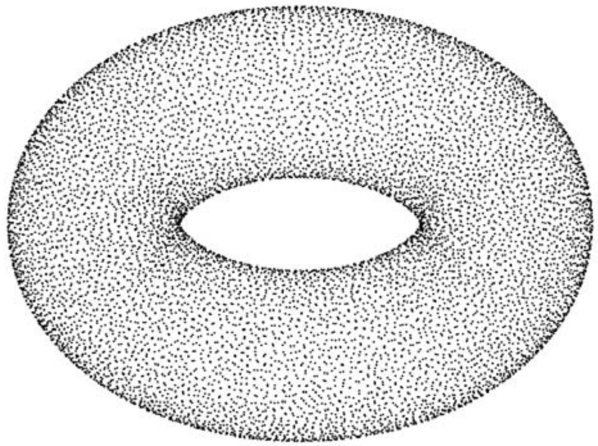
Cut Locus and Topology from Point Data

Tamal K. Dey Kuiyu Li
The Ohio State University

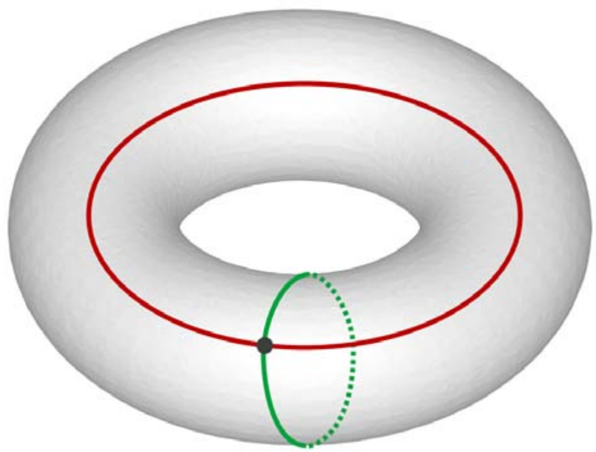


Problem

- Input: a point cloud ***P*** sampled from manifold ***M*** embedded in a higher dimensional Euclidean space $\boldsymbol{R}^d$

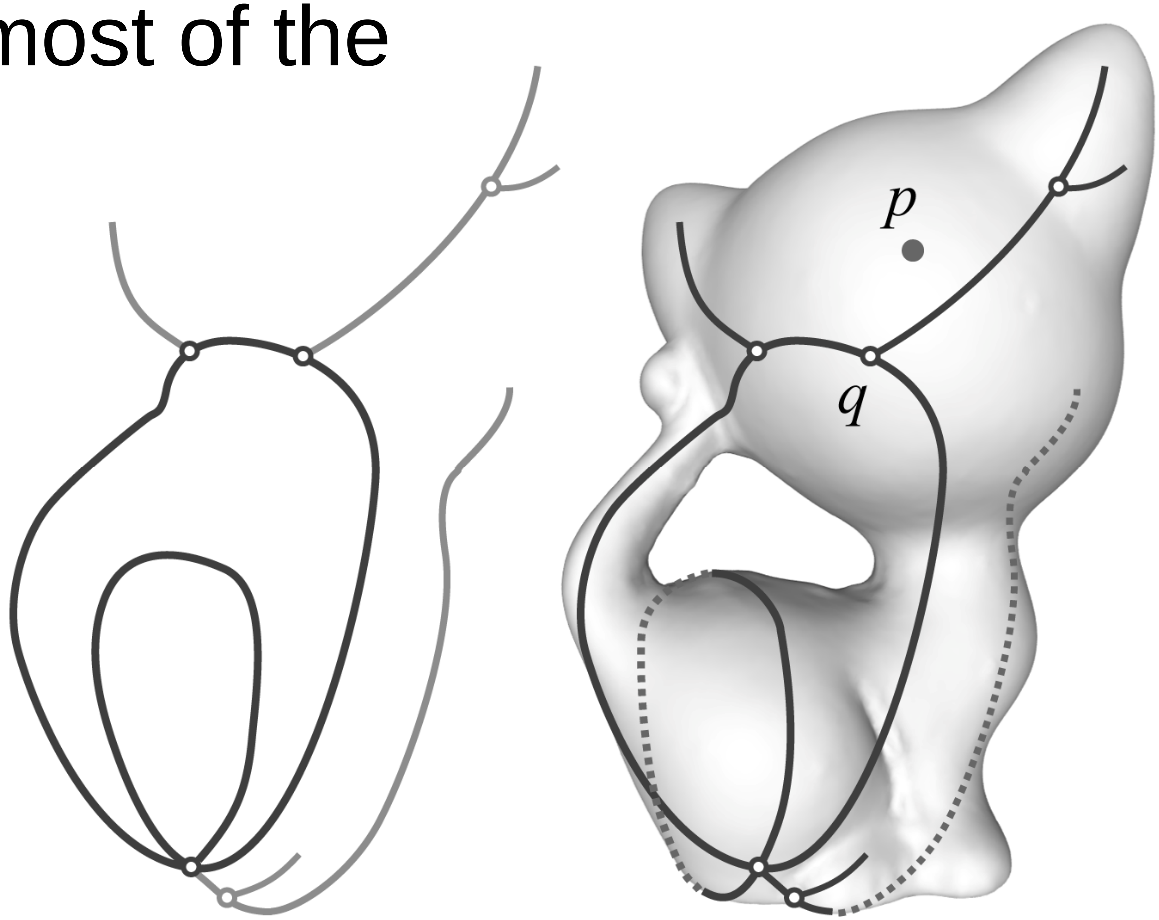


- Goal: extract topology of ***M*** from ***P***



Cut Locus in ***M***

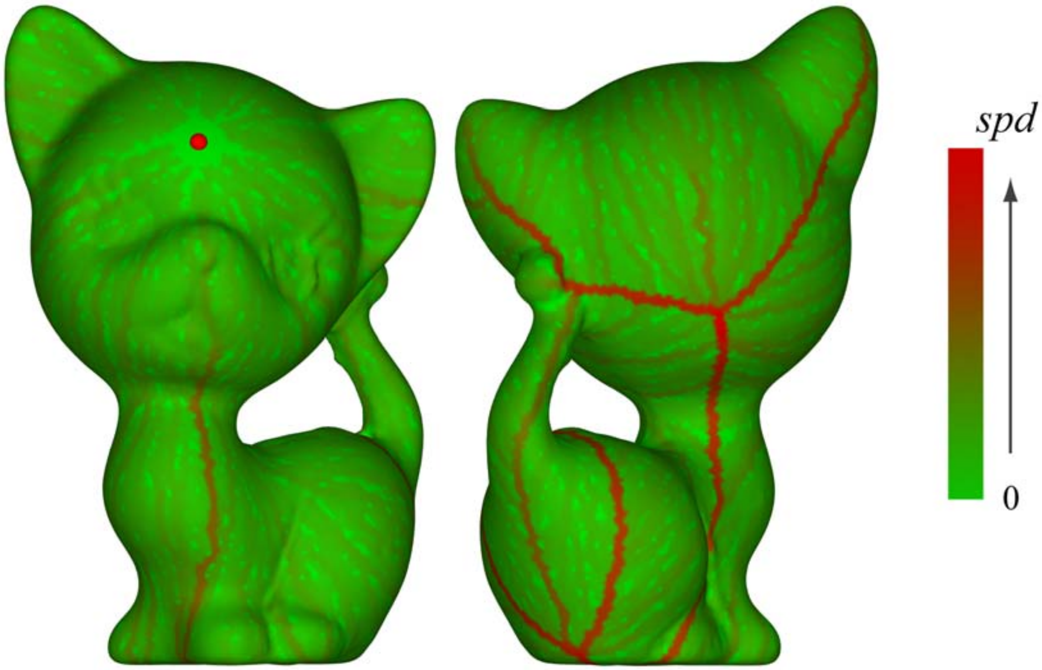
- Observation: Cut locus of ***M*** retains most of the topology



Cut Locus in *P*

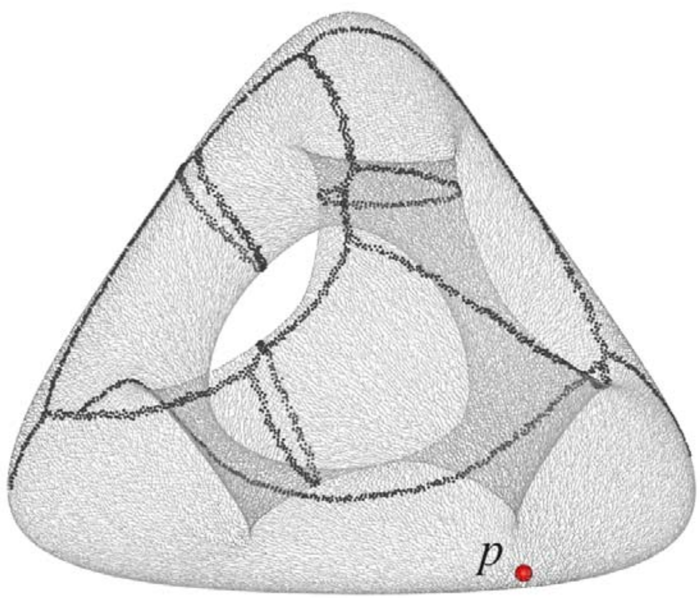
- Use a small subset ***P′*** of ***P*** to approximate the cut locus of ***M***

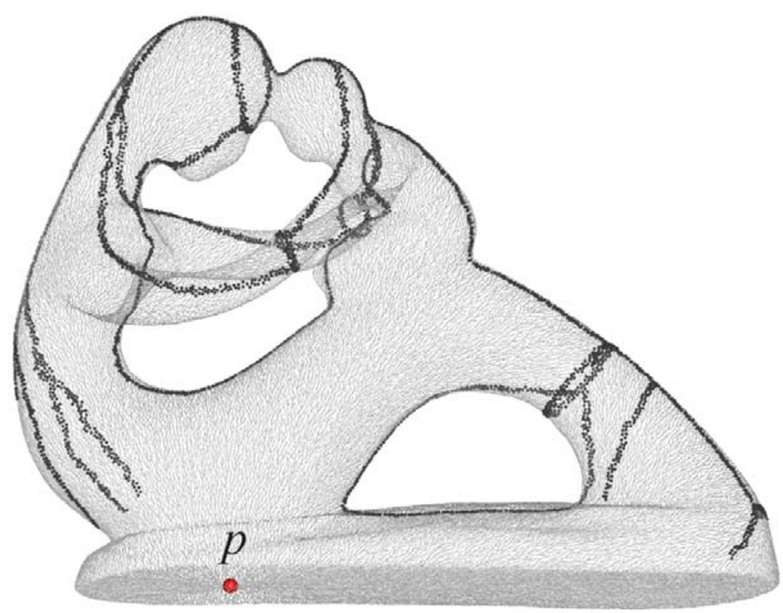
- ***P′*** is computed using the notion of **geodesic spread** function



Topology

- Use the method of [Chazal-Oudot 08] on ***P′*** to compute the Betti numbers of ***M***
- Computationally efficient because ***P′*** is much smaller

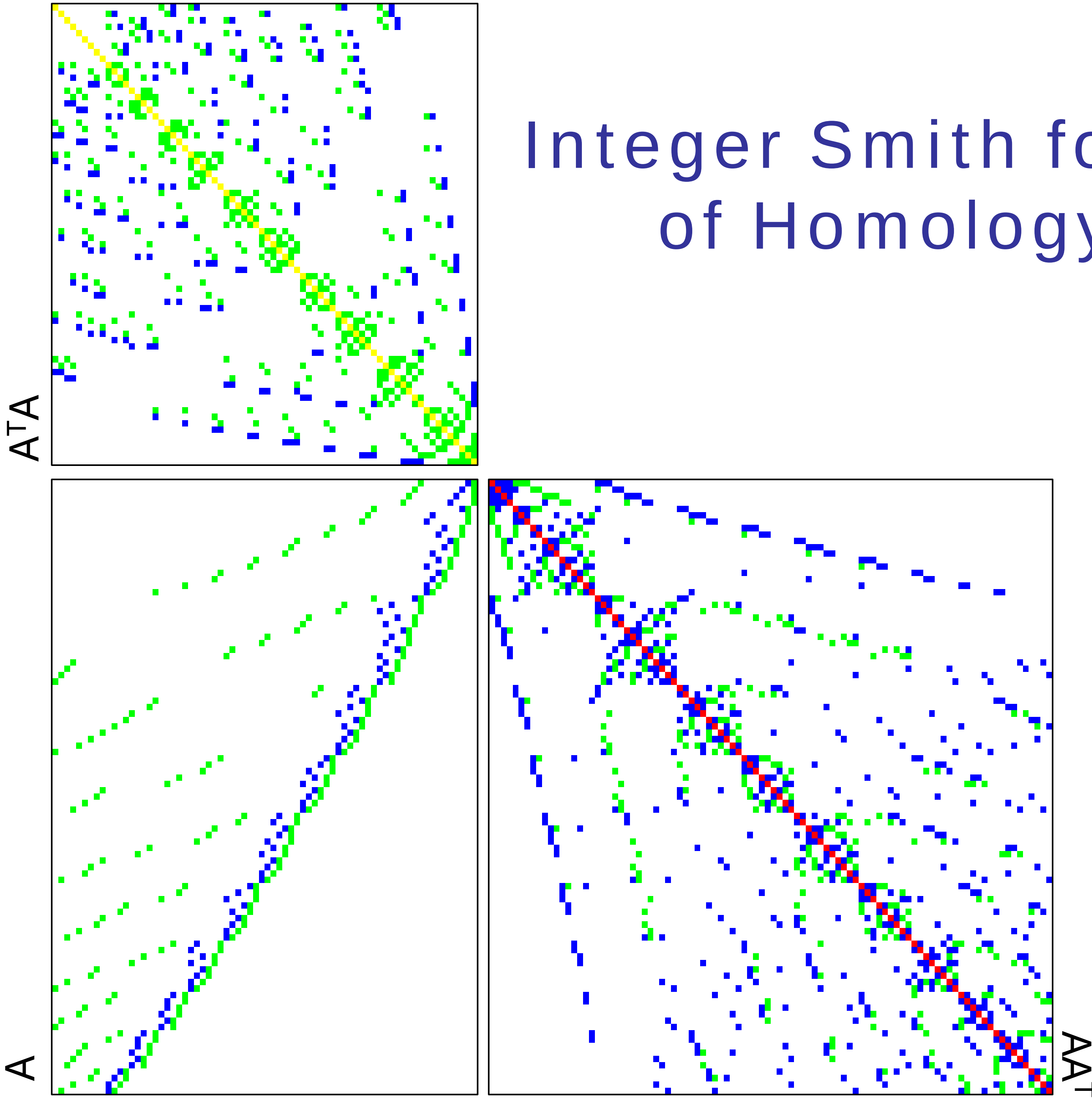




Paper

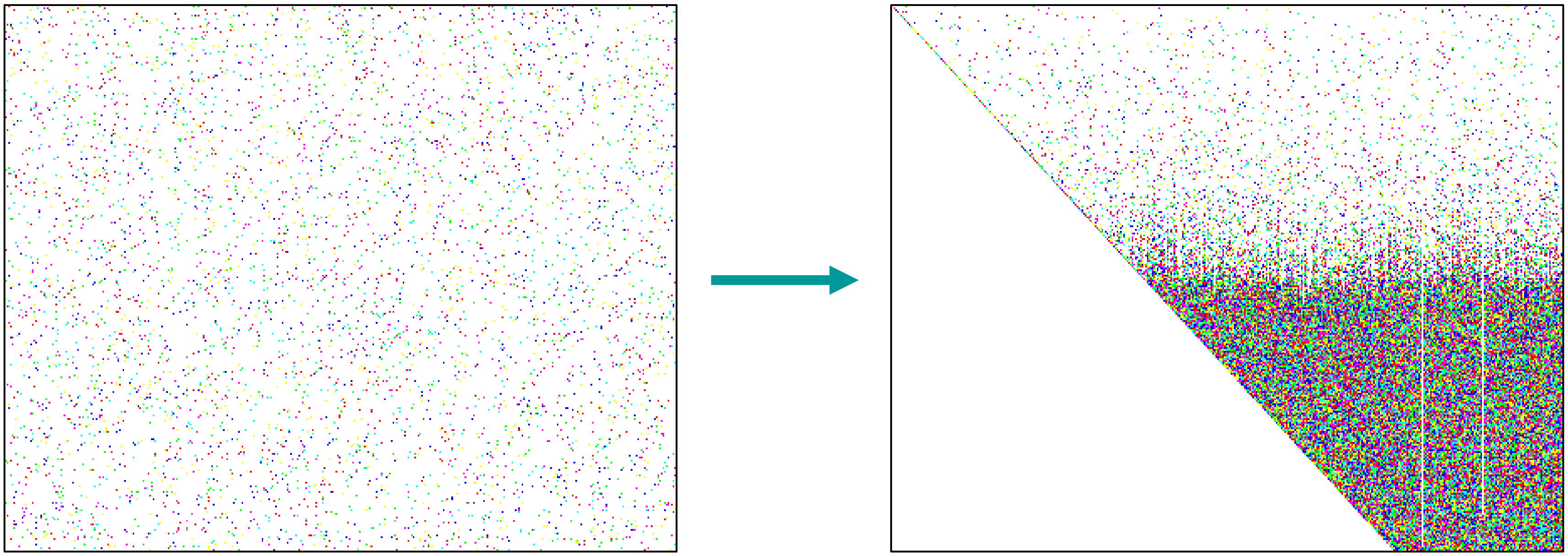
Cut locus and topology from surface point data. 2009. Tamal K. Dey, Kuiyu Li. Proc. 25th Ann. Sympos. Comput. Geom. (SoCG09), 125-134

Integer Smith form, ranks of Homology matrices



- Large dimensions
- Very sparse
- Small invariant factors
- Minimal polynomial of A^TA has very small degree

Direct or iterative methods ?



Fast at start, Fat at last

y 2 F^n → A → Ay 2 F^m

Slowly but Surely

Integer coefficient growth

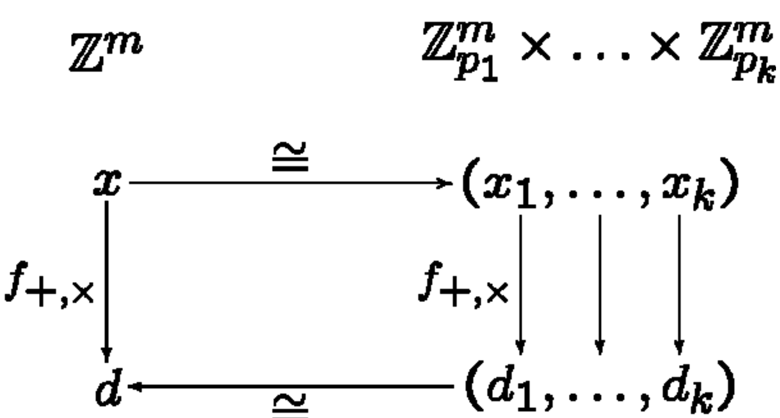
A:= [[2, 1, -3, 7, -4, 0, 9, 1, 6, -8],
[6, -2, -8, -7, 4, -5, 5, 2, 4, -9],
[-1, -7, 2, 9, 2, 3, -1, 6, -8, 7],
[3, -2, 0, -4, 8, 1, 2, 4, 6, 6],
[-4, -6, -3, -7, -8, 2, 4, -1, -2, -9],
[8, 7, 6, -3, -9, 3, 4, 6, 3, 6],
[-6, -8, 3, -3, -1, -5, 6, 2, 2, -5],
[-5, -2, -9, -6, -1, -3, 5, 8, 8, -1],
[2, 3, 8, -9, 3, -1, 8, -6, 7, -7],
[-8, 3, 7, 4, 3, 2, 2, -9, -2, -4]];

and b:= [-2, -9, -3, 8, 0, -2, -6, 8, 5, -7];

→ Solution of Ax=b:

x = **[** -154378467, 166174189,
556790464, 72535956,
258019417, 499902615,
-930948611, 1007780694,
356546217, -1001276698
] **/** 55181746

Chinese remaindering, Hensel lifting

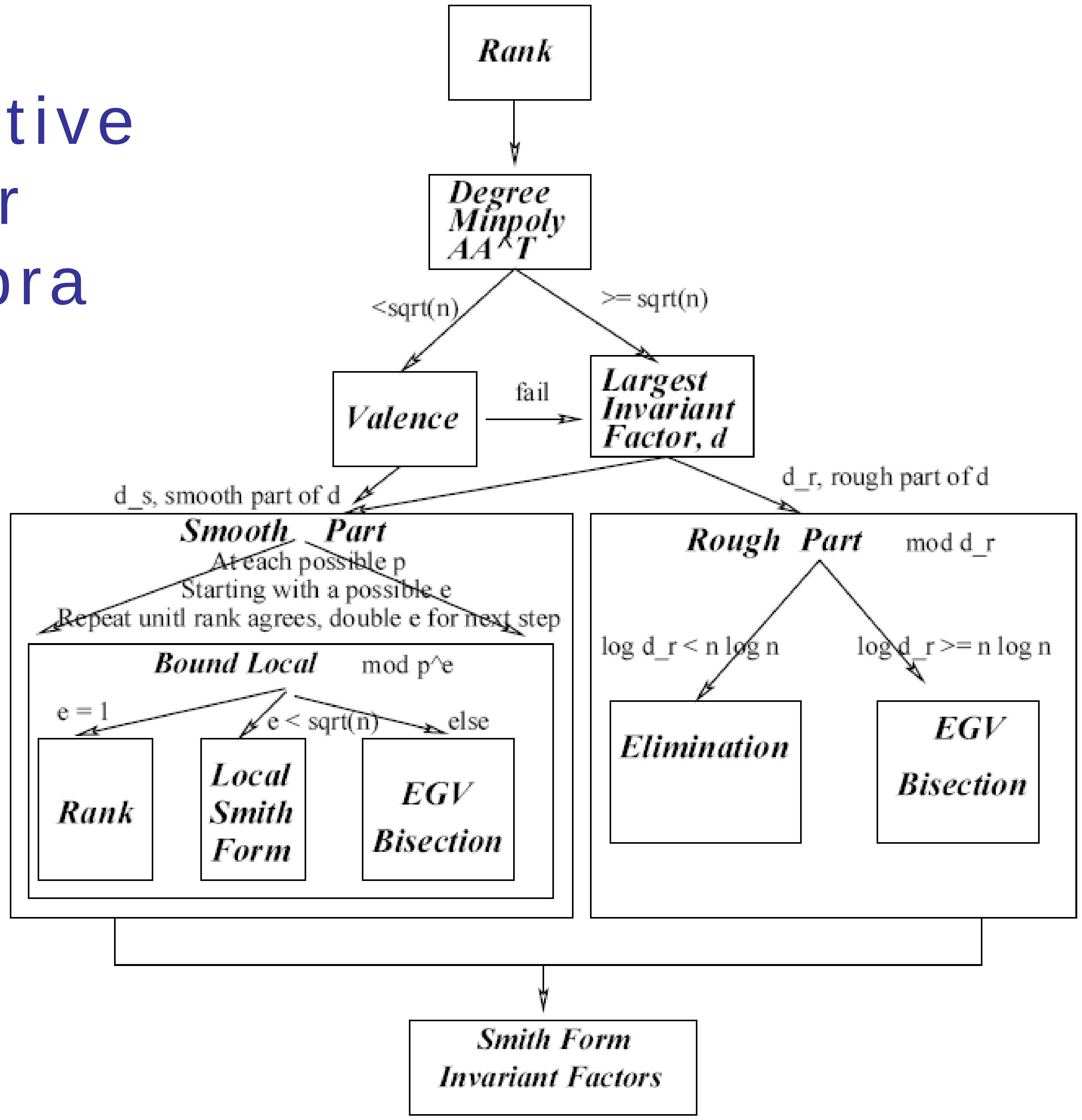


1. PLUQ(A) mod p — n^w
2. Iterate k times (k < n/2 log(n))
   - 2 triangular solve Ax_i ´ b_i mod p — $2n^2$
   - bi+1 = (bi-Axi)/p over Z — $4n^2\log(n)$
3. $x = x_0 + p\,x_1 + p^2\,x_2 + \ldots + p^k\,x_k$ — k^2
4. Rational reconstruction — nk^2

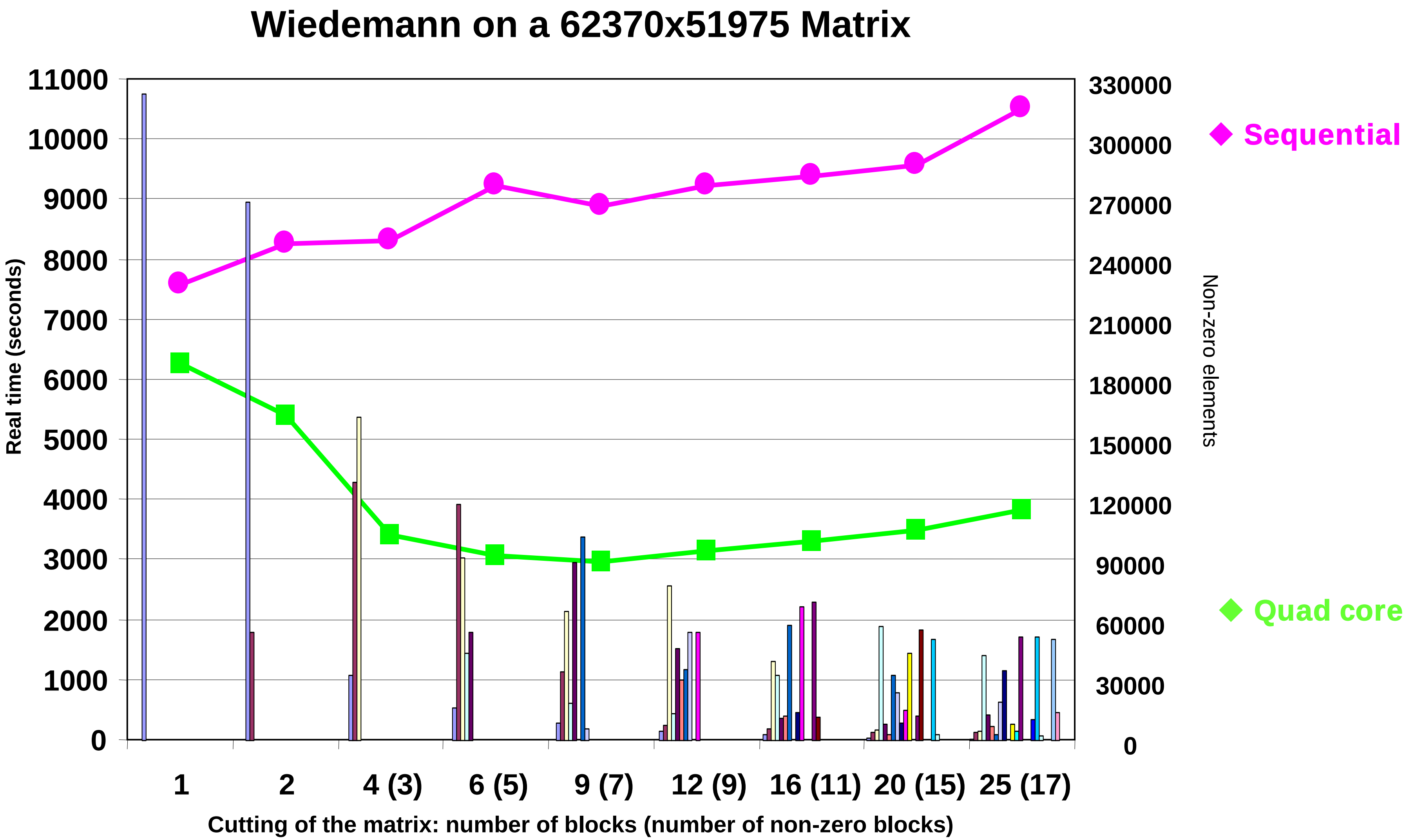
O (Pb £ k + mk^2), instead of O(Pb £ k^2)

O (n^w+ $n^3\log^2(n)$), instead of O($n^{w+1}\log(n)$)

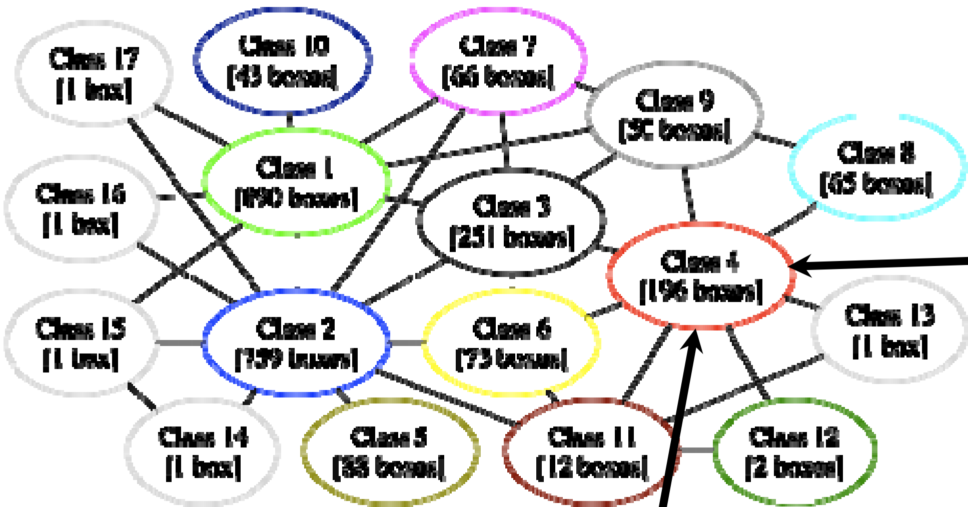
Adaptive linear algebra



Scalability: multicore algorithms

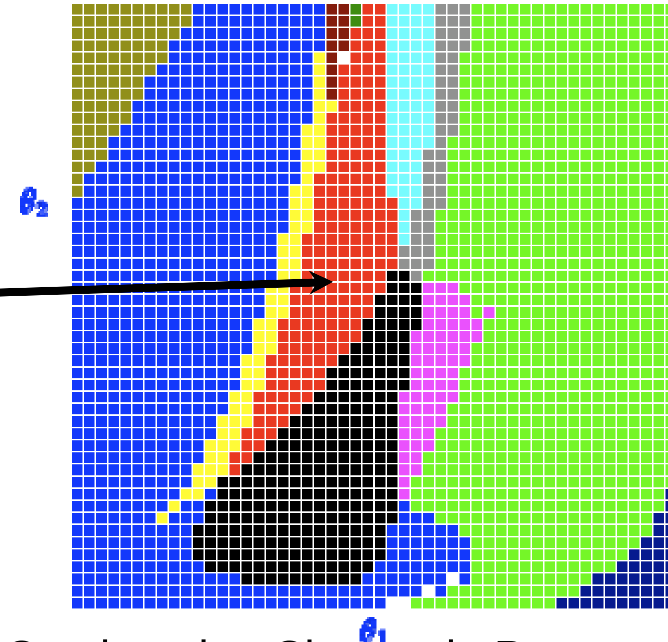


A Two Parameter Population Model: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-0.1(x_1+x_2)} \\ 0.7 x_1 \end{bmatrix}$

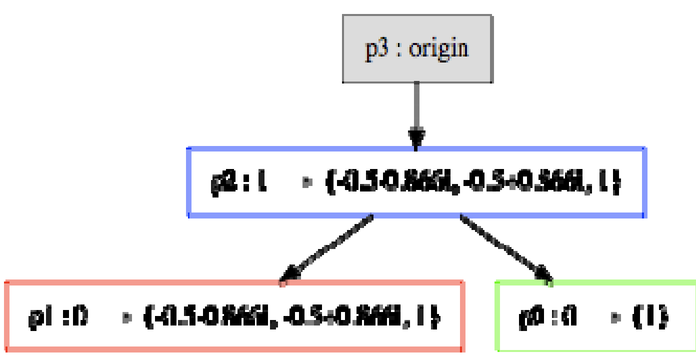


The database of global dynamics:

$(\theta_1, \theta_2) \in [8, 37] \times [3, 50]$



Continuation Classes in Parameter Space



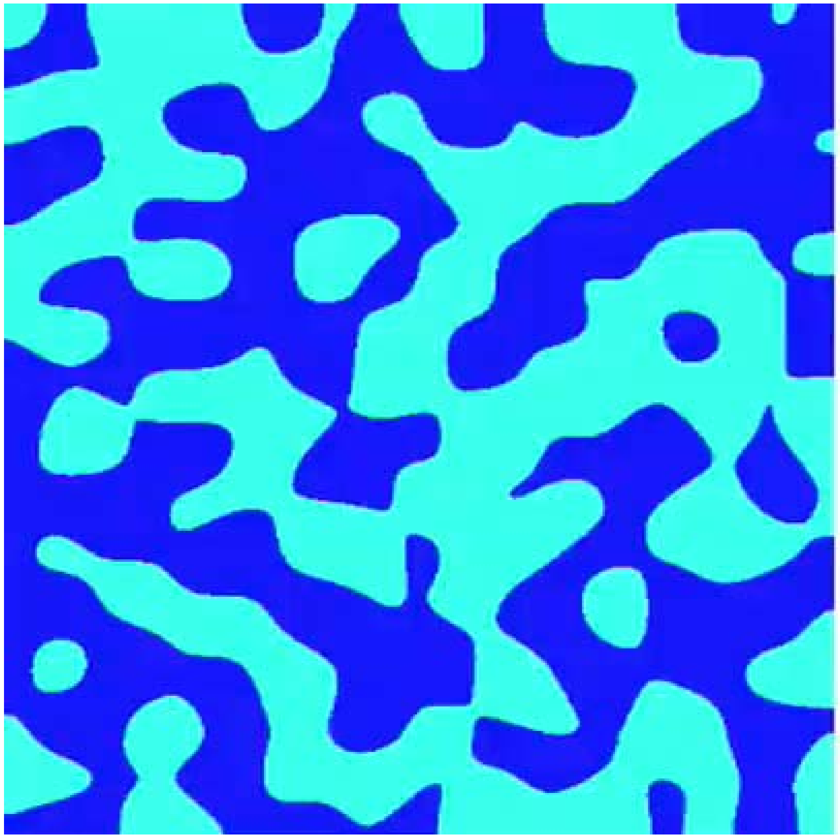
Global Dynamics of Class 4

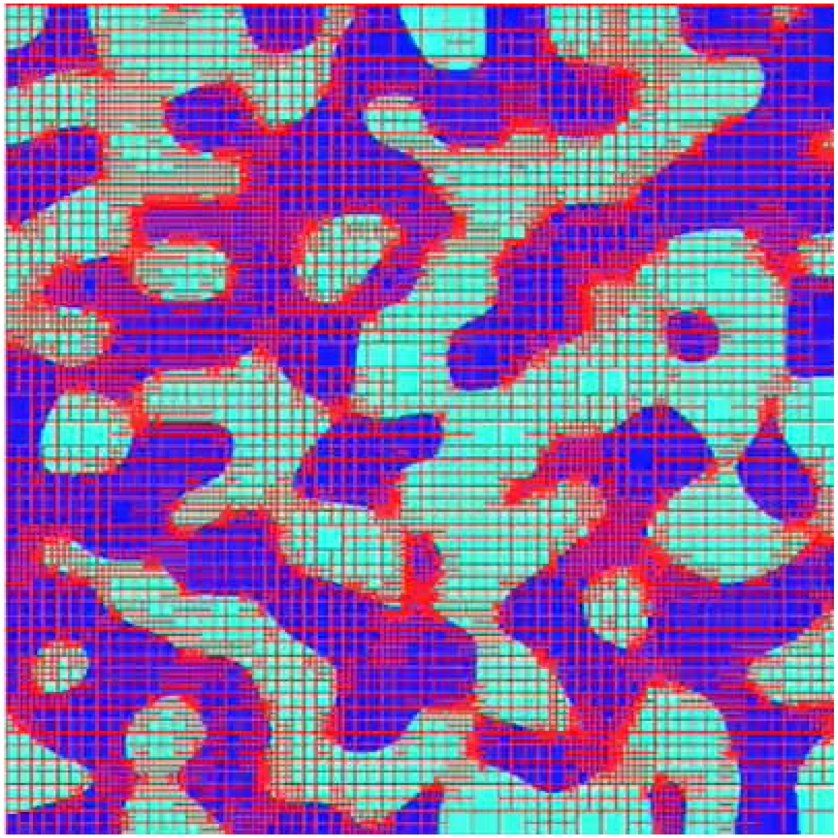
A Database Schema for the Analysis of Global Dynamics of Multiparameter Systems, Z. Arai, W. Kalies, H. Kokubu, K. Mischaikow, H. Oka, P. Pilarczyk, SIADS, 2009

Dynamics at a particular parameter value in red region.

An adaptive cubical verification algorithm for the homology of evolving nodal domains

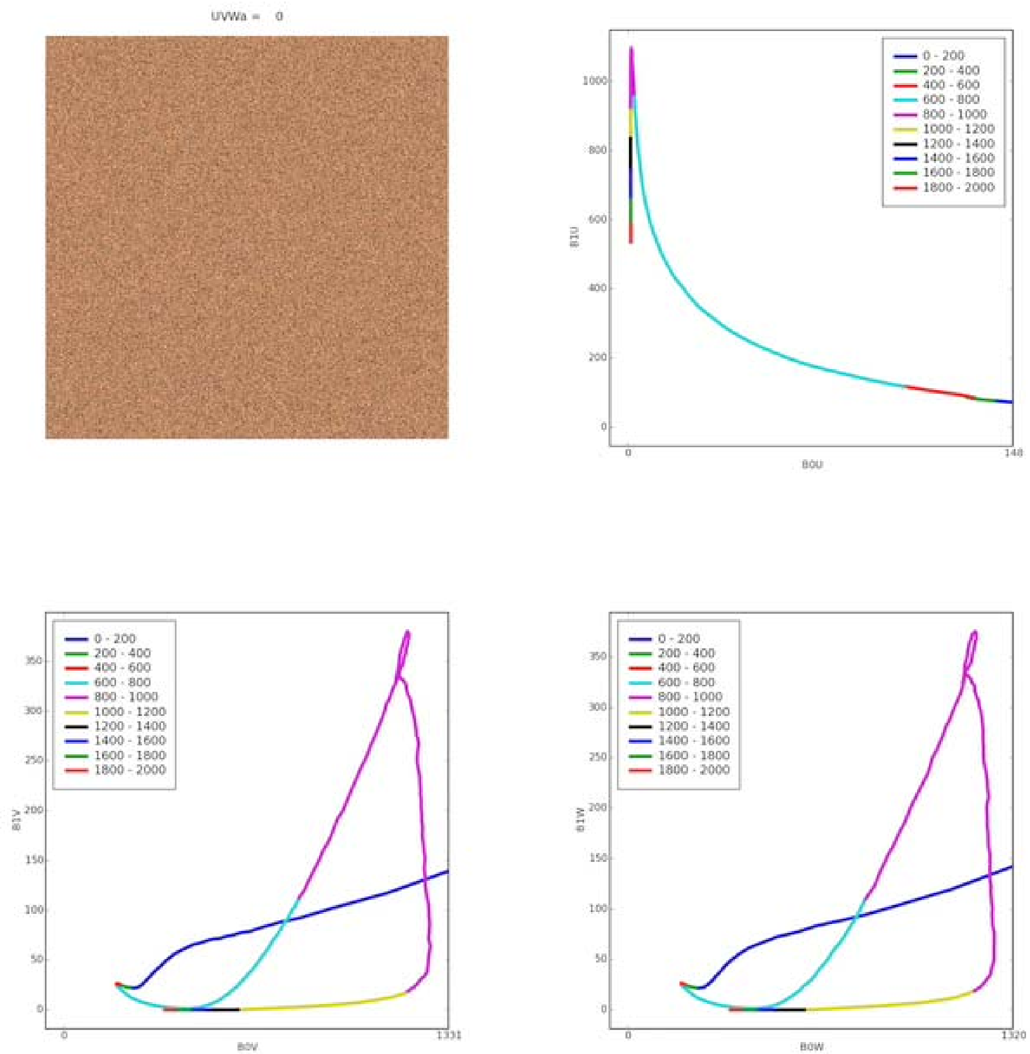
Day, Kalies, Wanner (2009)

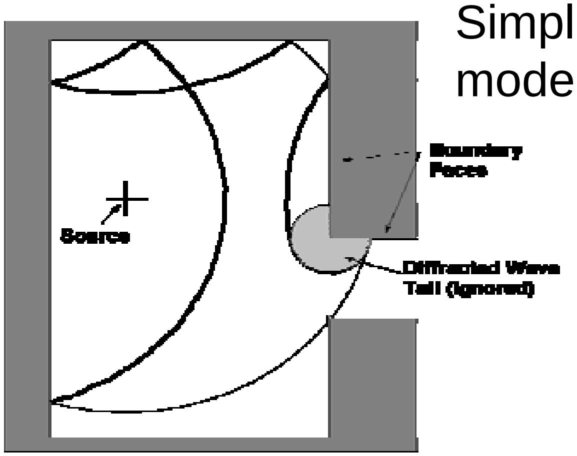




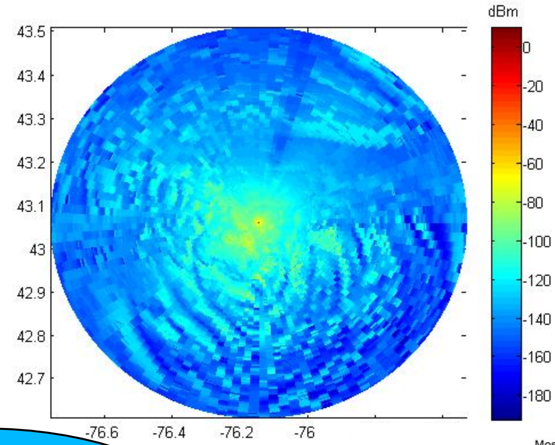
Pattern morphology evolution in ternary alloys

Corrigan, Mischaikow, Wanner (2009)

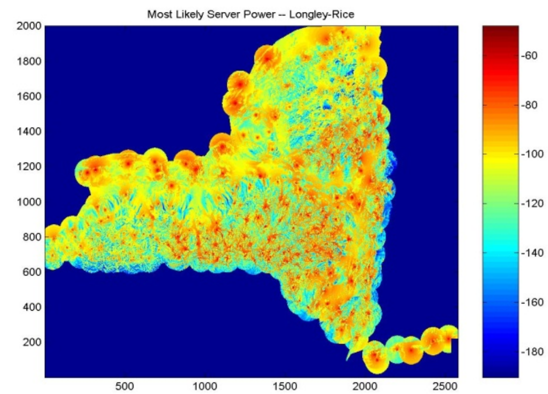




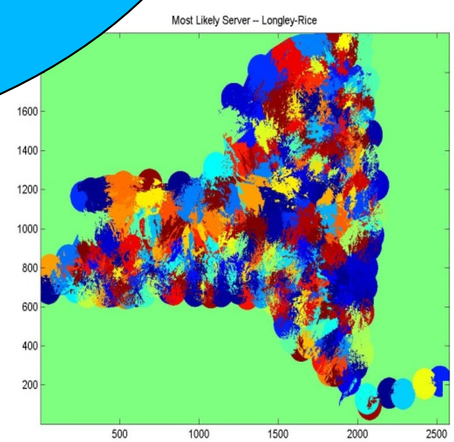
Simplified wave models



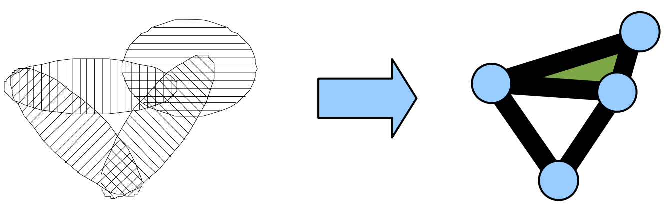
Propagation modeling for transmitters



Topology for communications network performance and capacity analysis



Full network and interference modeling

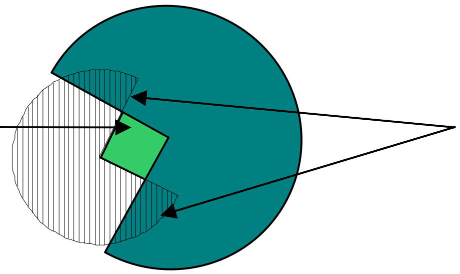


Interference complexes and sheaf-based approaches for network analysis

M. Robinson
Penn

Discrete Opportunistic Topological Imaging

Obstacle

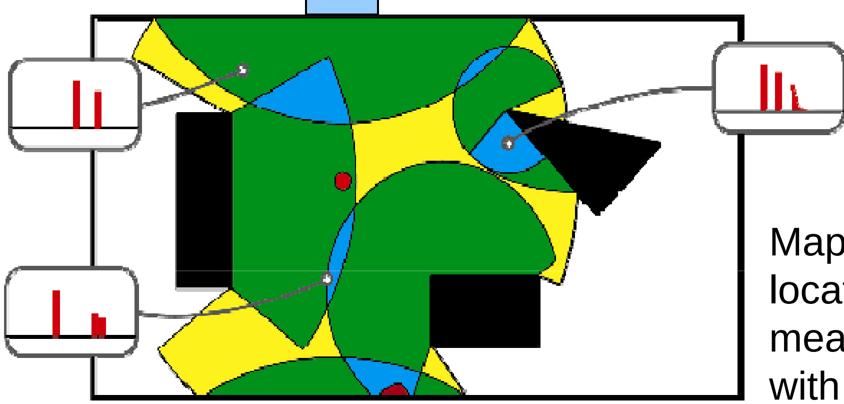


Disconnected intersection region

Regions of impact for opportunistic transmitters forms a cover

Refinement → Clustering →

Recovers homotopy type, geometric information of propagation environment



Map of space of receiver locations to space of measurements is an embedding with enough transmitters (Whitney)

Localization of receivers

M. Robinson

Penn

A Floer theory for semilinear parabolic equations

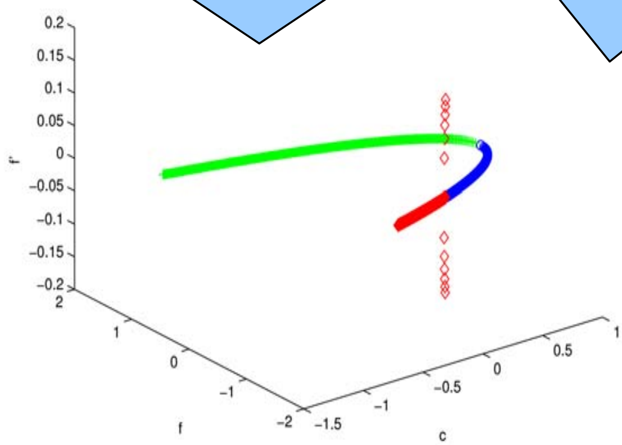
Semilinear parabolic equation

Dynamical properties of global solutions

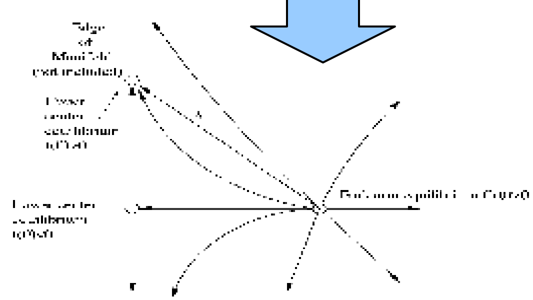
Equilibrium analysis

Evolution of stability

A topological index



Bifurcation diagram

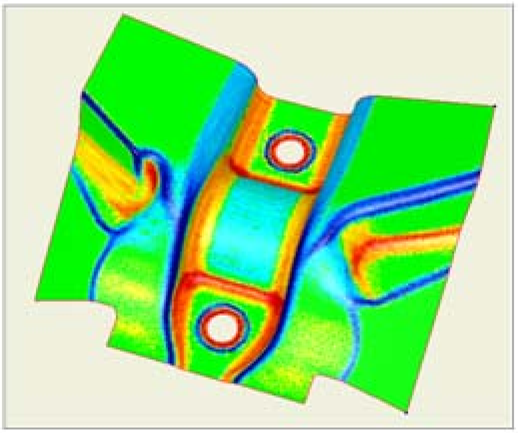


Cellular complex of global solutions

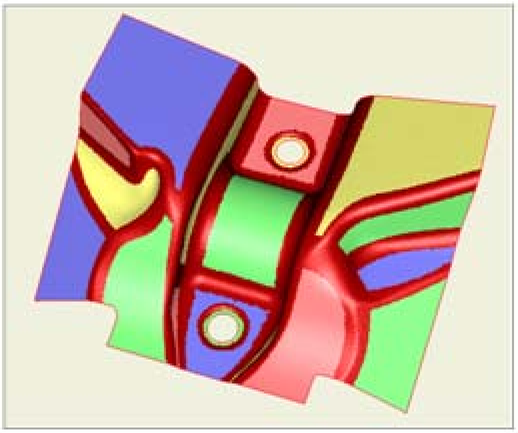
M. Robinson

Penn

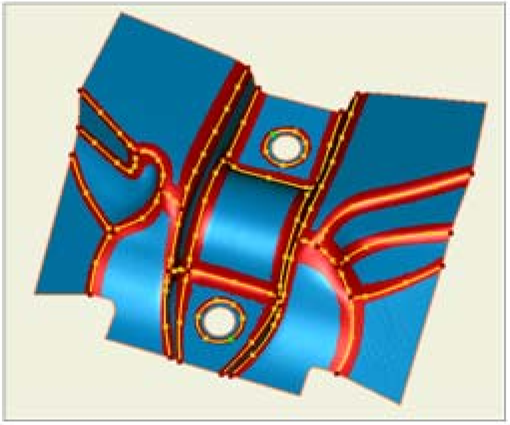
Steps in Functional Decomposition / 1



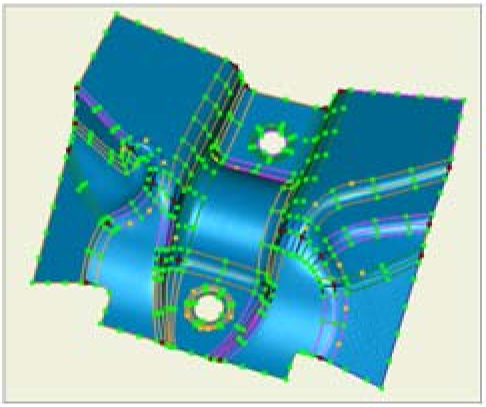
a. Numerical Curvature Map



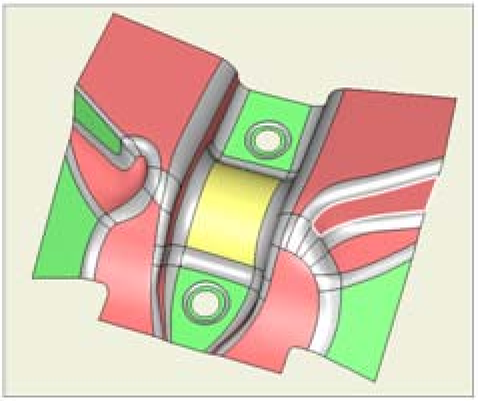
b. Regions and Separator Sets



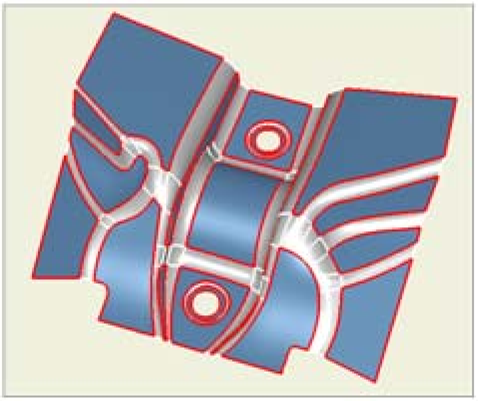
c. Contour extraction



d. Extending Contours

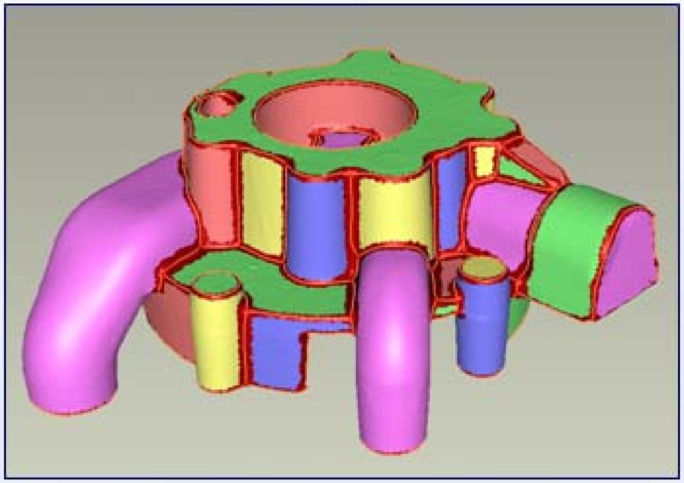


e. Classification

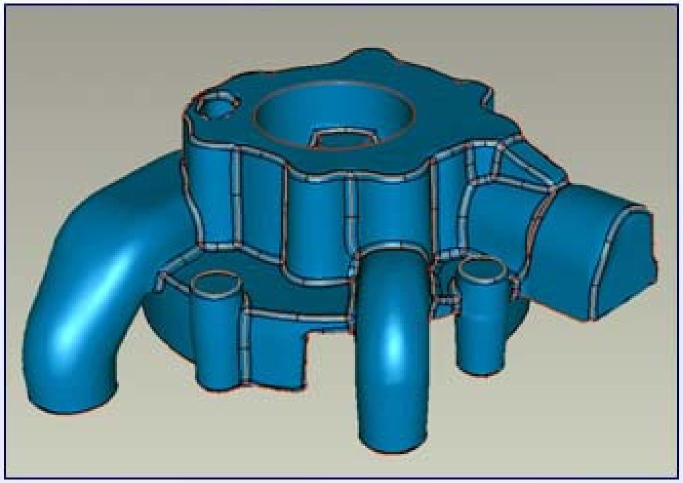


f. Trimmed Primaries

6. Automatic (rapid) surface generation II.



Segmented regions, separator sets and feature skeleton (contours)



Thickened feature skeleton and connecting features

Segmentation using Morse theory

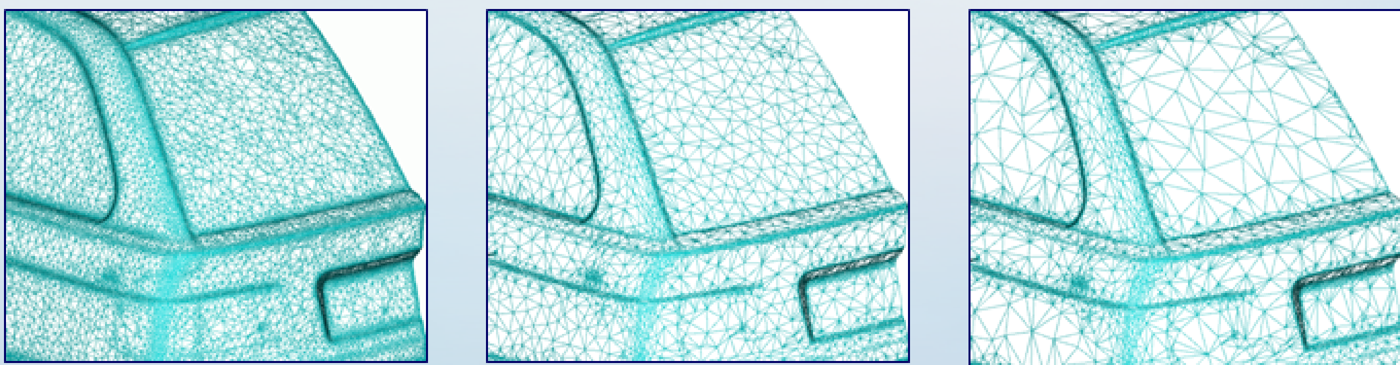
- smooth function over a manifold
- critical points – minimum, saddle, maximum
- piecewise linear Morse complex
- topological persistence and simplification

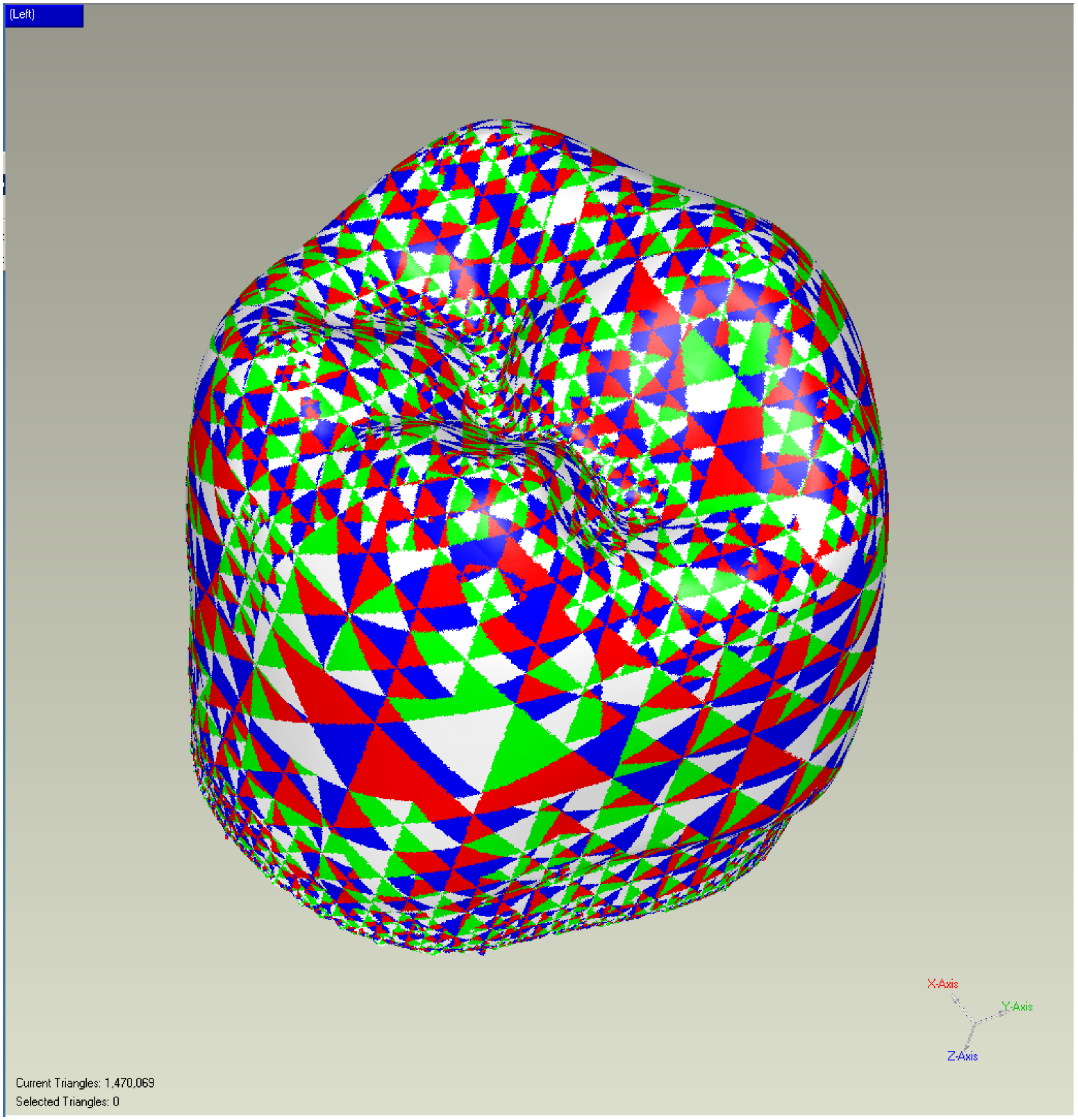
Edelsbrunner et al 02, Facello et al. 04, Edelsbrunner 05

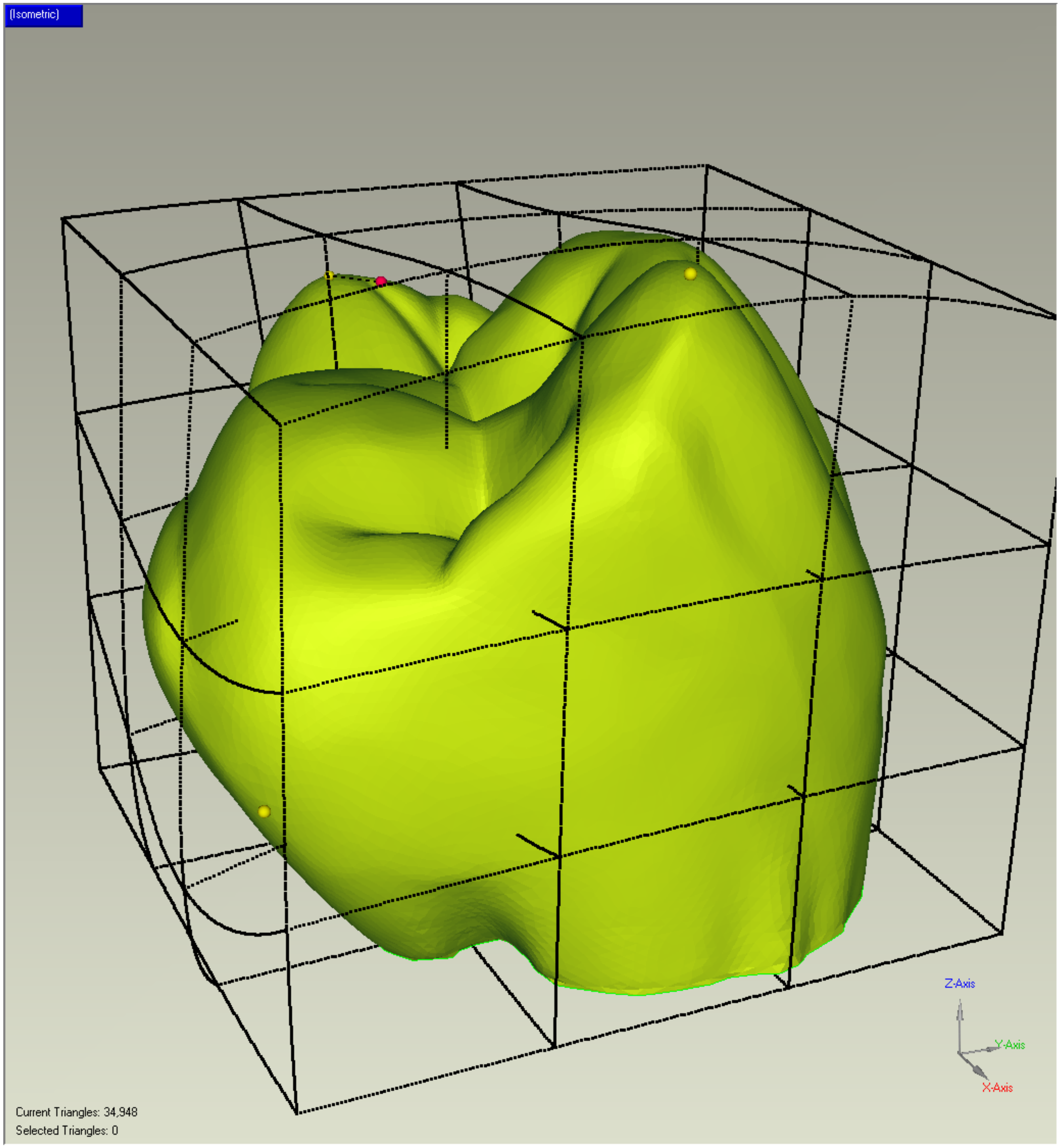
Triangulation/Decimation

well-established techniques

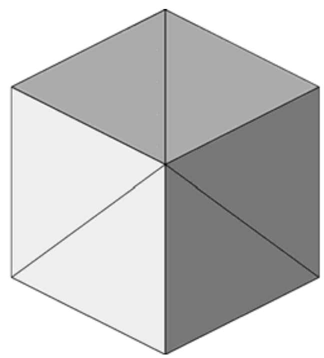
Hoppe et al. 92; Edelsbrunner, Mücke 94; Kos 01; Owen 98; Taubin et al. 98;...

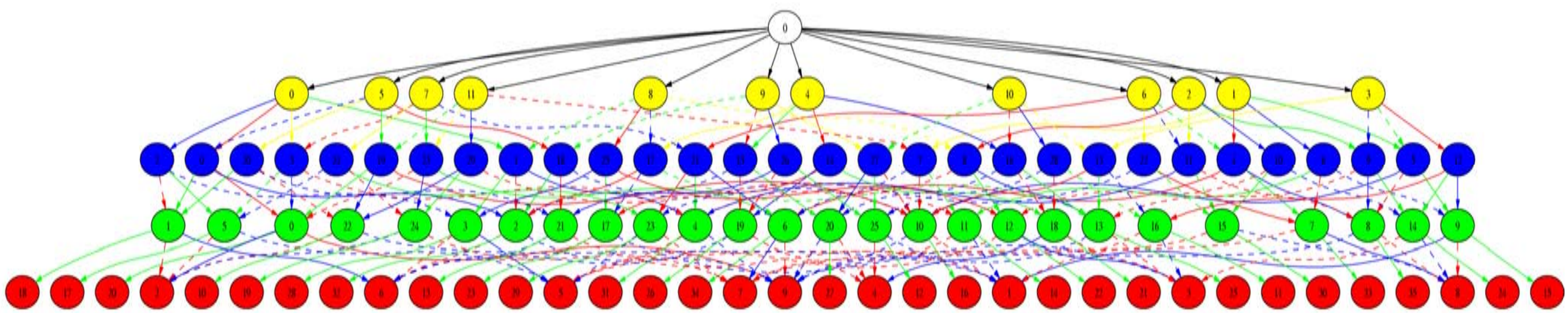



(Left)
X-Axis
Y-Axis
Z-Axis
Current Triangles: 1,470,069
Selected Triangles: 0

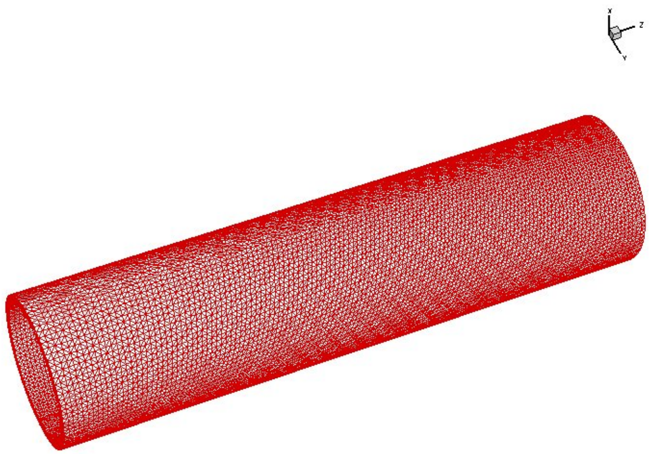

(Isometric)
Z-Axis
Y-Axis
X-Axis
Current Triangles: 34,948
Selected Triangles: 0

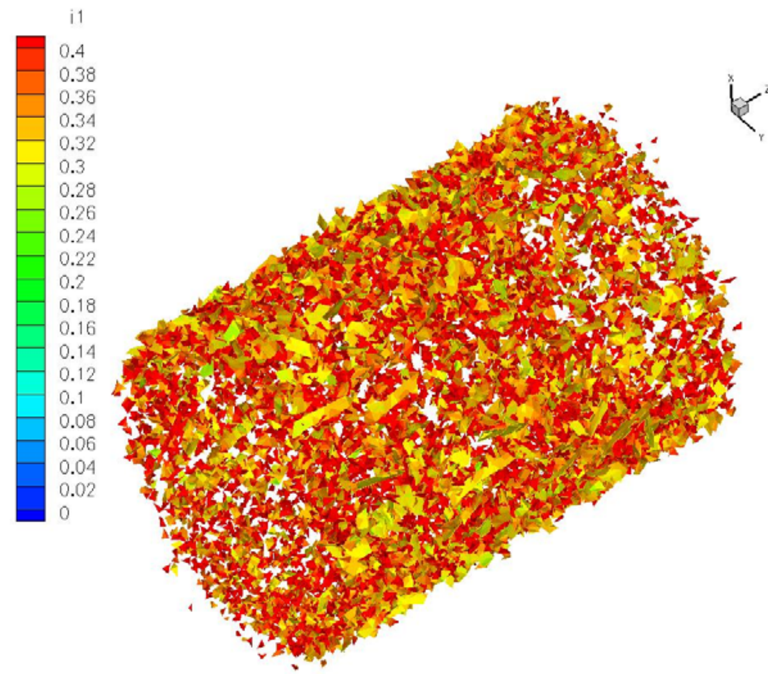
Mesh of Cube as Graph



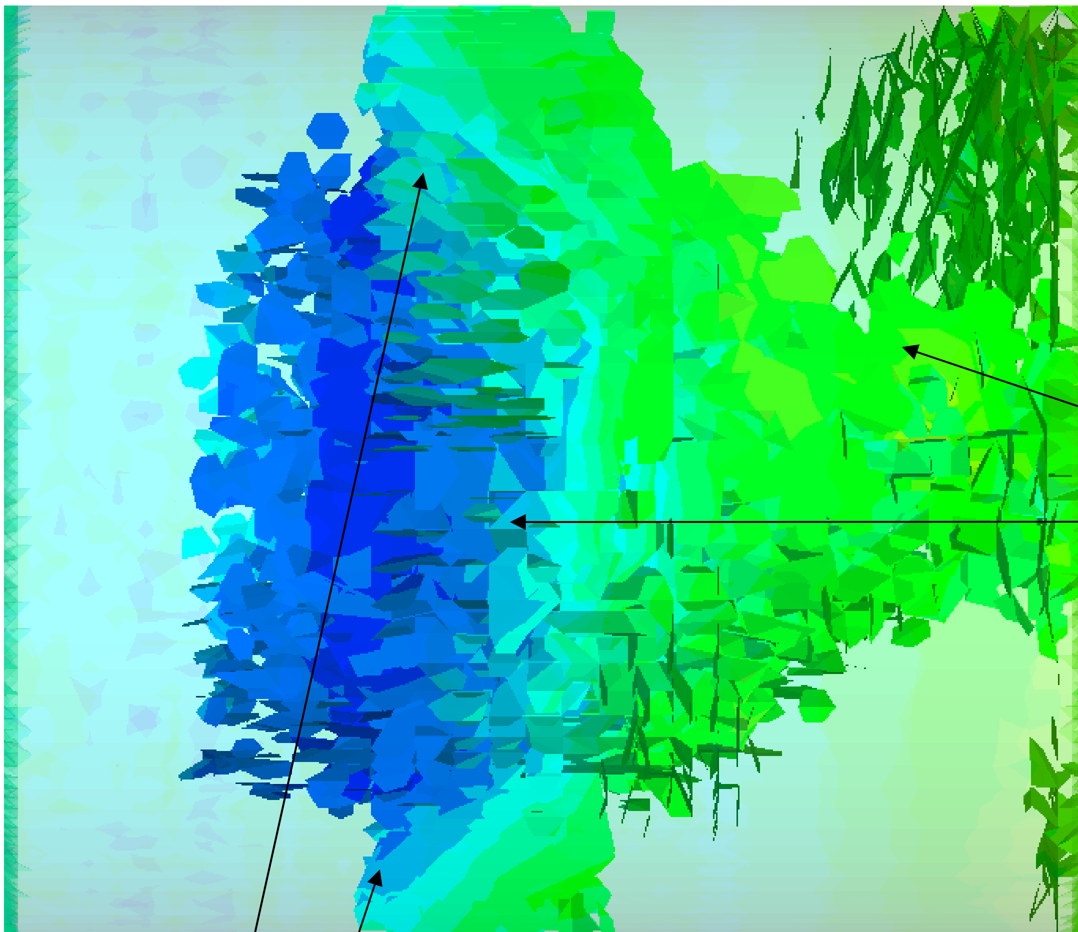


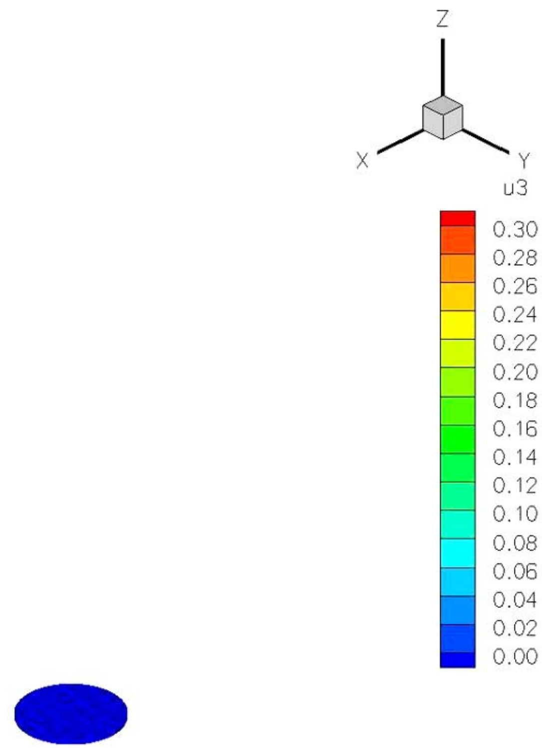
Testing of Graph Fracture



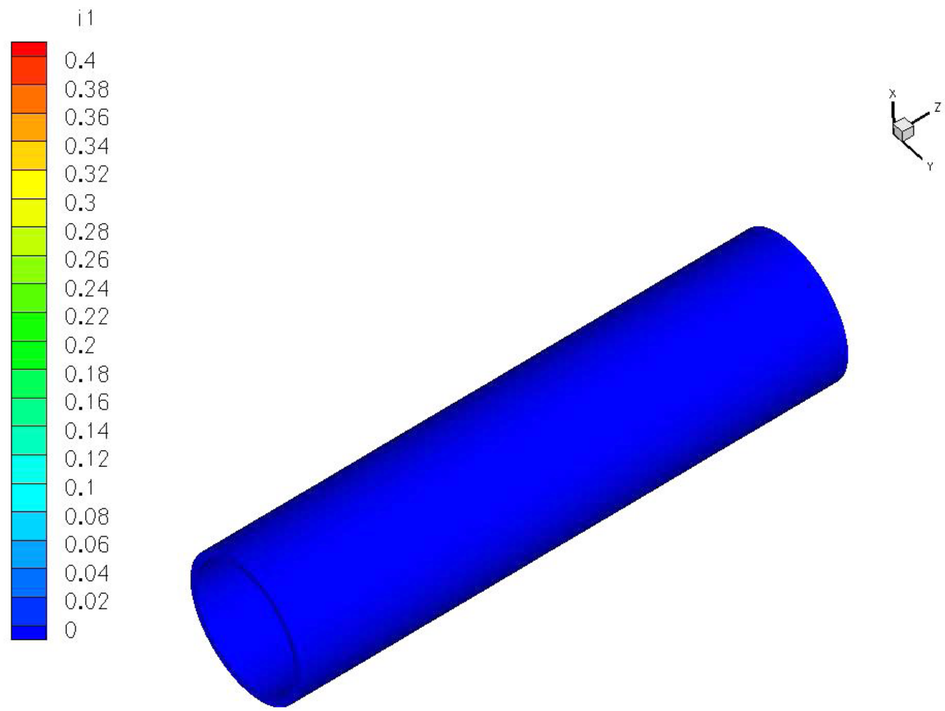


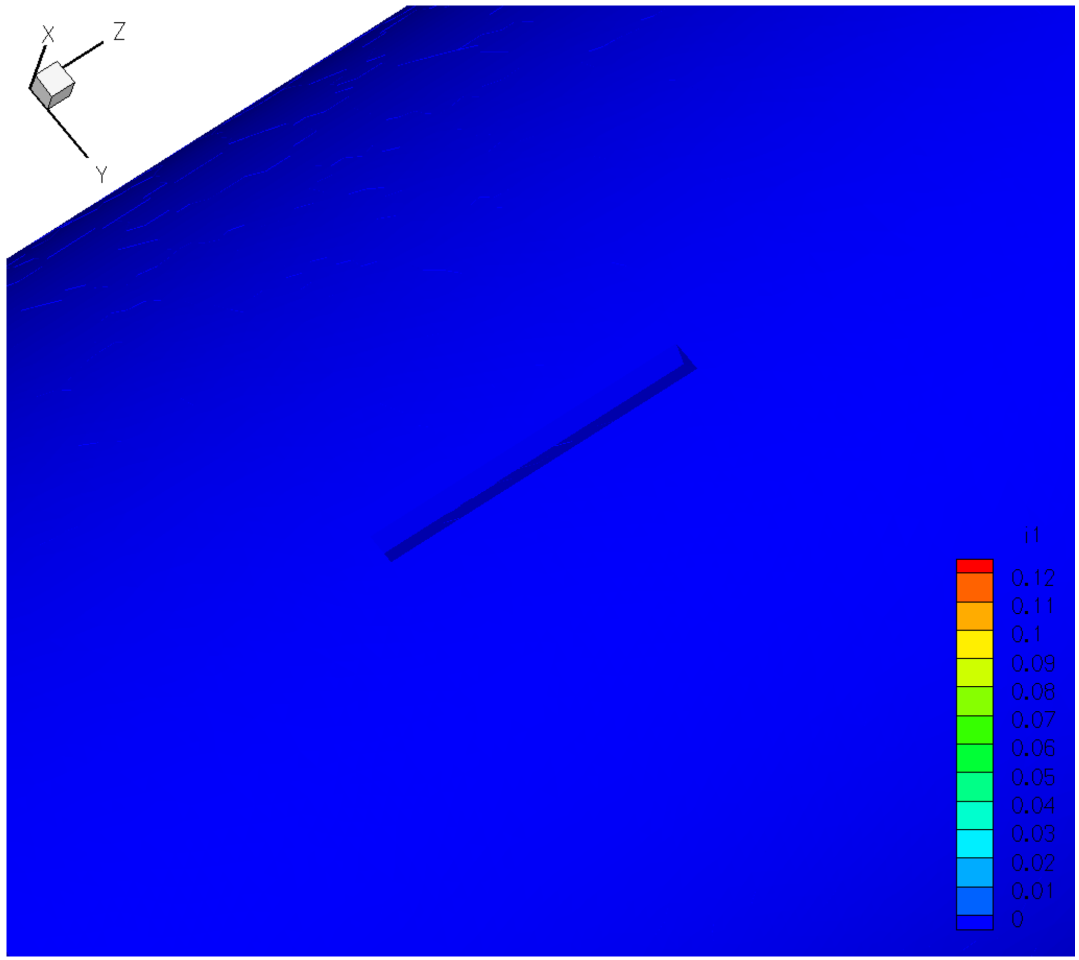
Fracture in Gypsum Cylinder





Z
X
Y
u3
0.30
0.28
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00


i1
0.4
0.38
0.36
0.34
0.32
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
X
Z
Y


X
Z
Y
i1
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0.05
0.04
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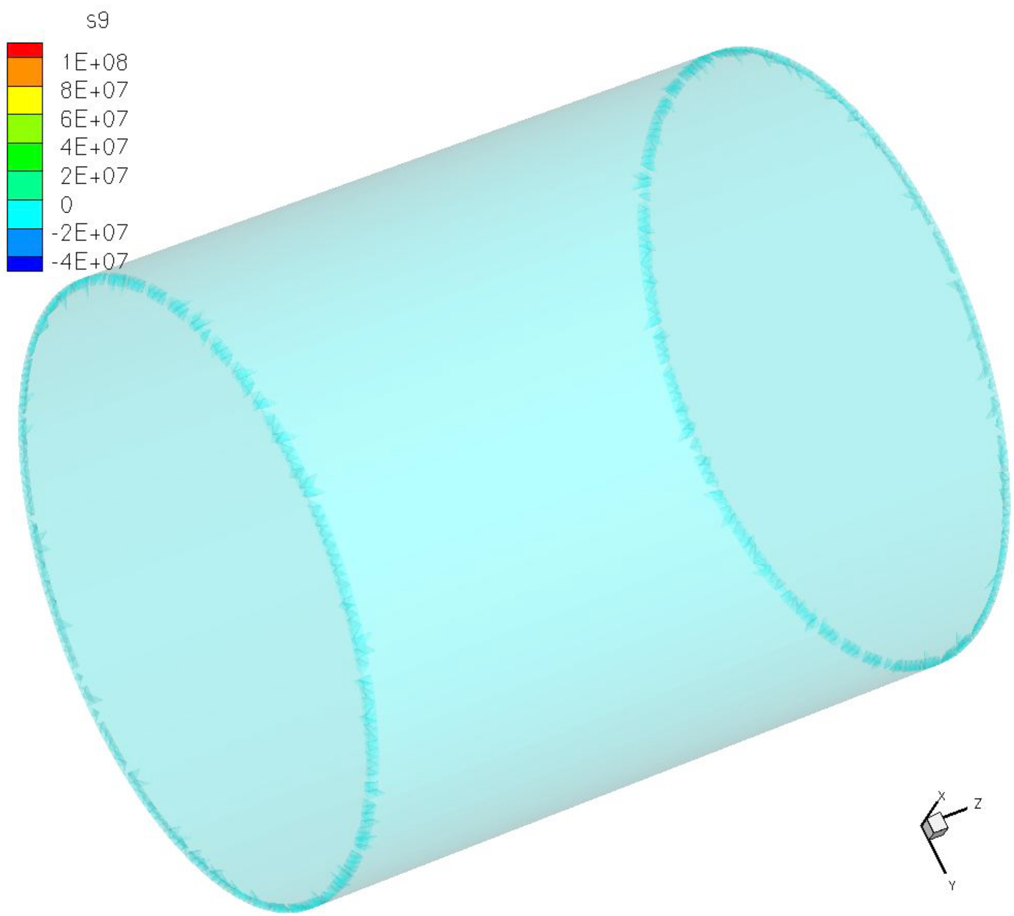
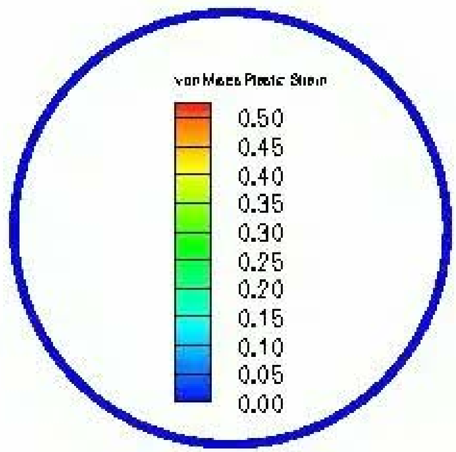
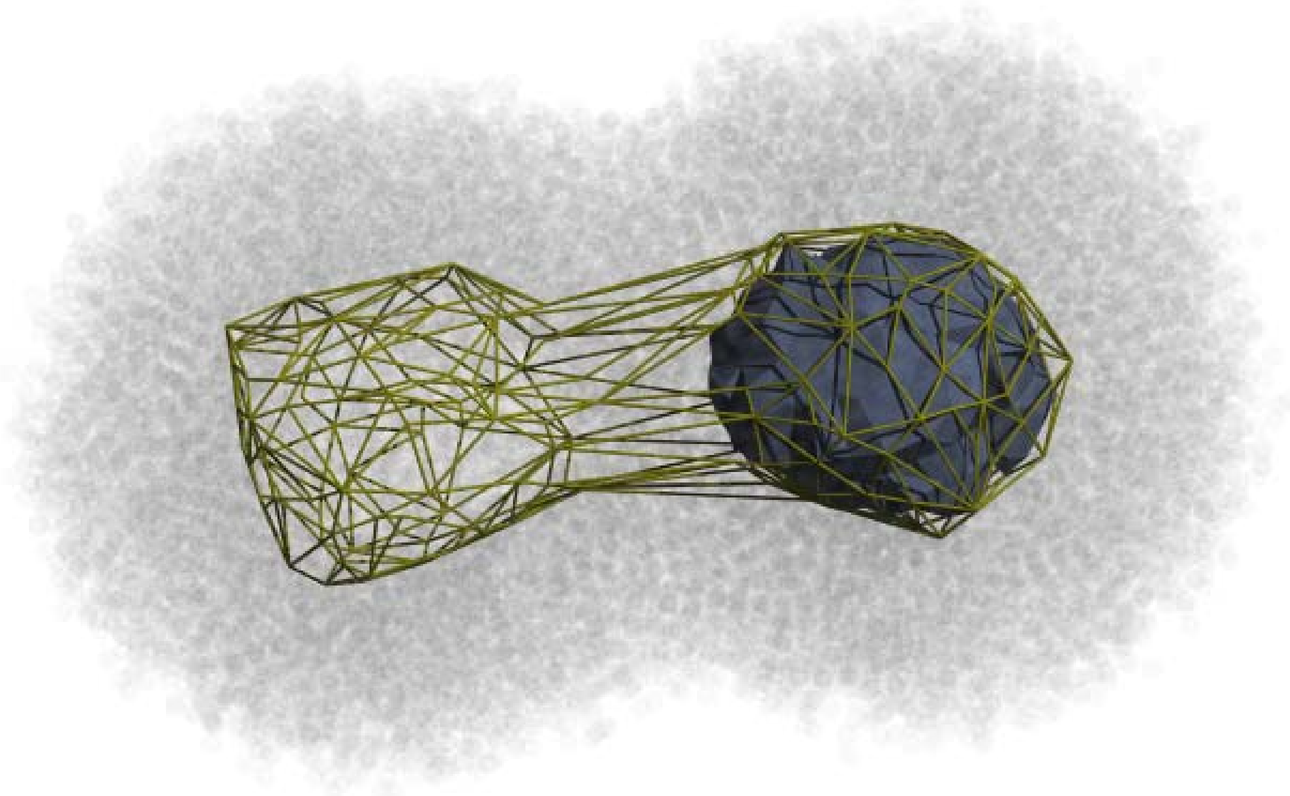
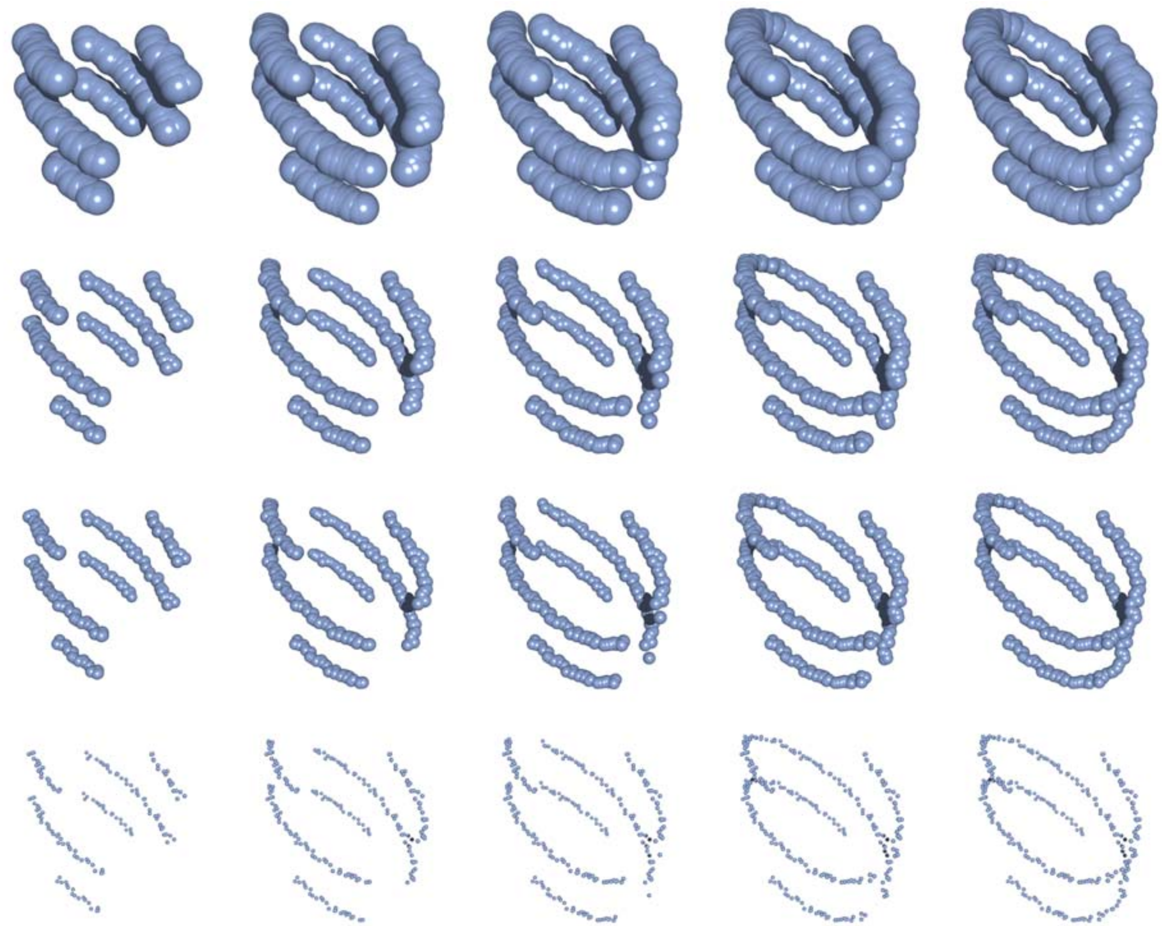
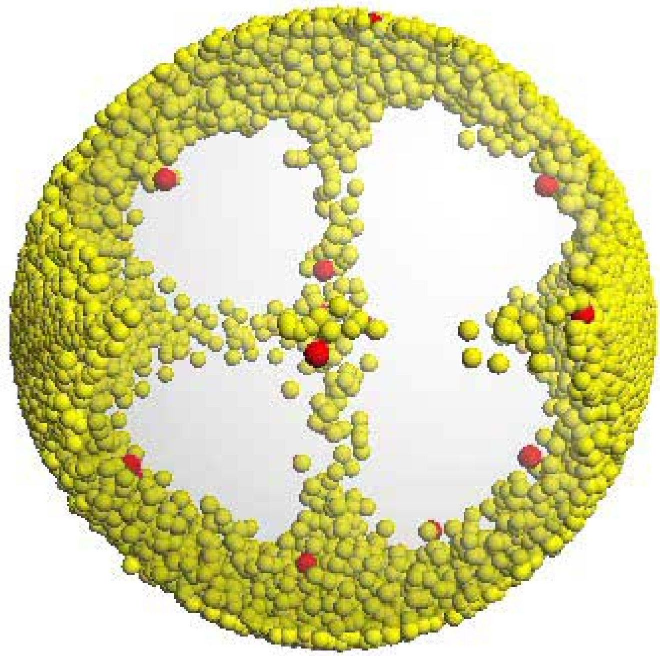
0.02
0.01
0

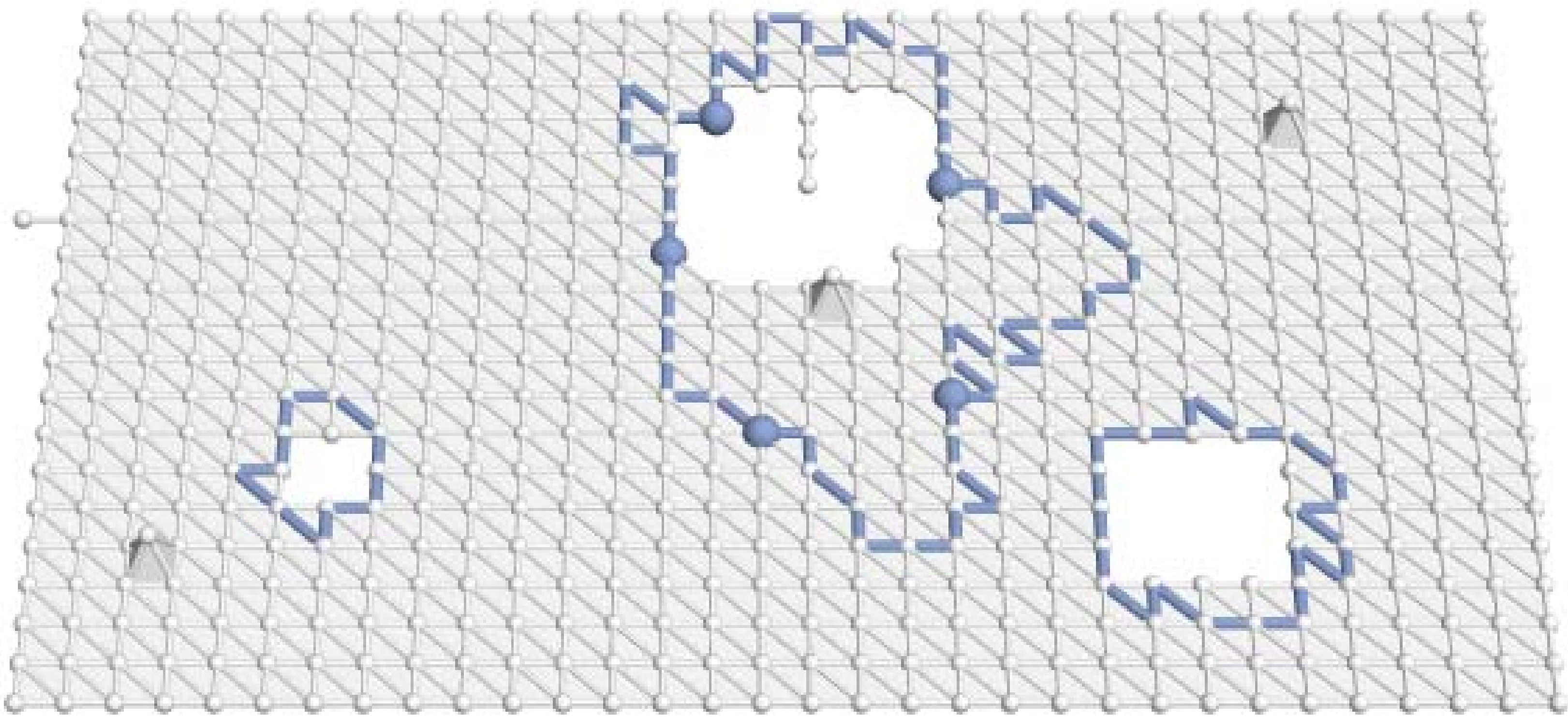

s9
1E+08
8E+07
6E+07
4E+07
2E+07
0
-2E+07
-4E+07
X
Z
Y

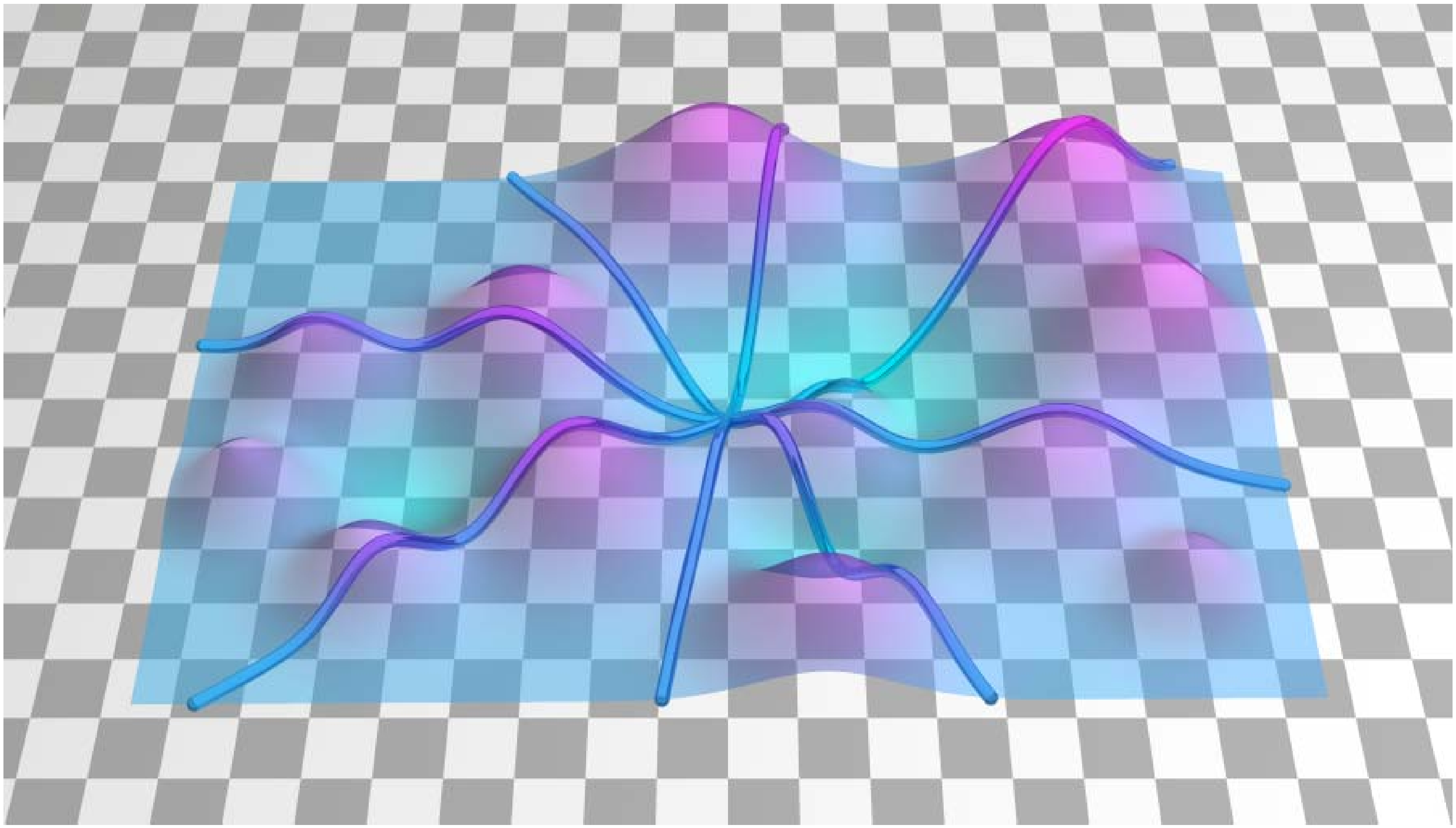

0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00



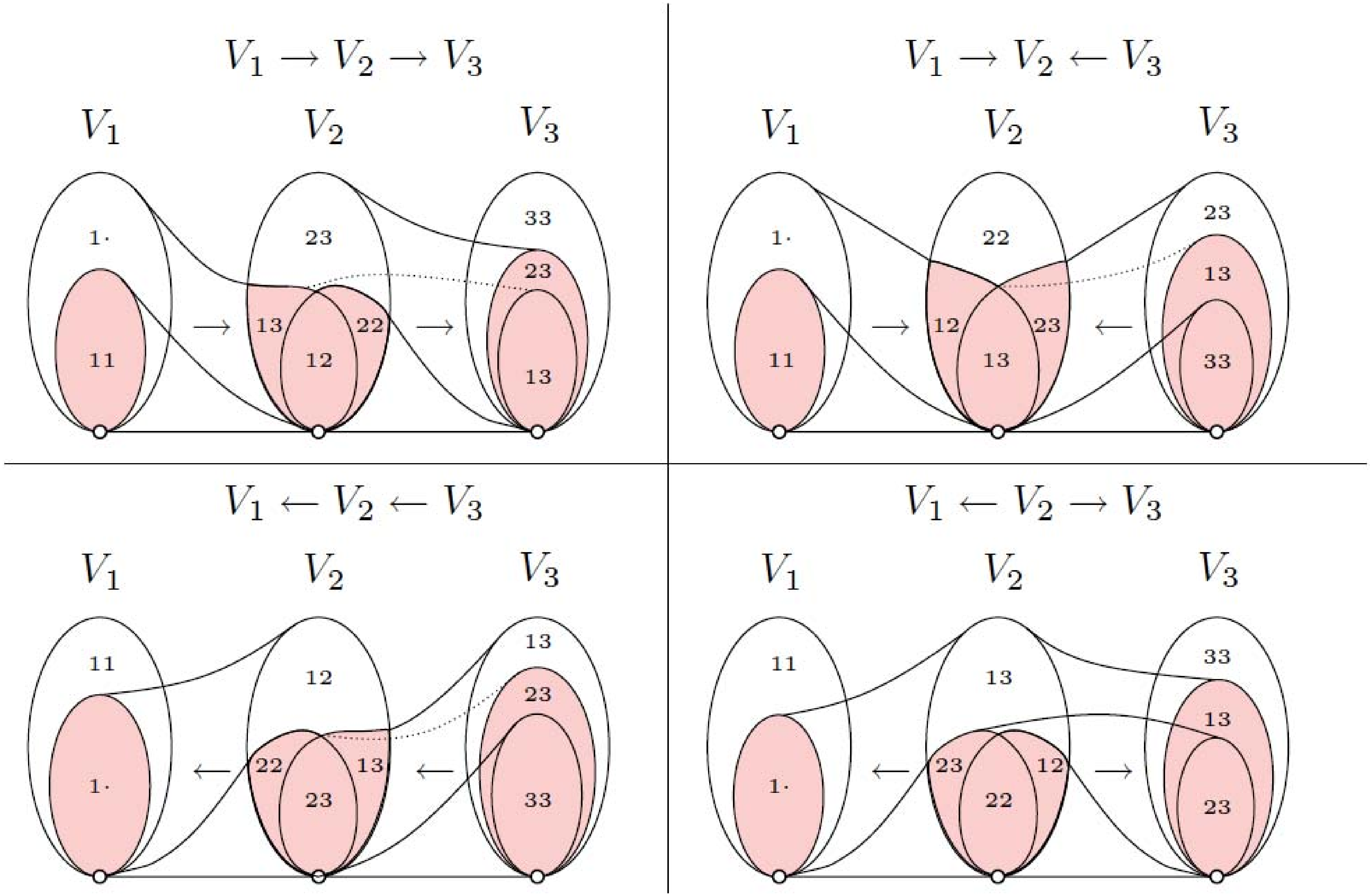




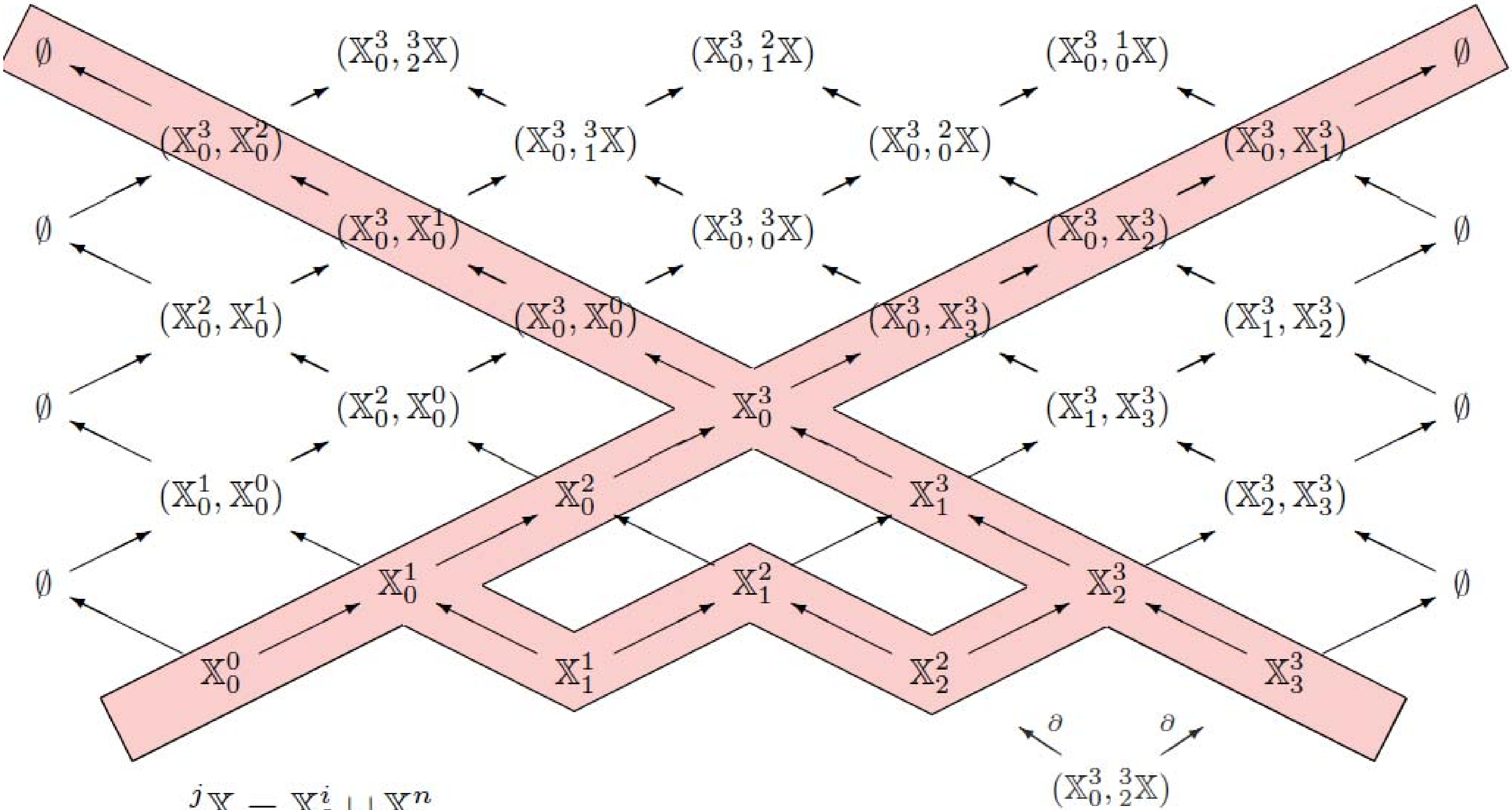




Zigzag Combinations



Pyramid



$${}_i^j\mathbb{X} = \mathbb{X}_0^i \cup \mathbb{X}_j^n$$

Approximating Shortest Homology Loops

Presenter: Tamal K. Dey

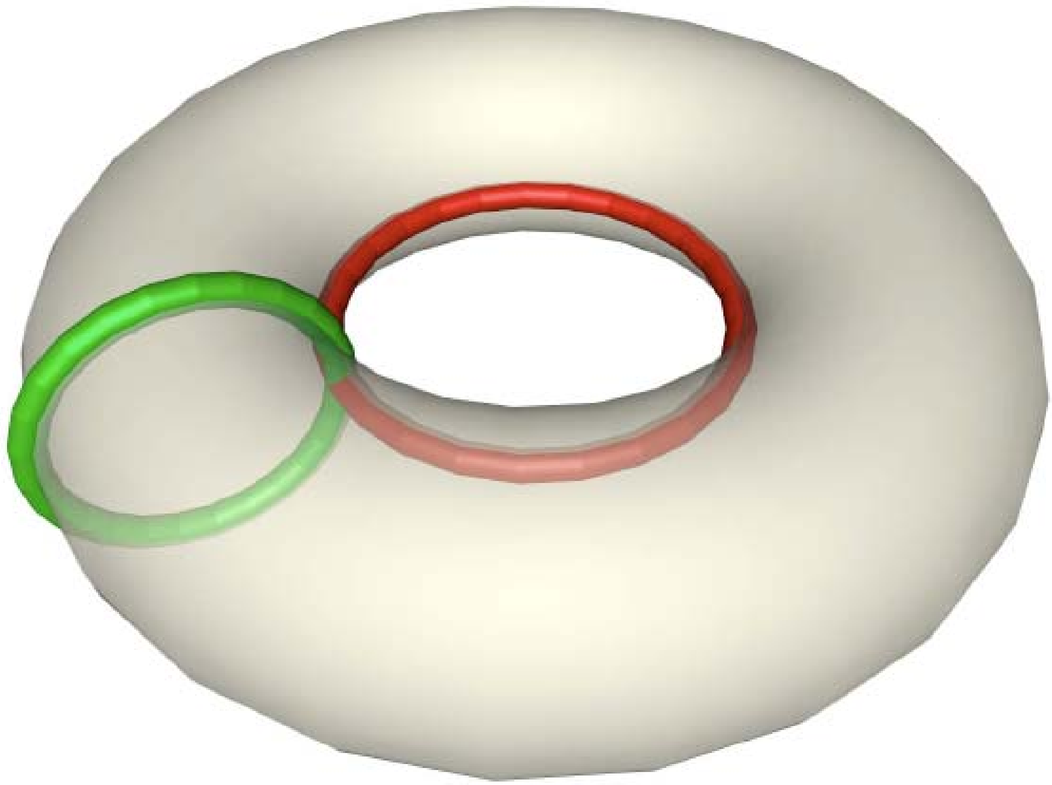
Department of CS&E, The Ohio State University

Joint work with Jian Sun and Yusu Wang

Abstract

- Inference of topology and geometry of a hidden manifold from its point data is a fundamental problem.
- We present an algorithm to compute a set of loops from point data that approximate a **shortest** basis of the homology group $\mathsf{H}_1(M)$ of the sampled manifold M.
- We also present a polynomial time algorithm for computing a shortest basis of $\mathsf{H}_1(M)$ for any finite simplicial complex embedded in an Euclidean space.

Example



Main Results: Theorem 1.3

- Let $M \subset \mathbb{R}^d$ be a smooth, closed manifold with l as the length of a shortest basis of $\mathsf{H}_1(M)$. Let $k = rank\ \mathsf{H}_1(M)$, $\rho(M)$ and $\rho_c(M)$ be *reach* and *convexity radius* of M, and $c(n, \alpha)$ be the worst-case size of any α-Rips complex with n vertices.
- Given a set $P \subset M$ of n points which is an ε-sample of M and $4\varepsilon \leq \alpha \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(M), \rho_c(M)\}$, one can compute a set of loops G in $O(n^3 + knc(n, \alpha)^3)$ time where

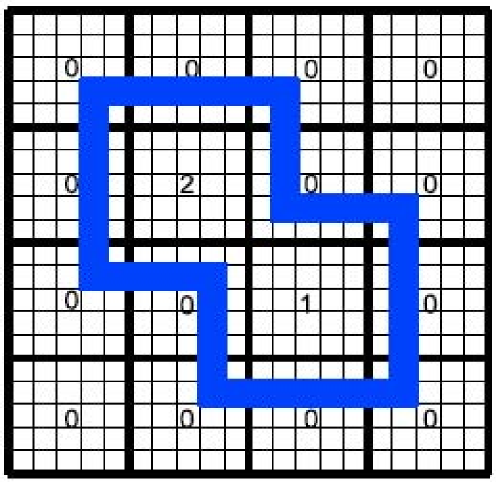
$$\frac{1}{1 + \frac{4\alpha^2}{3\rho^2(M)}} l \leq \mathrm{Len}(G) \leq (1 + \frac{4\varepsilon}{\alpha}) l.$$

Main Results: Theorem 1.4

- Let $\mathcal{K}$ be a finite simplicial complex with positive weights on edges.
- A shortest basis for $\mathsf{H}_1(\mathcal{K})$ can be computed in $O(kn^4)$ time where $k = rank\ \mathsf{H}_1(\mathcal{K})$ and $n = |\mathcal{K}|$.

Detection and Classification of Critical Components

Subdivide a cubical grid $\mathscr{A}$ into five cubes in each dimension. Build an isolating neighbourhood N by taking two layers of subdivided cubes around $|\mathscr{A}|$. The outermost layer is the outer boundary, and will be divided into an upper level set L_p, a lower level set L_n and a level set L_z. The algorithm has two passes: first, cubes are identified as singular and consolidated into connected components; second, the components are identified as regular or critical, and classified.



Algorithm 1 (Detecting singular cubes)

For each elementary full cube Q
build N, L_p, L_n
$H := H_*(N,L_p) = 0$ *and* $H_*(N,L_n) = 0$
    if $H =$ *TRUE* **then** *Q is ordinary*
        else *Q is singular*
    endif

Algorithm 2 (Sorting components in $\mathbb{R}^d$)

For each singular cube Q, $\mathscr{C} := \{Q\}$
while $P \in bd\,(\mathscr{C}) \cap X$ *is singular*
    $\mathscr{C} := \mathscr{C} \cup \{P\}$
    build $bd\,(\mathscr{C})$
endwhile
build N, L_p, L_n, L_z
do
    $H := H_*(N,L_p) = 0$ *and* $H_*(N,L_n) = 0$
        if $H =$ *TRUE* **then** $\mathscr{C}$ *is a regular component*
        else if $L_p = L_z = \emptyset$
            then $\mathscr{C}$ *is a maximum component*
        else if $L_n = L_z = \emptyset$
            then $\mathscr{C}$ *is a minimum component*
        else $\mathscr{C}$ *is a saddle component*
    endif
break

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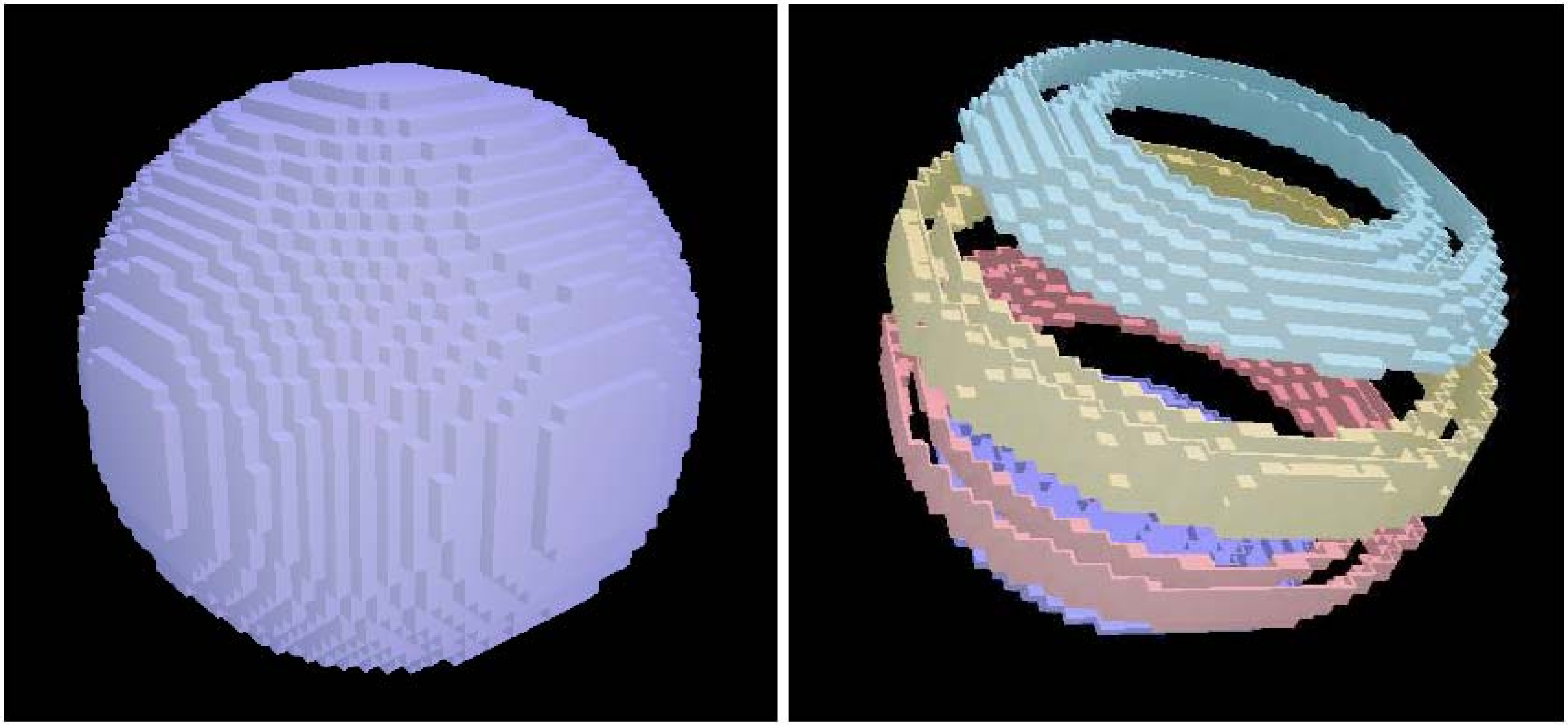
Madjid Allili, Bishop's University

Critical Sphere



N and slices of L_p for the minimal sphere of $f(x,y,z) = (x^2+y^2+z^2-5)^2$

Domain: $[-3,3]^3$ discretized in 51^3 3-cubes

Image: Discretized over 8192 levels

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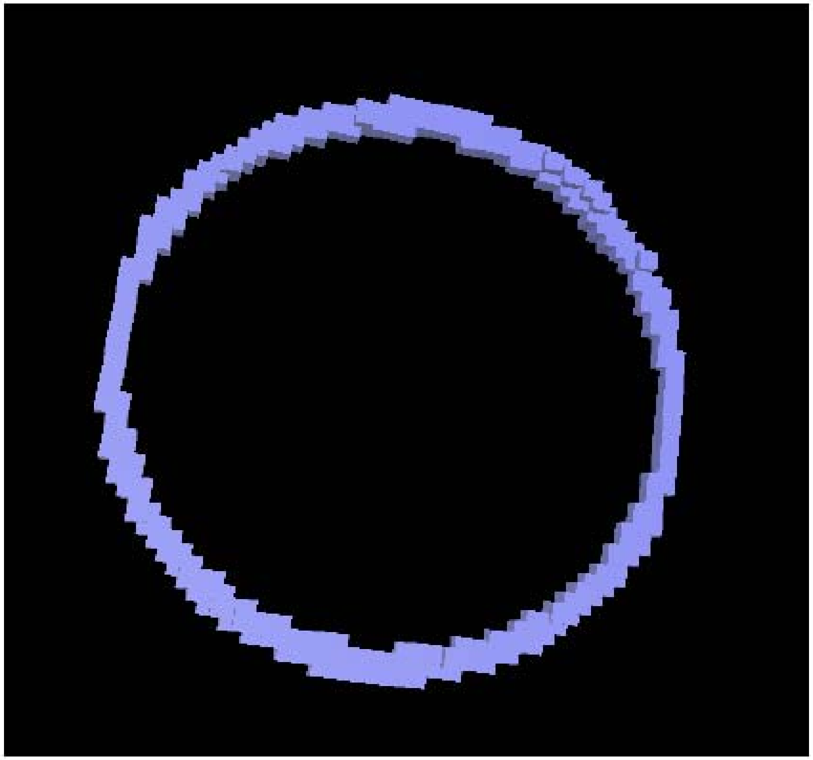
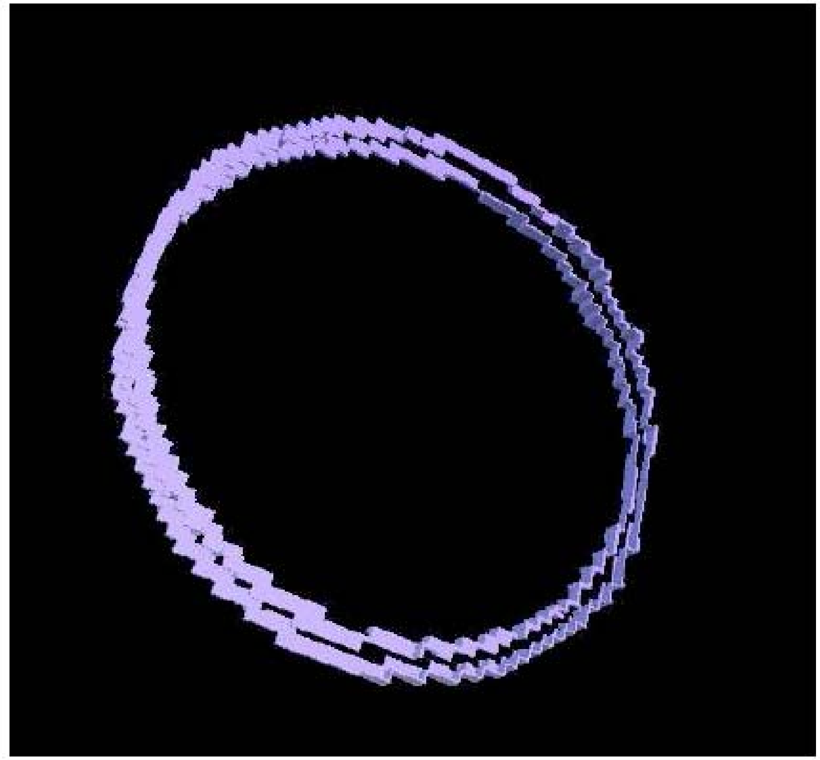
Critical Torus





N and L_p for the function in toric coordinates $f(r,\theta,\phi) = r^2(1-2\sin^2(\theta-\phi/2))$

Domain: $[-3,3]^2 \times [-4/5,4/5]$ discretized in $101^2 \times 27$ 3-cubes

Image: Discretized over 16384 levels

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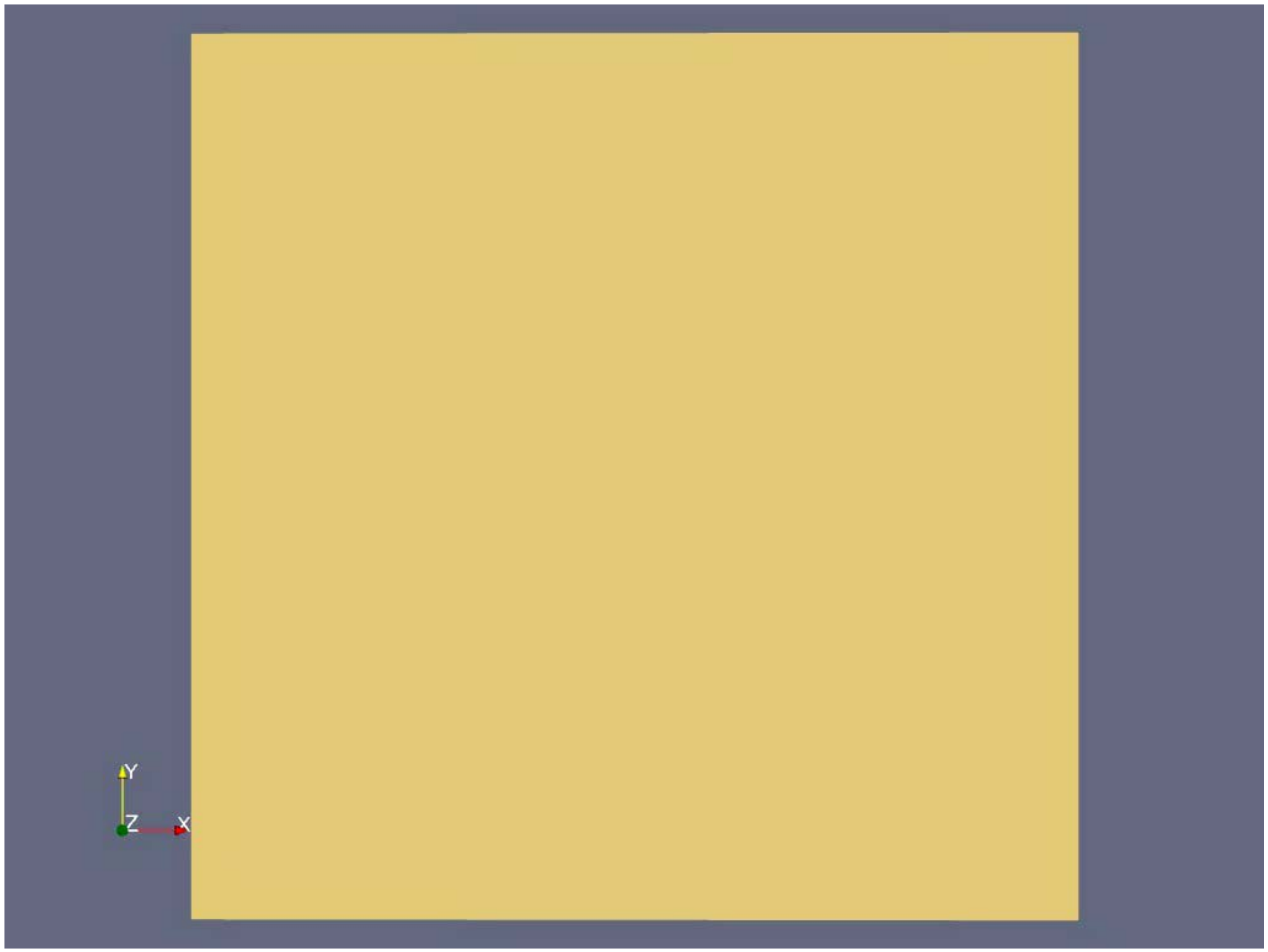
Y
Z
X