# Approximating Shortest Homology Loops

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- We present an algorithm to compute a set of loops from point data that approximate a **shortest** basis of the homology group $\mathsf{H}_1(M)$ of the sampled manifold $M$.
- We also present a polynomial time algorithm for computing a shortest basis of $\mathsf{H}_1(\mathcal{K})$ for any finite simplicial complex $\mathcal{K}$ embedded in an Euclidean space.

# Example

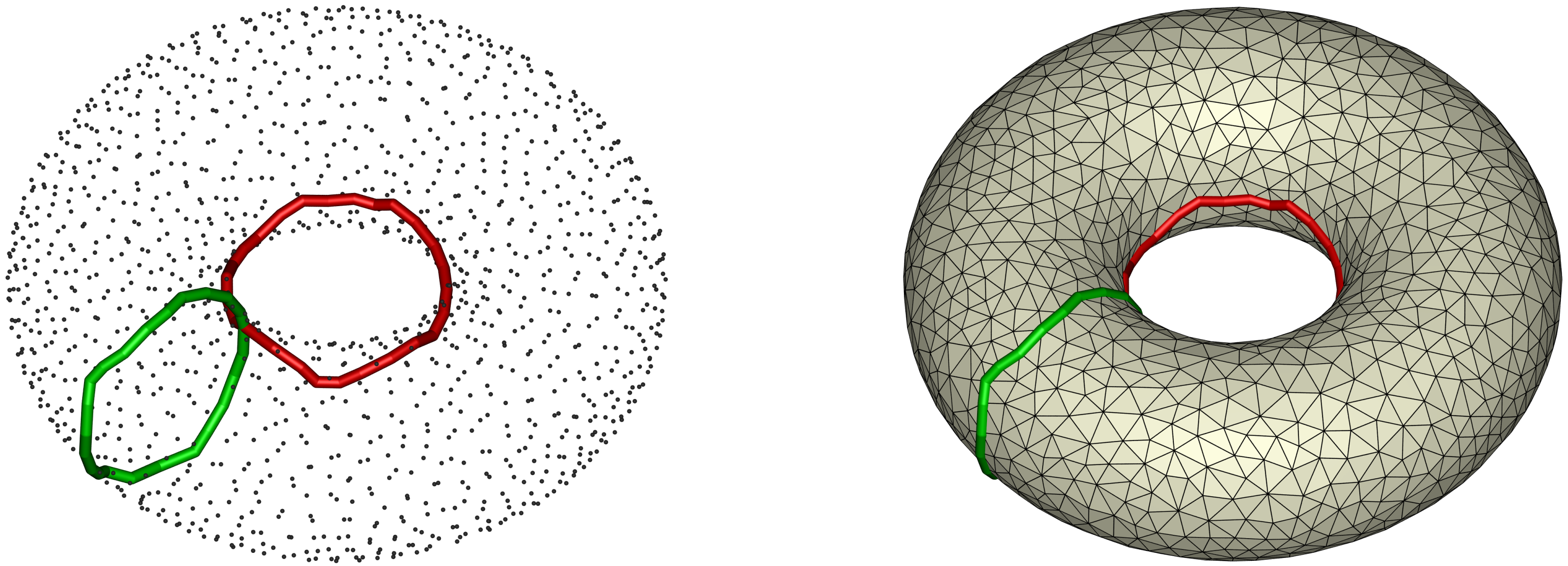

Figure 1: Shortest basis of $\mathsf{H}_1(M)$ for the torus.

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- 2-manifold triangulations [EW05], localized homology [CF07,ZC08].
- We combine *greedy characterization* [EW05], *persistence* [ELZ02,ZC05], *intertwined complexes* [CO08] and new observations.

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- A **shortest basis** of $\mathsf{H}_1(\mathbb{T})$ is a basis with minimal length.

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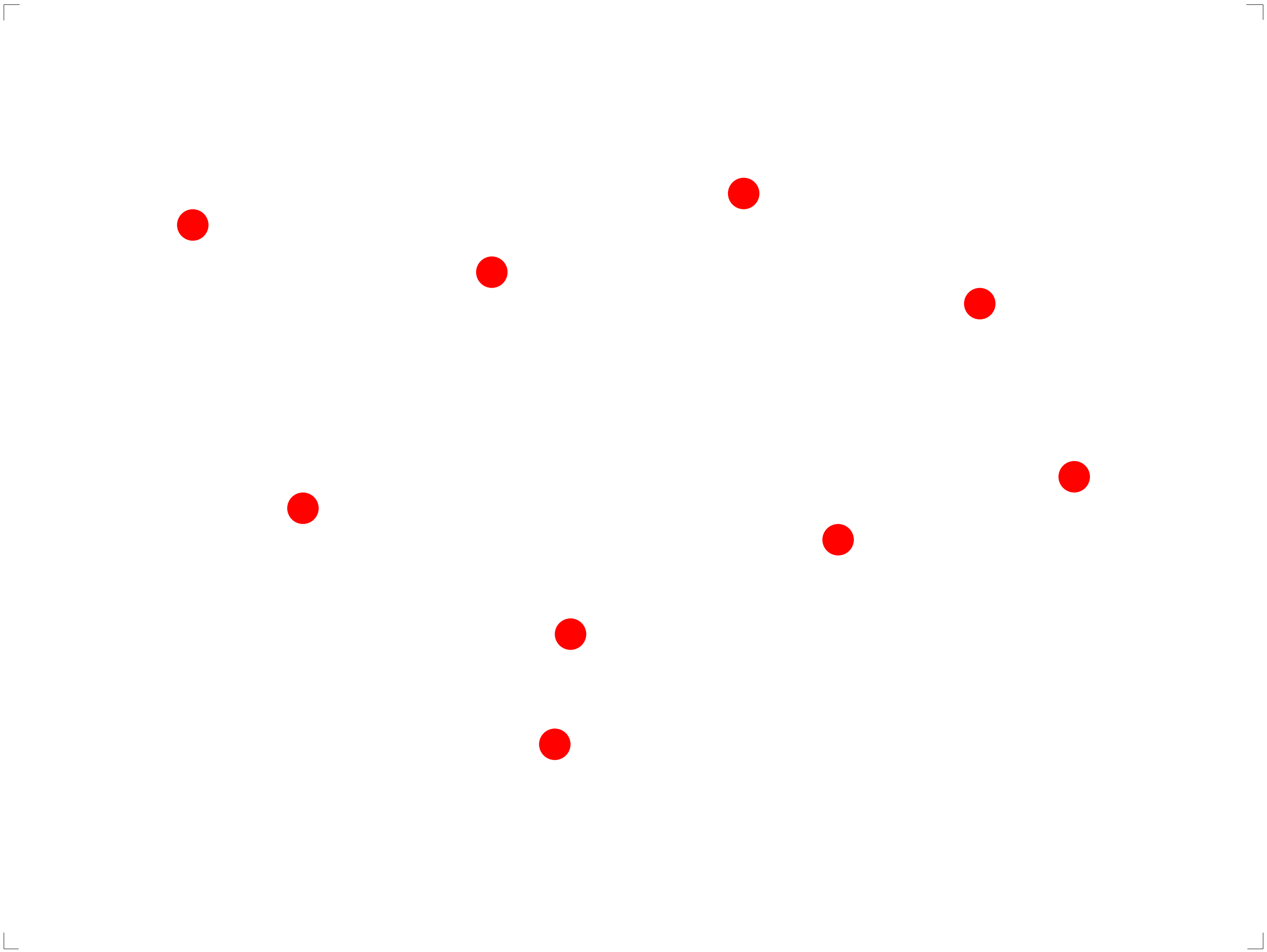
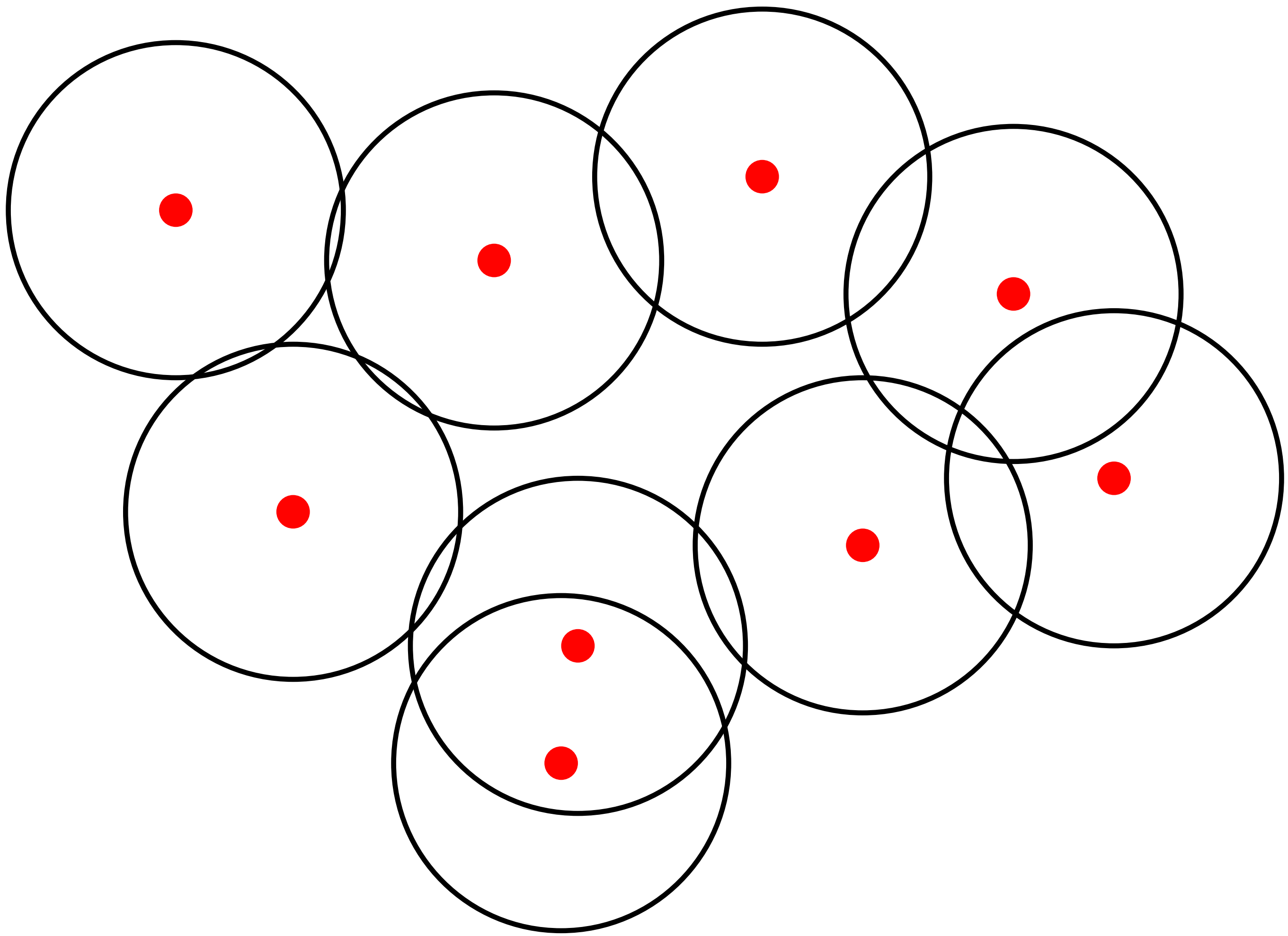
- The **Rips complex** $\mathcal{R}^\alpha(P)$ is a simplicial complex where a simplex $\sigma \in \mathcal{R}^\alpha(P)$ if and only if $\mathrm{Vert}(\sigma)$ are within pairwise Euclidean distance of $\alpha$.
- **Proposition 1** For any finite set $P \subset \mathbb{R}^d$ and any $\alpha \geq 0$, one has $\mathcal{C}^{\frac{\alpha}{2}}(P) \subseteq \mathcal{R}^\alpha(P) \subseteq \mathcal{C}^\alpha(P)$.

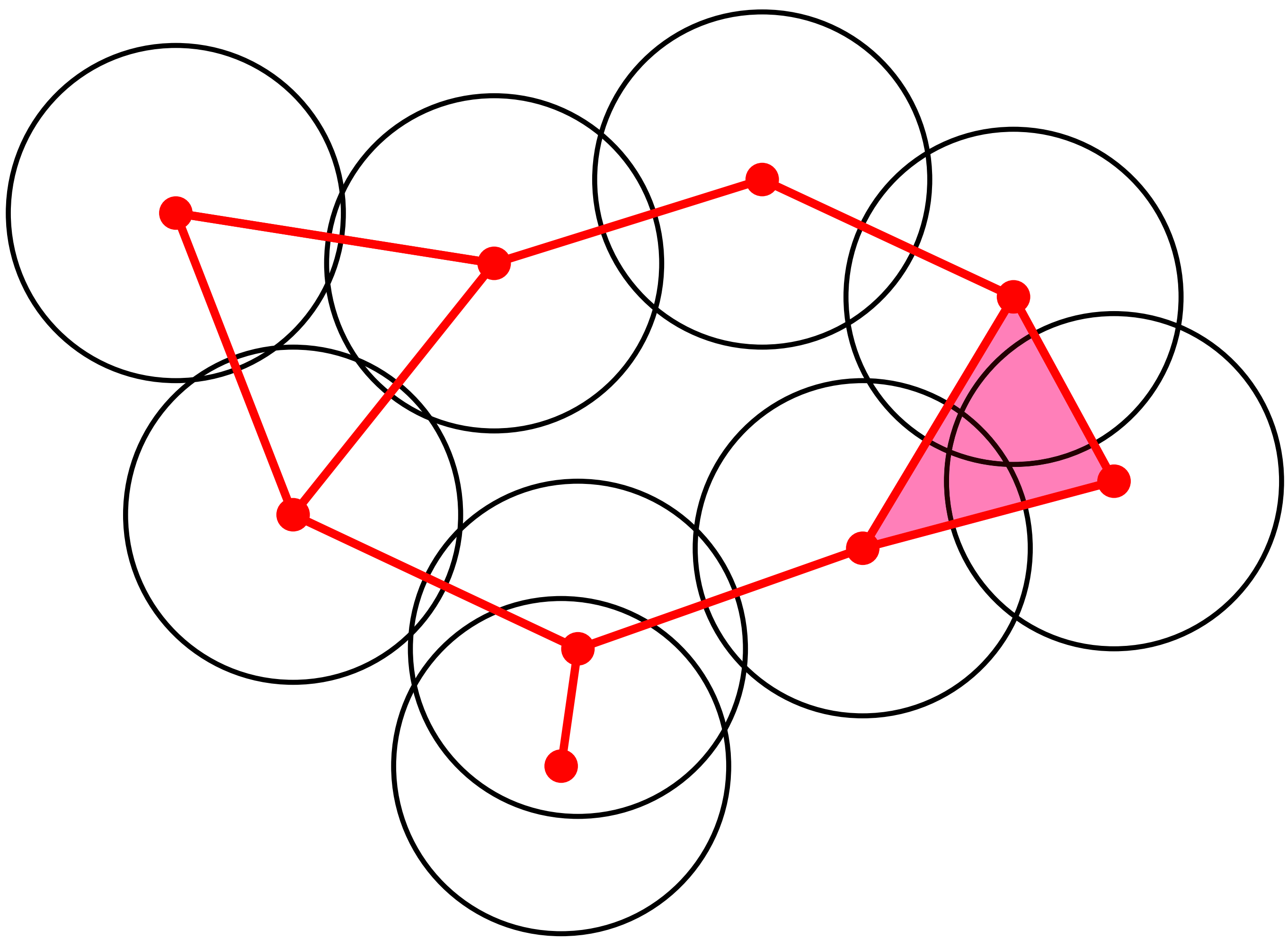
# Point set $P$



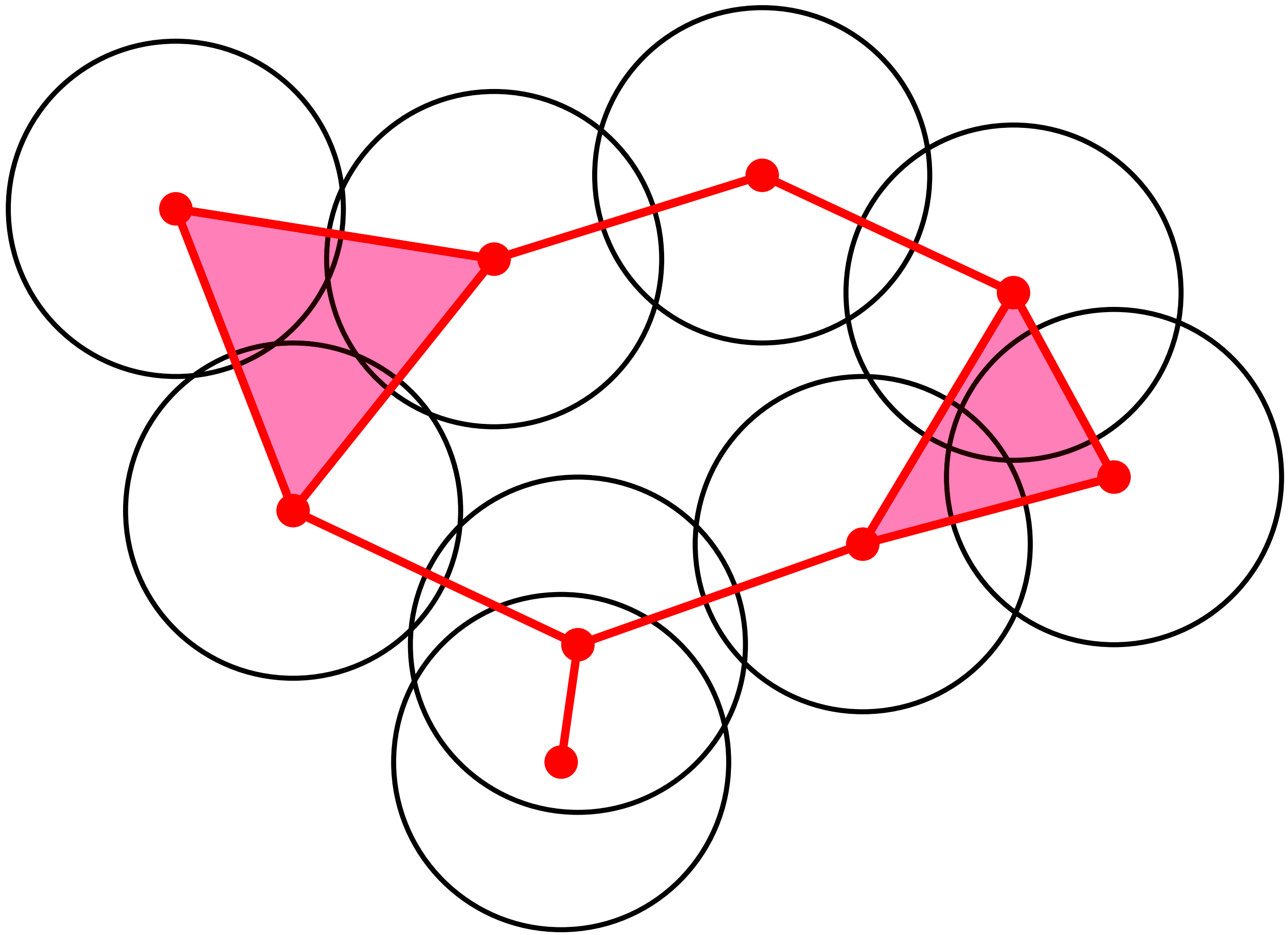
# Balls $B(p, \alpha)$ for $p \in P$



# Čech complex $\mathcal{C}^{\alpha}(P)$



# Rips complex $\mathcal{R}^{2\alpha}(P)$



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- If $p$ and $q$ are close enough, the minimizing geodesic is unique, which we denote as $\gamma(p, q)$.
- Minimizing geodesics induce a distance metric $d_M : M \times M \to \mathbb{R}$.

# Euclidean and geodesic lengths

If $d(p,q) \leq \rho(M)/2$, one has

$$d_M(p,q) \leq (1 + \frac{4d^2(p,q)}{3\rho^2(M)})d(p,q).$$

- Here $d(p,q)$ is the Euclidean distance.
- $\rho(M)$ is the **reach** defined as the minimum distance between $M$ and its medial axis.

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- There is a $\alpha_p > 0$ for each $p \in M$ where $\alpha_p$ is the supremum $\alpha$ so that $B_M(p, \alpha)$ is convex (the minimizing geodesics between any two points in $B_M(p, \alpha)$ lie in $B_M(p, \alpha)$).

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- **Convexity radius** of $M$: $\rho_c(M) = \inf_{p \in M} \alpha_p$.
- $P$ is an $\varepsilon$-sample of $M$ if $B(x, \varepsilon) \cap P \neq \emptyset$ for each $x \in M$.

# Theorem 1

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- Given a set $P \subset M$ of $n$ points which is an $\varepsilon$-sample of $M$ and $4\varepsilon \leq \alpha \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(M), \rho_c(M)\}$, one can compute a set of loops $G$ in $O(n^3 + knc(n,\alpha)^3)$ time where

$$\frac{1}{1 + \frac{4\alpha^2}{3\rho^2(M)}} l \leq \mathrm{Len}(G) \leq (1 + \frac{4\varepsilon}{\alpha}) l.$$

Here $c(n,\alpha)$ is the size of the 2-skeleton of a Rips complex $\mathcal{R}^{2\alpha}(P)$.

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- A shortest basis for $\mathsf{H}_1(\mathcal{K})$ can be computed in $O(kn^4)$ time where $k = rank\ \mathsf{H}_1(\mathcal{K})$ and $n = |\mathcal{K}|$.

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- The **greedy set** $G$ is an *ordered* set of loops $\{g_1, ..., g_k\}$ satisfying the following conditions:

  $g_1$ is the shortest loop in $\mathcal{L}$ nontrivial in $\mathsf{H}_1(\mathcal{K})$;

  $g_{i+1}$ is the shortest loop in $\mathcal{L}$ independent of $g_1, ..., g_i$.

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- Let $C_p$ be the set of all canonical loops with respect to $p$:

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- **Proposition 4** The greedy set chosen from $\cup_{p \in P} G_p$ is a shortest basis of $\mathsf{H}_1(\mathcal{K})$.

# Algorithm 2

**Proposition 5** CANONGEN$(p, \mathcal{K})$ outputs $G_p$.

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3: Run the persistence algorithm based on the following filtration of $\mathcal{K}$: vertices in $P = \mathrm{Vert}(\mathcal{K})$, tree edges in $T$, non-tree edges in the canonical order, triangles in $\mathcal{K}$. Return the set of canonical loops associated with $k = \mathrm{rank}(\mathsf{H}_1(\mathcal{K}))$ edges unpaired after the algorithm.

# Algorithm 3

SPGEN($\mathcal{K}$)

For each $p \in P = \mathrm{Vert}(\mathcal{K})$, set $G_p :=$ CANONGEN$(p, \mathcal{K})$.
Sort all loops in $\cup_p G_p$ by lengths in the increasing order.
Let $g_1, ..., g_{k|P|}$ be this sorted list. Initialize $G := \{g_1\}$.
**for** $i := 2$ to $k|P|$, **do**
    **if** $|G| = k$, **then**
        Exit the for loop.
    **else if** $g_i$ is independent of loops in $G$, **then**
        Add $g_i$ to $G$.
    **end if**
**end for**
Return $G$.

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**Theorem 3** SHORTLOOP$(P, \alpha)$ computes a shortest basis for the persistent homology group $\mathsf{H}_1(\mathcal{C}^{\alpha/2}(P))$.

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3: Compute the shortest basis for $\mathsf{H}_1(\mathcal{K})$.

4: Return first $k$ loops from the computed basis where $k$ is the rank of the $\mathsf{H}_1(\mathcal{R}^{\alpha}(P)) \rightarrow \mathsf{H}_1(\mathcal{R}^{2\alpha}(P))$.

# Bounding Lengths: Proposition 1

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- Let $g$ be a geodesic loop in $M$. There is a loop $\hat{g}$ in $\mathcal{C}^{\alpha/2}(P)$ so that $[h(\hat{g})] = [g]$ where $h$ is a homotopy equivalence and $Len(\hat{g}) \leq (1 + \frac{4\varepsilon}{\alpha})Len(g)$.

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- If $G = \{g_1, ..., g_k\}$ and $G' = \{g'_1, ..., g'_k\}$ are the generators of a shortest basis of $\mathsf{H}_1(M)$ and $\mathsf{H}_1(\mathcal{K})$ respectively, then we have $\mathsf{Len}(G') \leq (1 + \frac{4\varepsilon}{\alpha})\mathsf{Len}(G)$.

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- Let $P \subset M$ be an $\varepsilon$-sample and $4\varepsilon \leq \alpha \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(M), \rho_c(M)\}$.
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- Let $G$ and $G'$ be defined as in Proposition 1.
- We have Len$(G) \leq (1 + \frac{4\alpha^2}{3\rho^2(M)})$Len$(G')$.

# Theorem 4

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- Let $G$ and $G'$ be a shortest basis of $\mathsf{H}_1(M)$ and $\mathsf{H}_1(\mathcal{K})$ respectively.
- We have $\frac{1}{1+\frac{4\alpha^2}{3\rho^2(M)}}\mathsf{Len}(G) \leq \mathsf{Len}(G') \leq (1 + \frac{4\varepsilon}{\alpha})\mathsf{Len}(G)$.

# Thank you!