A Mathematical Tour of Geomagic Software

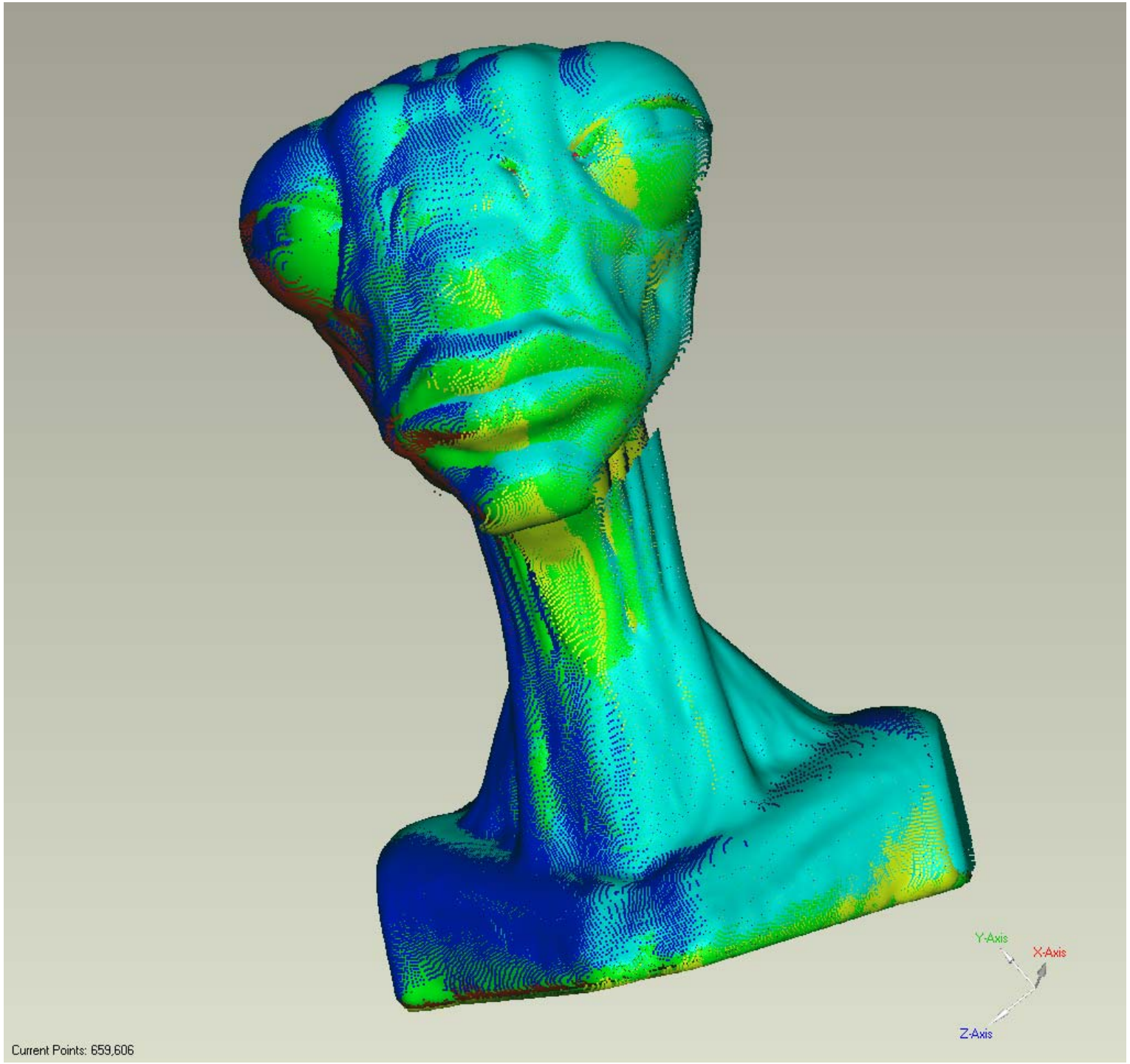
Yates Fletcher, Geomagic Inc.

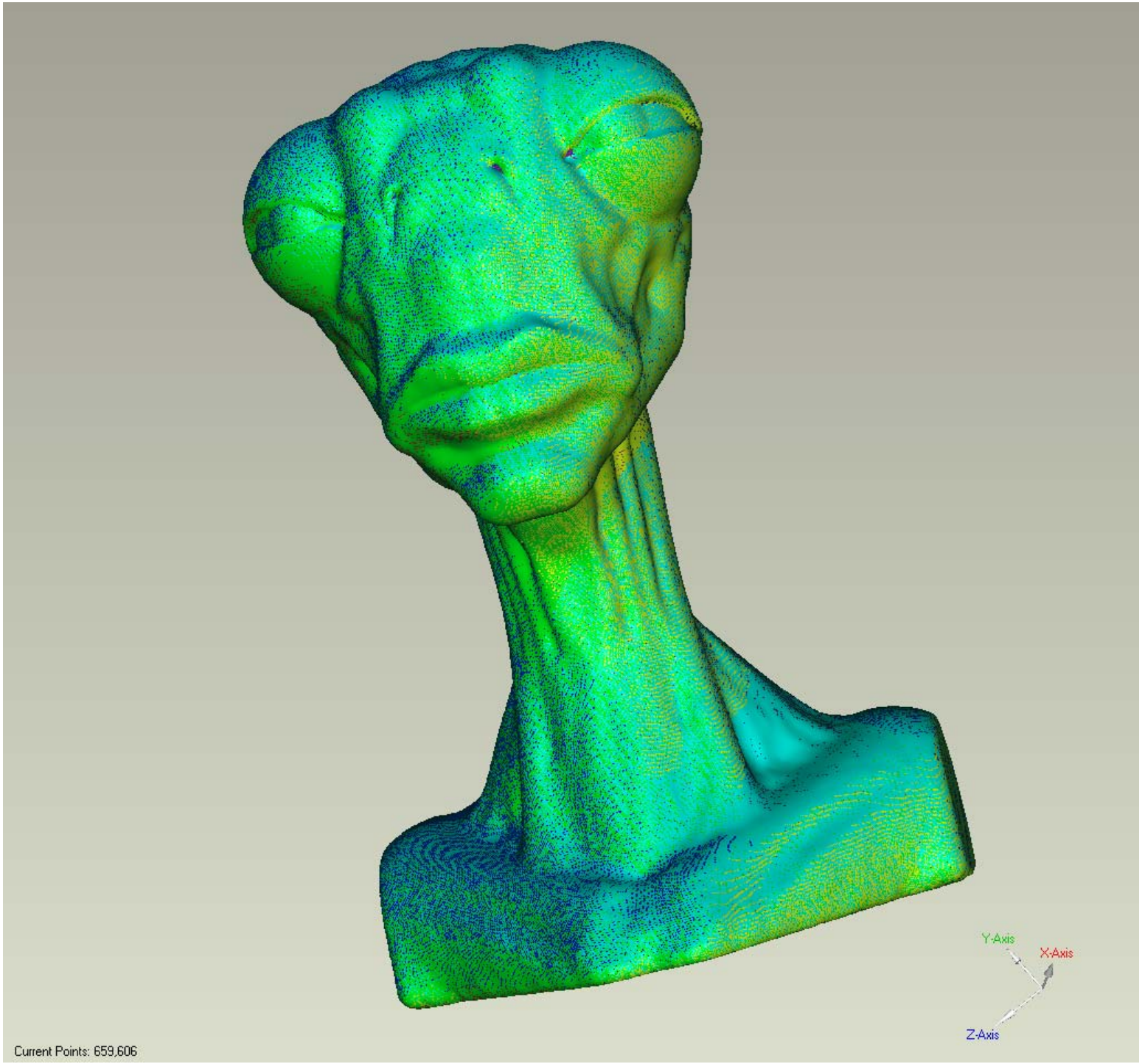
DSSP
digital shape sampling and processing

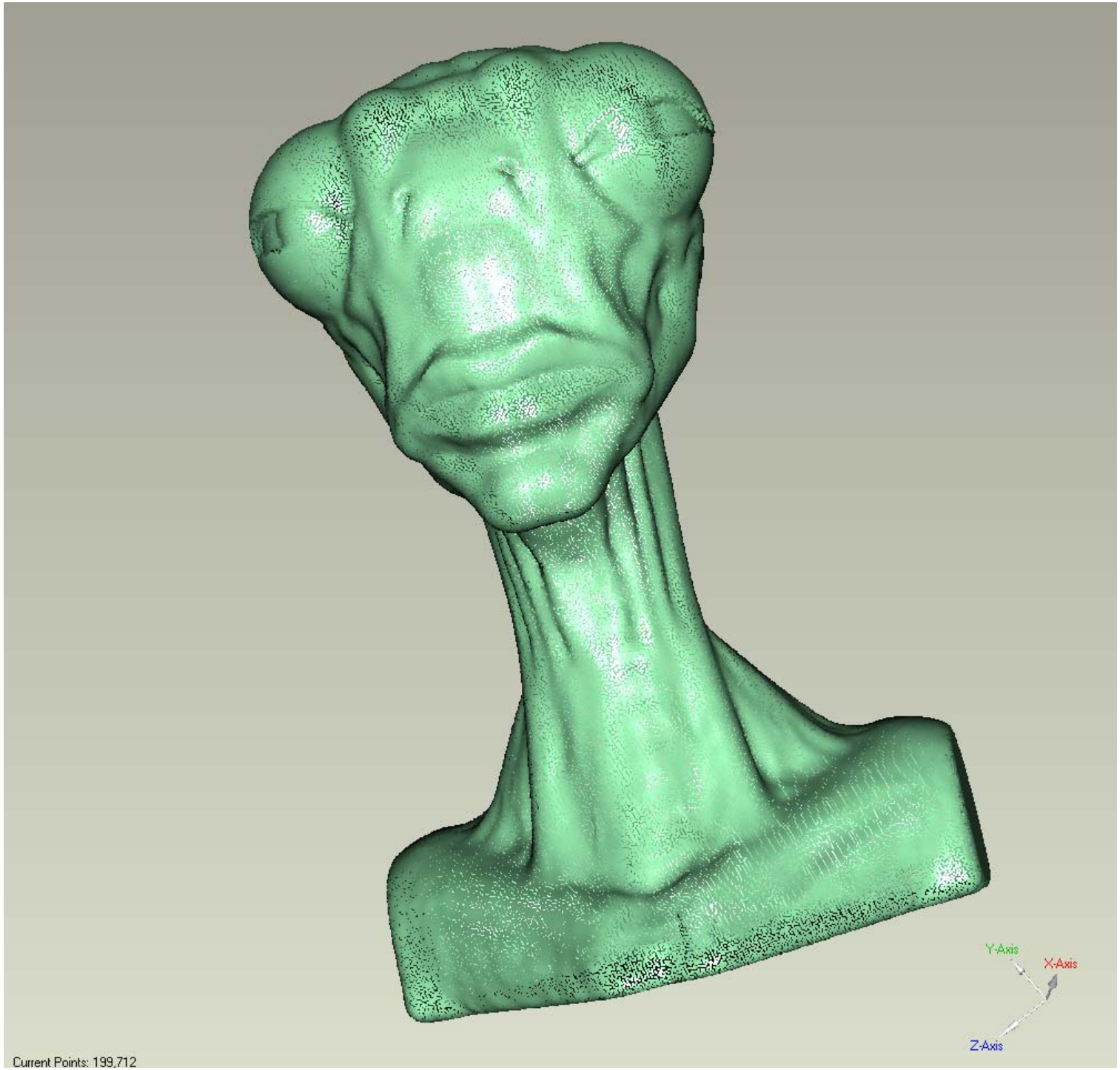
- Registration and cleanup
- Points to polygons – triangulation (wrap)
- Polygons to NURBs – capture shape
- Polygons to CAD – capture design intent
- Shape analysis and segmentation
- Shape warping
- Shape compression

Registration and Cleanup

- Bring all scans into the same coordinate system
- Remove outliers
- Prune for uniform density
- Reduce noise


Y-Axis
X-Axis
Z-Axis
Current Points: 659,606

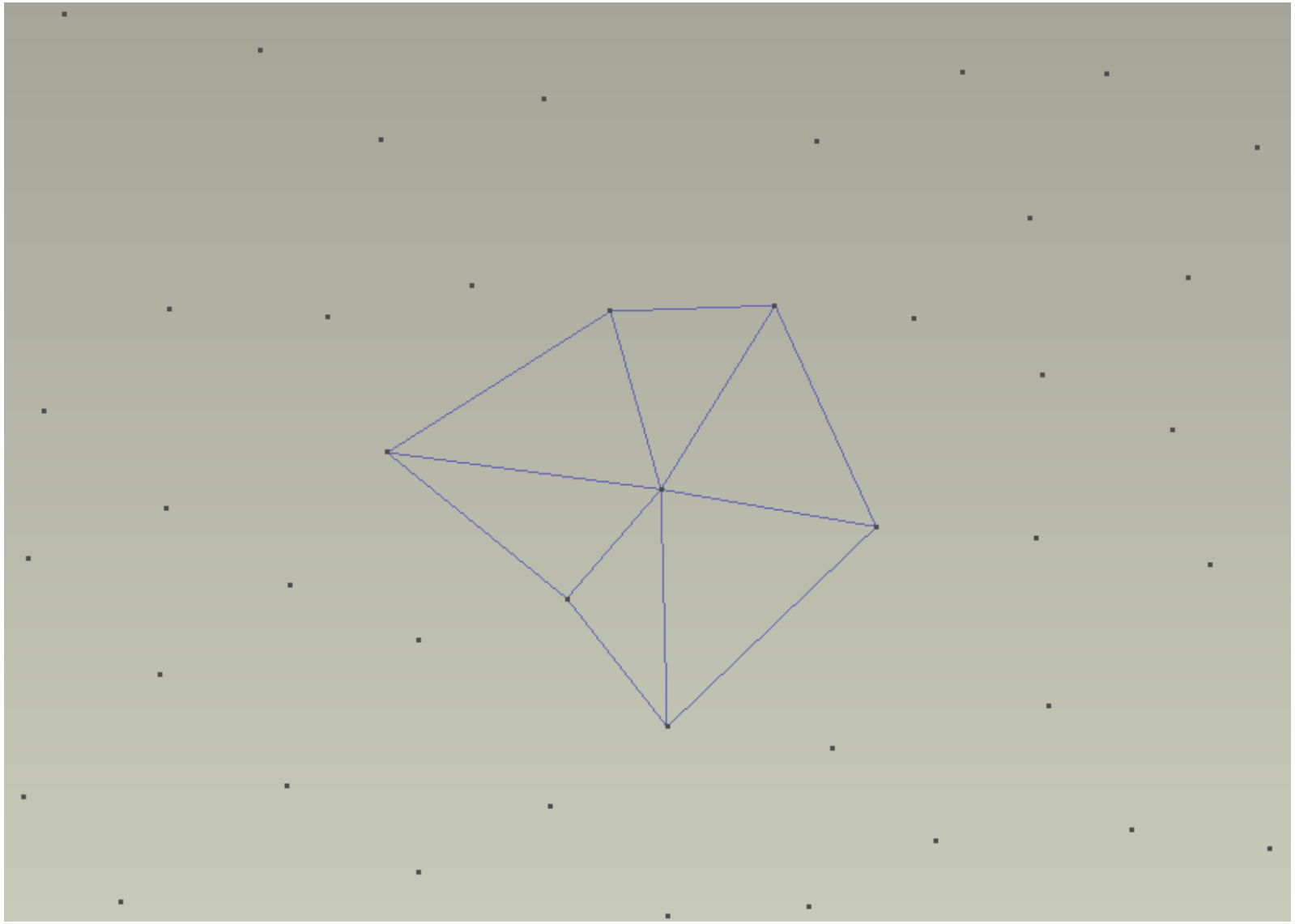

Y-Axis
X-Axis
Z-Axis
Current Points: 659,606

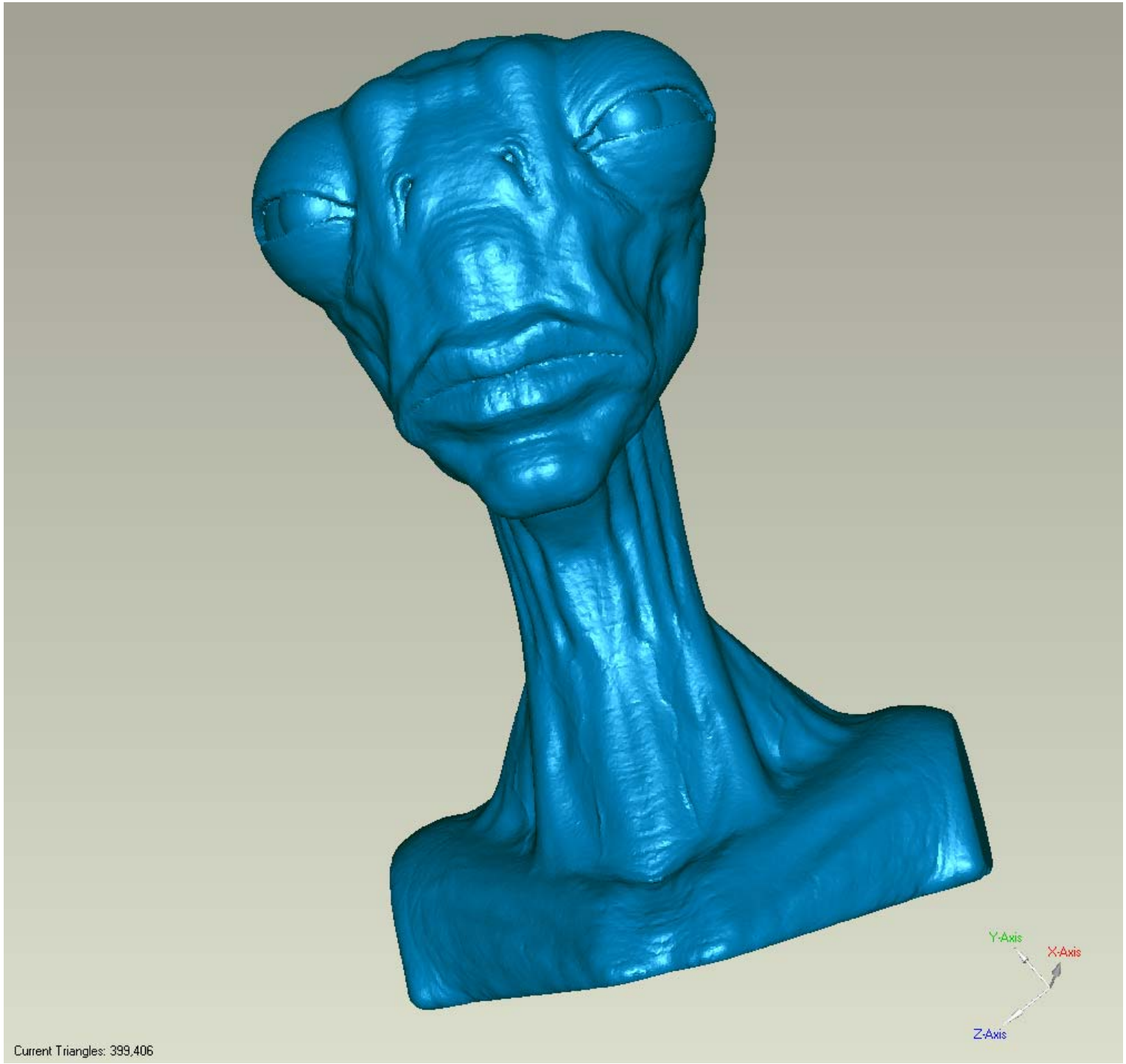

Y-Axis
X-Axis
Z-Axis
Current Points: 199,712

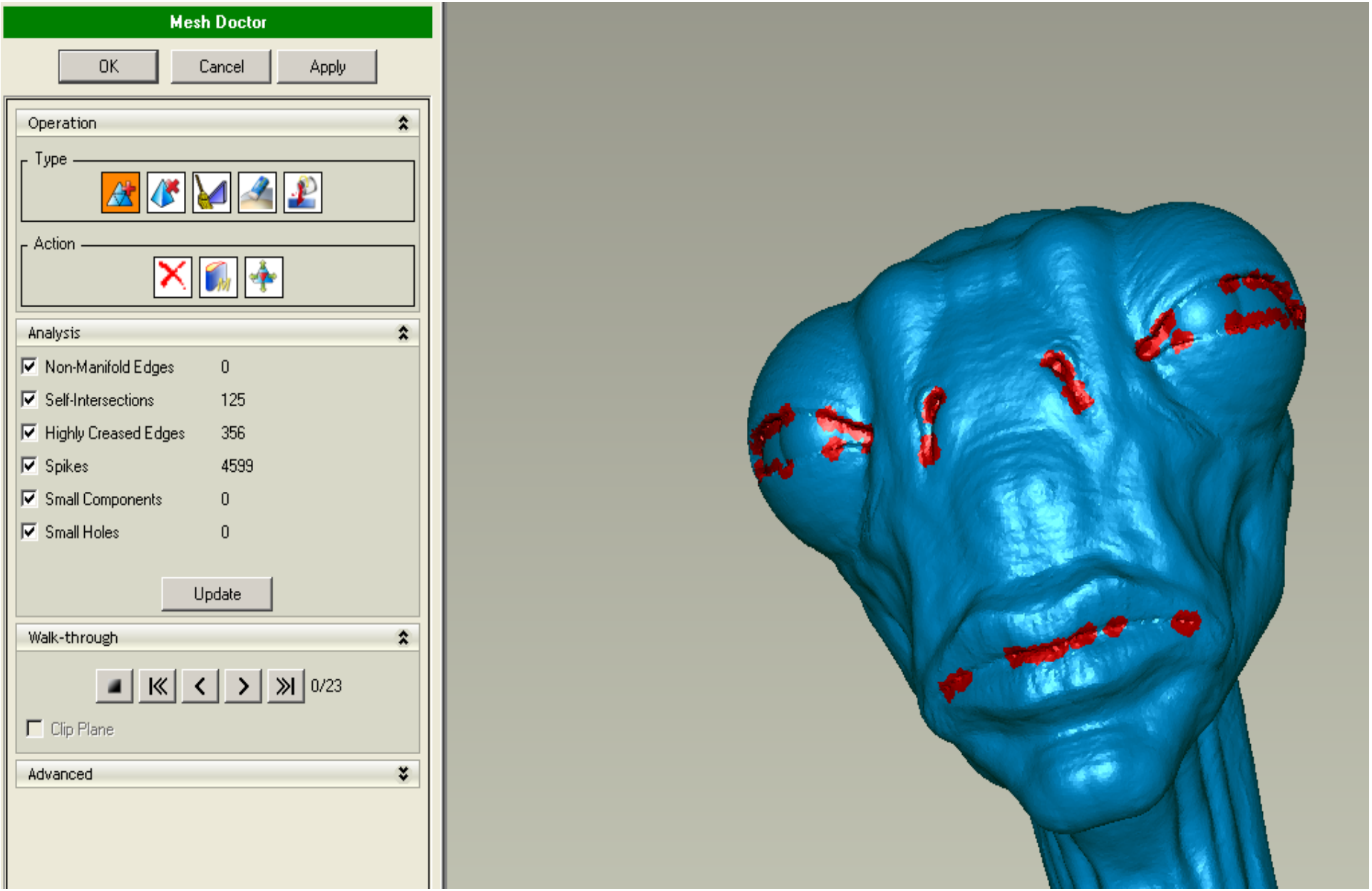
Triangulation (wrap)

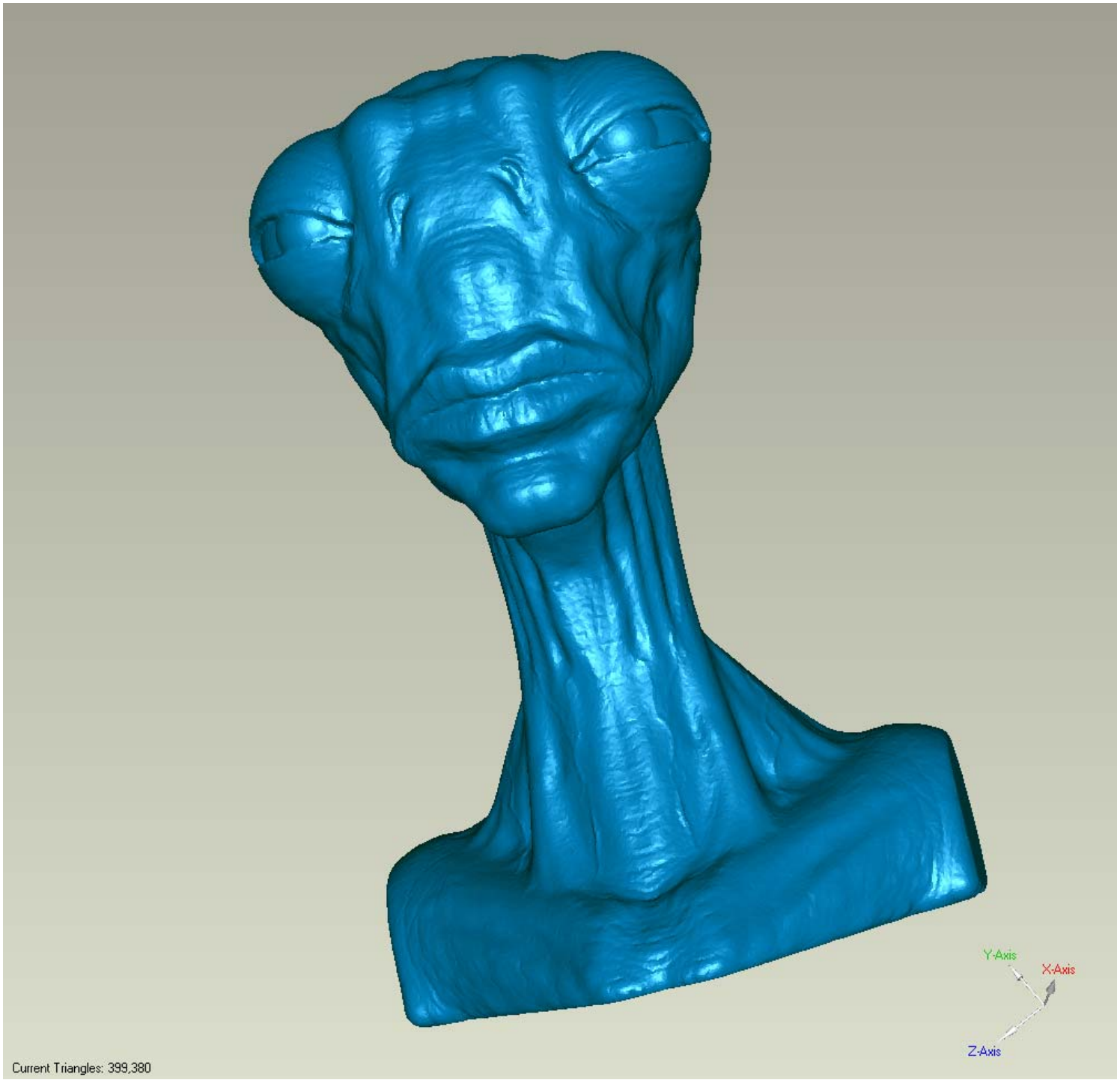
- For each point P in cloud
  - Project P and neighbors to plane along local normal
  - Compute Delaunay star of P
  - Add each triangle in star to master list
- Accept all triangles appearing three times in master list
- Fill holes

The Delaunay Star of a Point



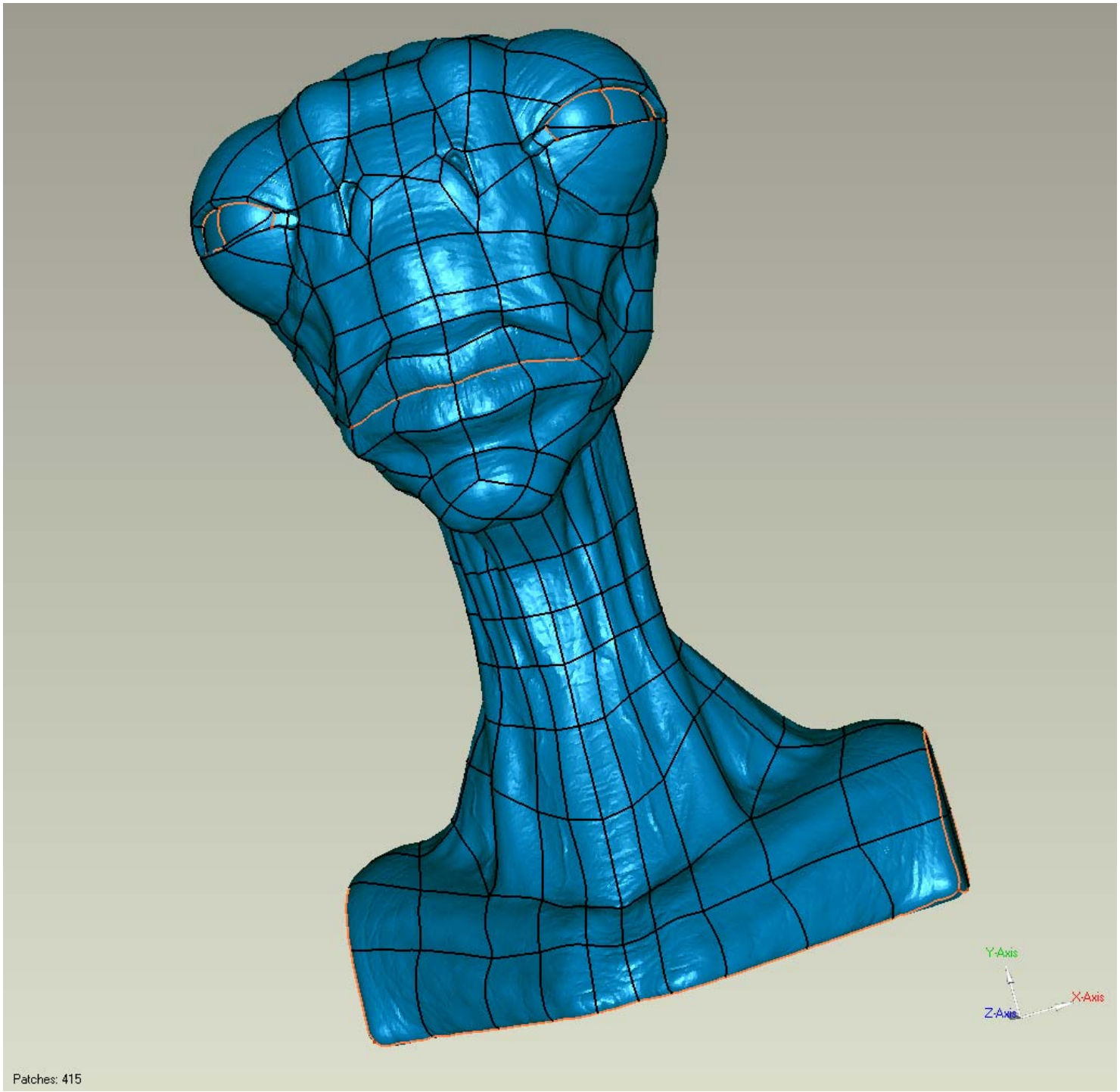

Y-Axis
X-Axis
Z-Axis
Current Triangles: 399,406

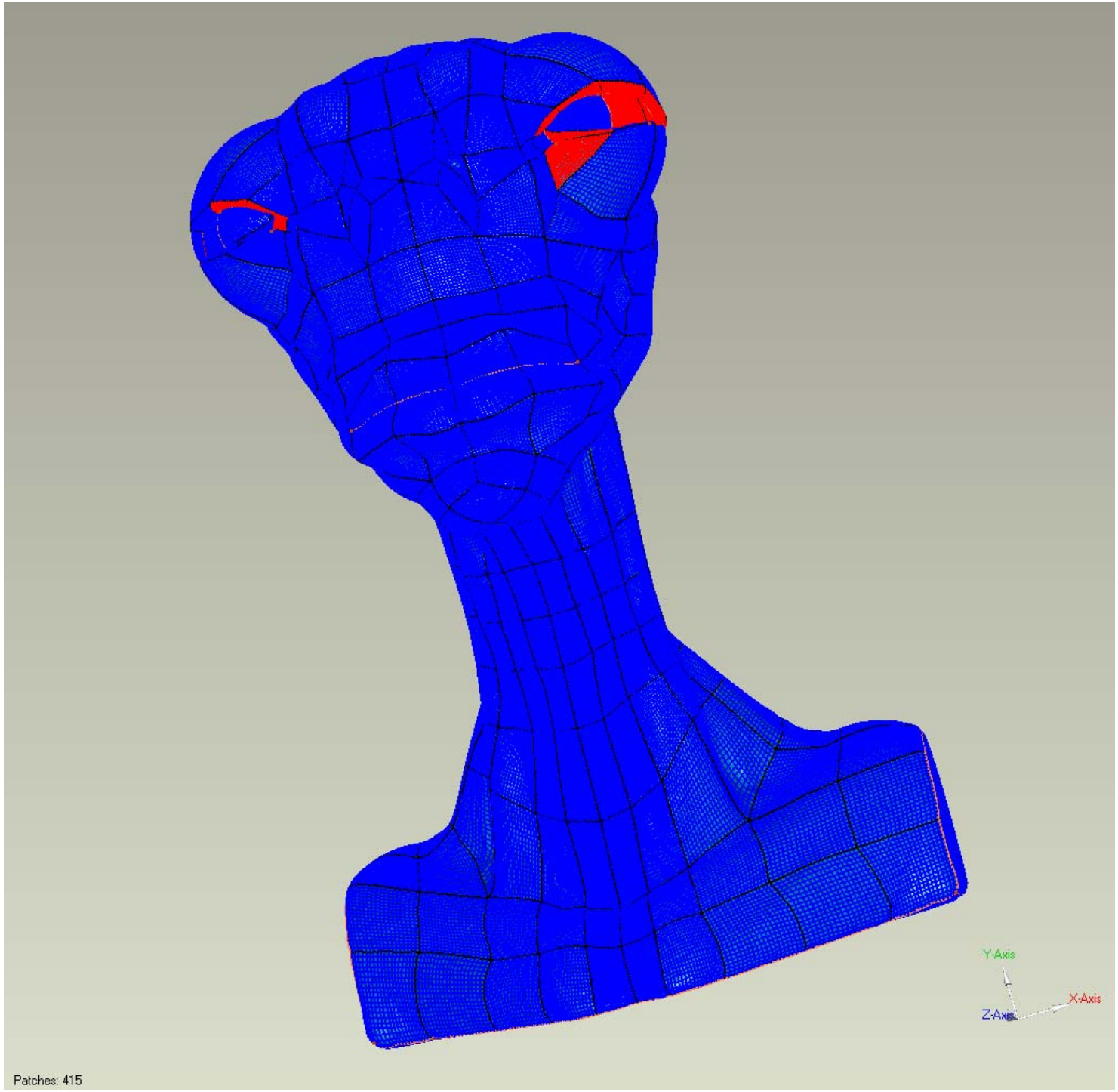

Mesh Doctor
OK
Cancel
Apply
Operation
Type
Action
Analysis
Non-Manifold Edges 0
Self-Intersections 125
Highly Creased Edges 356
Spikes 4599
Small Components 0
Small Holes 0
Update
Walk-through
0/23
Clip Plane
Advanced

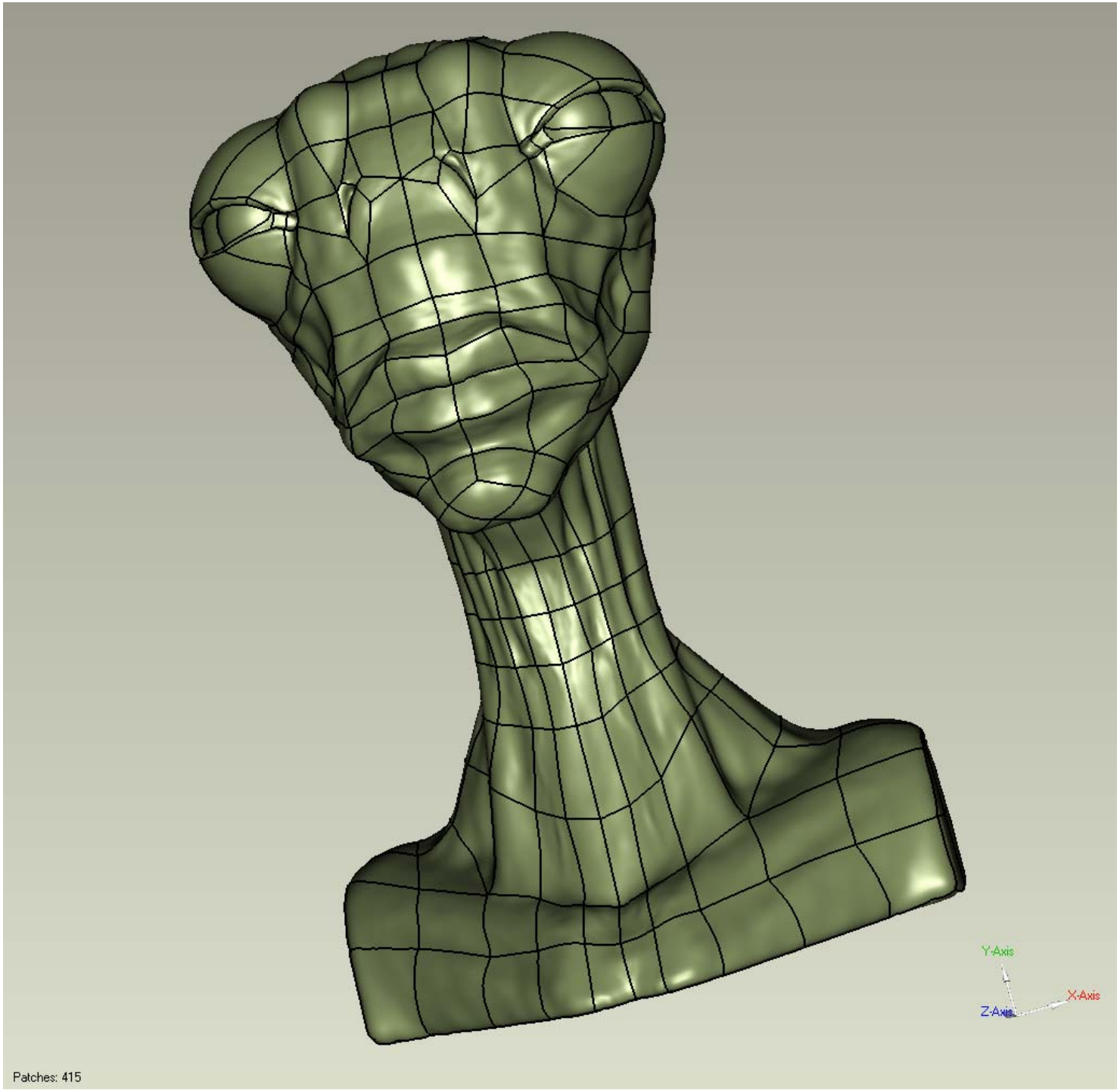

Y-Axis
X-Axis
Z-Axis
Current Triangles: 399,380

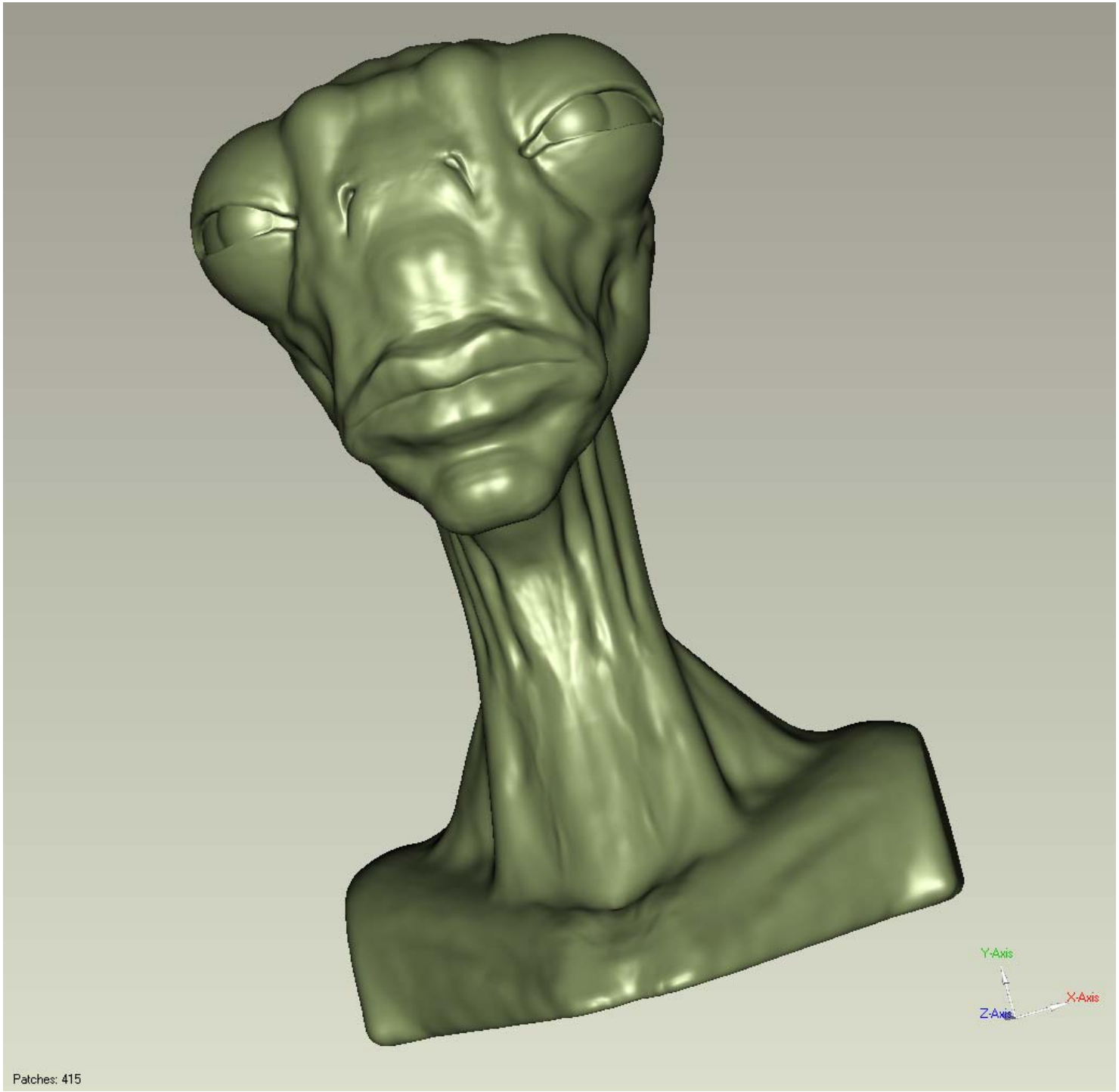
Polygons to NURBs

- Produce a network of quadrilateral patches over the geometry
- Sample the underlying geometry on a regular parameterized grid over each patch
- Construct a NURBs surface on each patch that is G2 continuous with its neighbors


Y-Axis
X-Axis
Z-Axis
Patches: 415

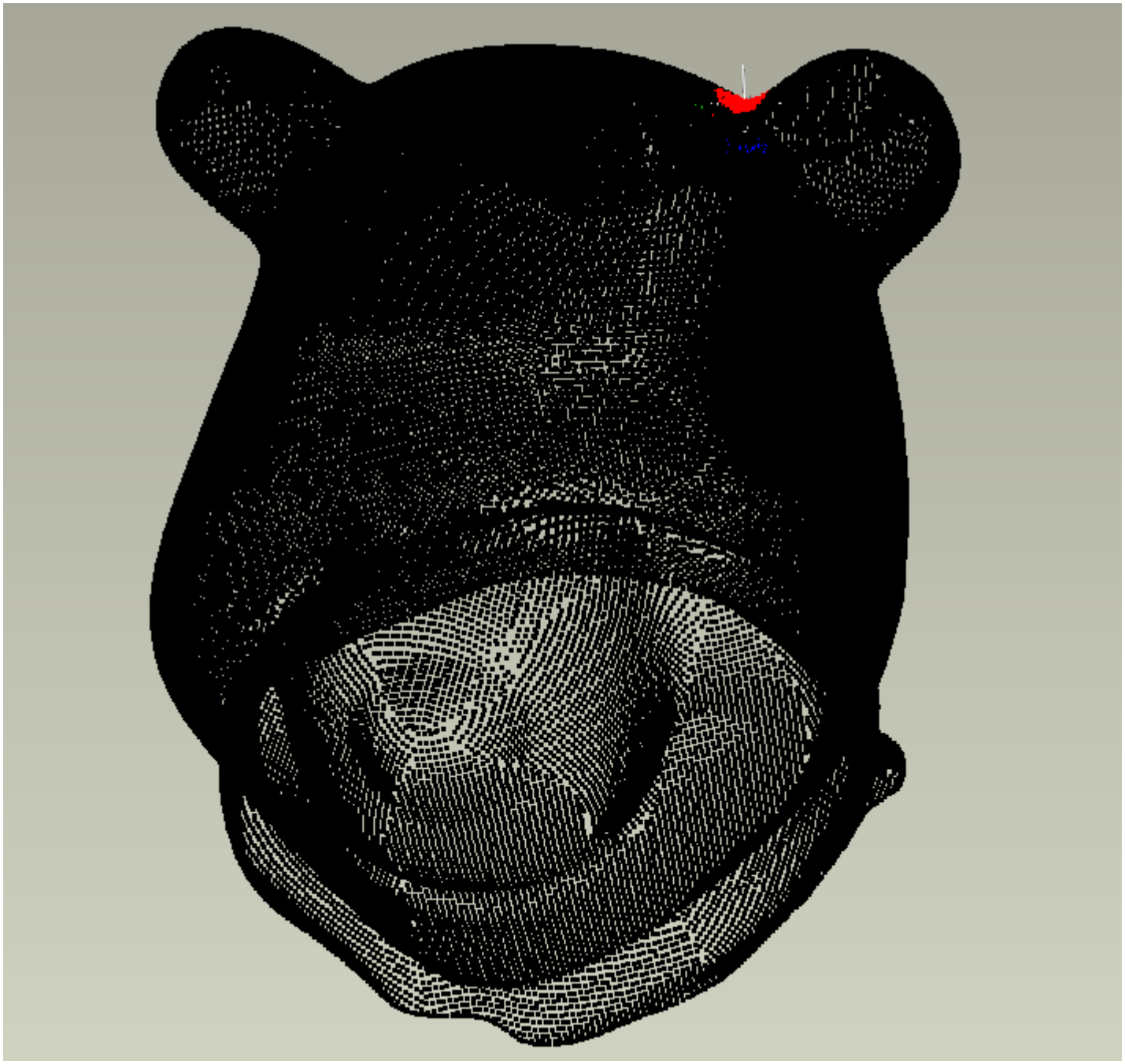

Y-Axis
X-Axis
Z-Axis
Patches: 415

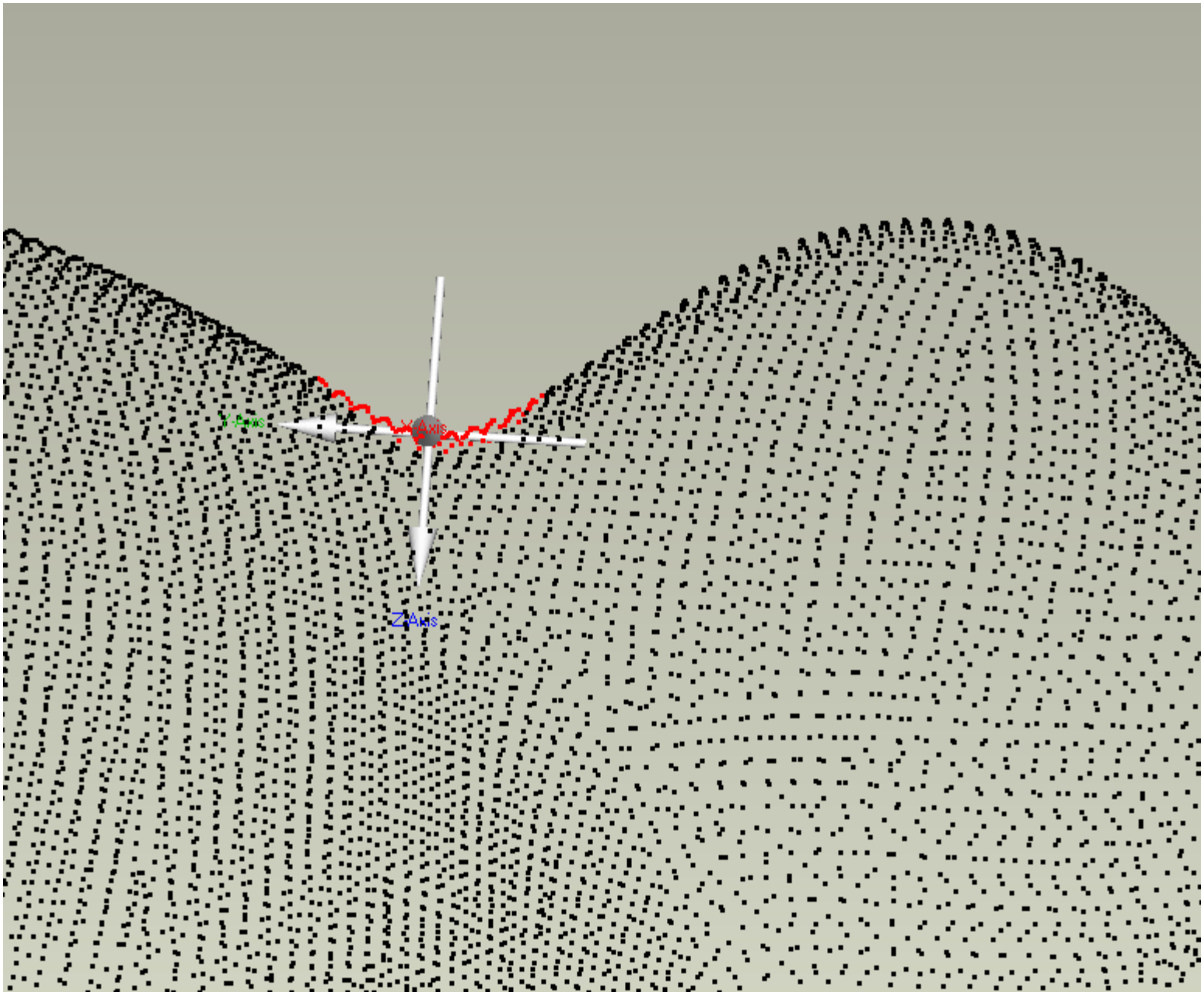

Y-Axis
X-Axis
Z-Axis
Patches: 415

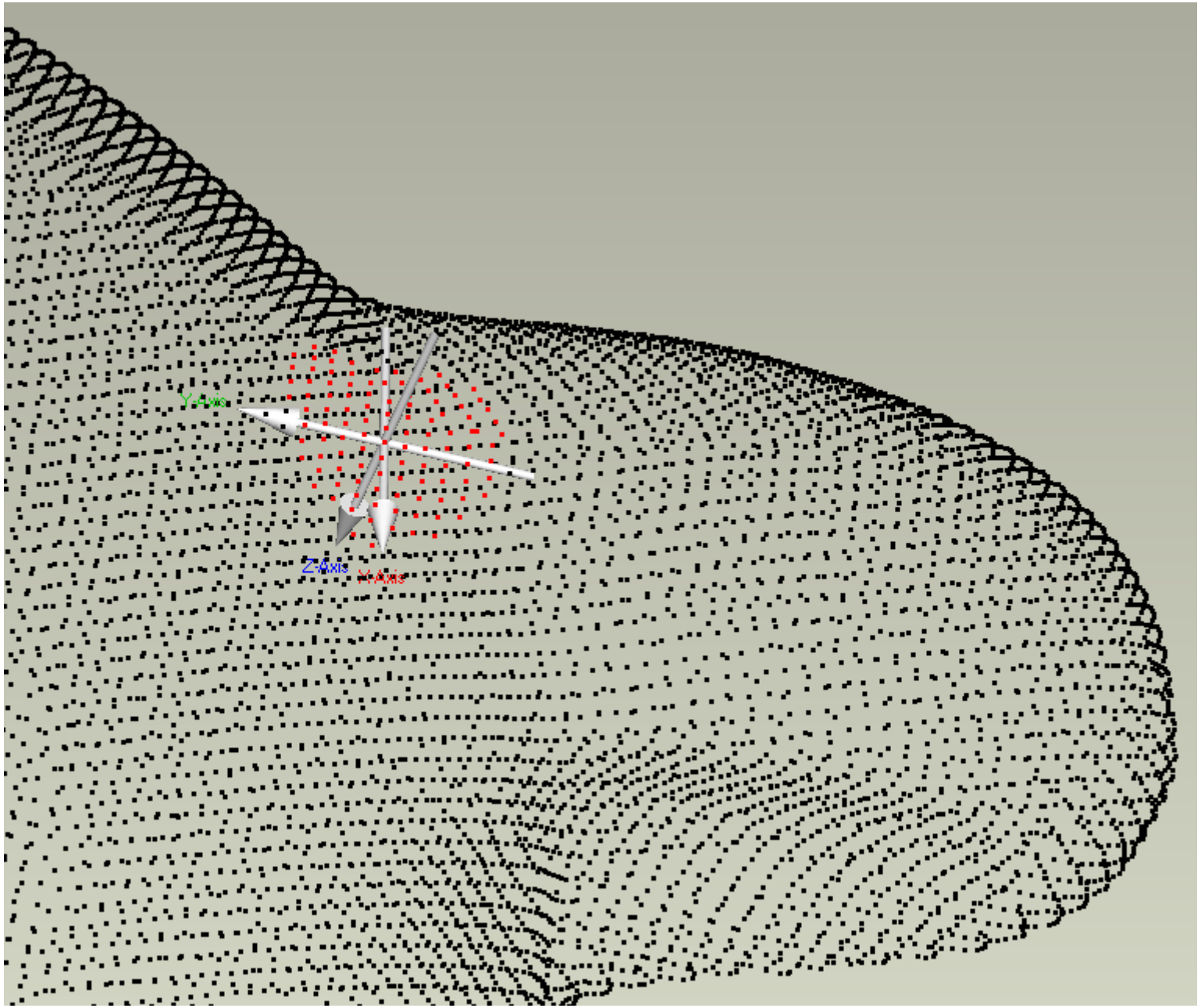

Y-Axis
X-Axis
Z-Axis
Patches: 415

Point Processing

- Local neighborhood – all points within *r* of given point
- Fast computation if data organized as an octree or k-d tree
- Local reference surface – quadratic (2nd order) surface to neighborhood points
- Local reference frame – principal axes of reference surface => normal, principal curvatures

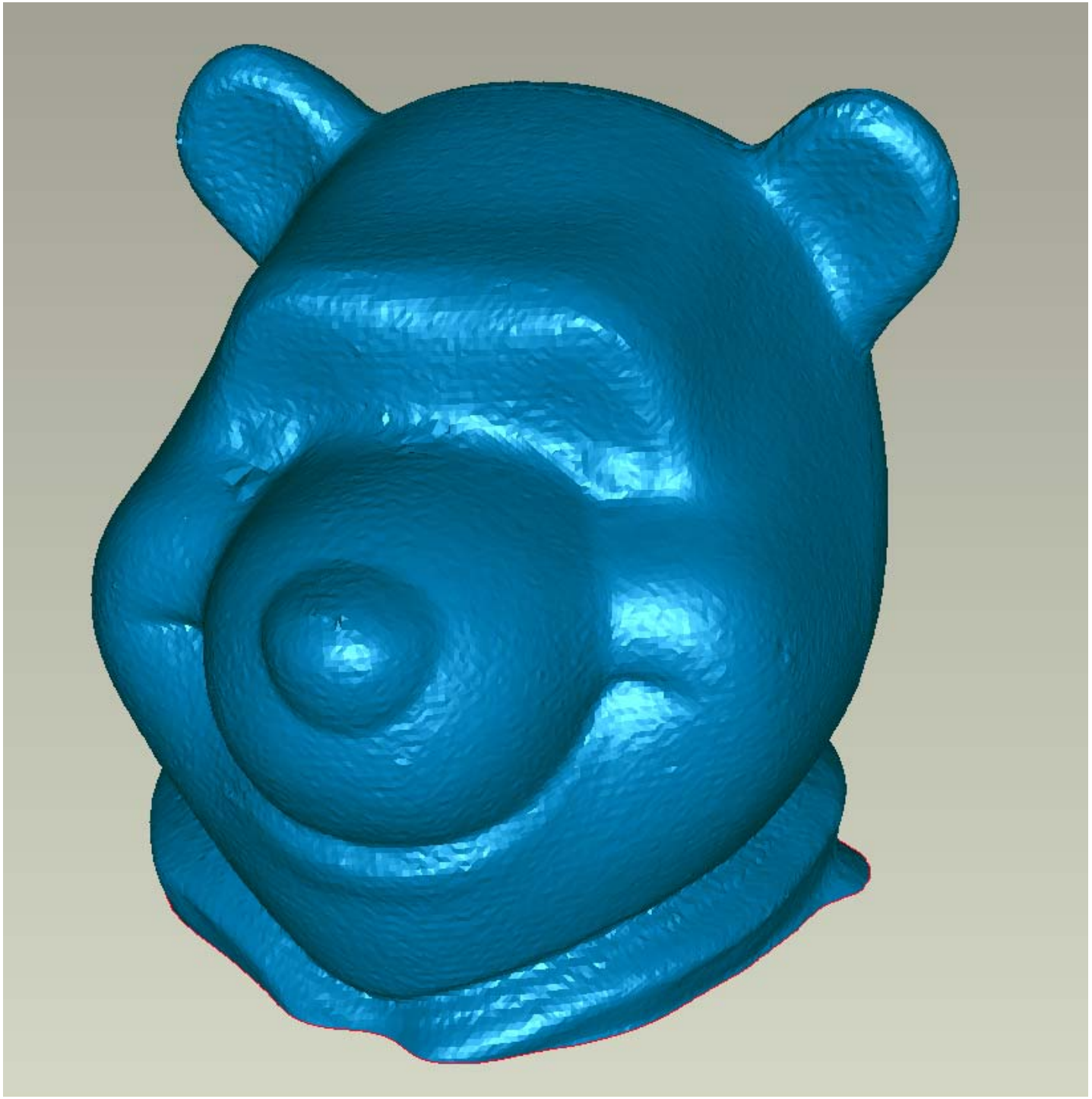


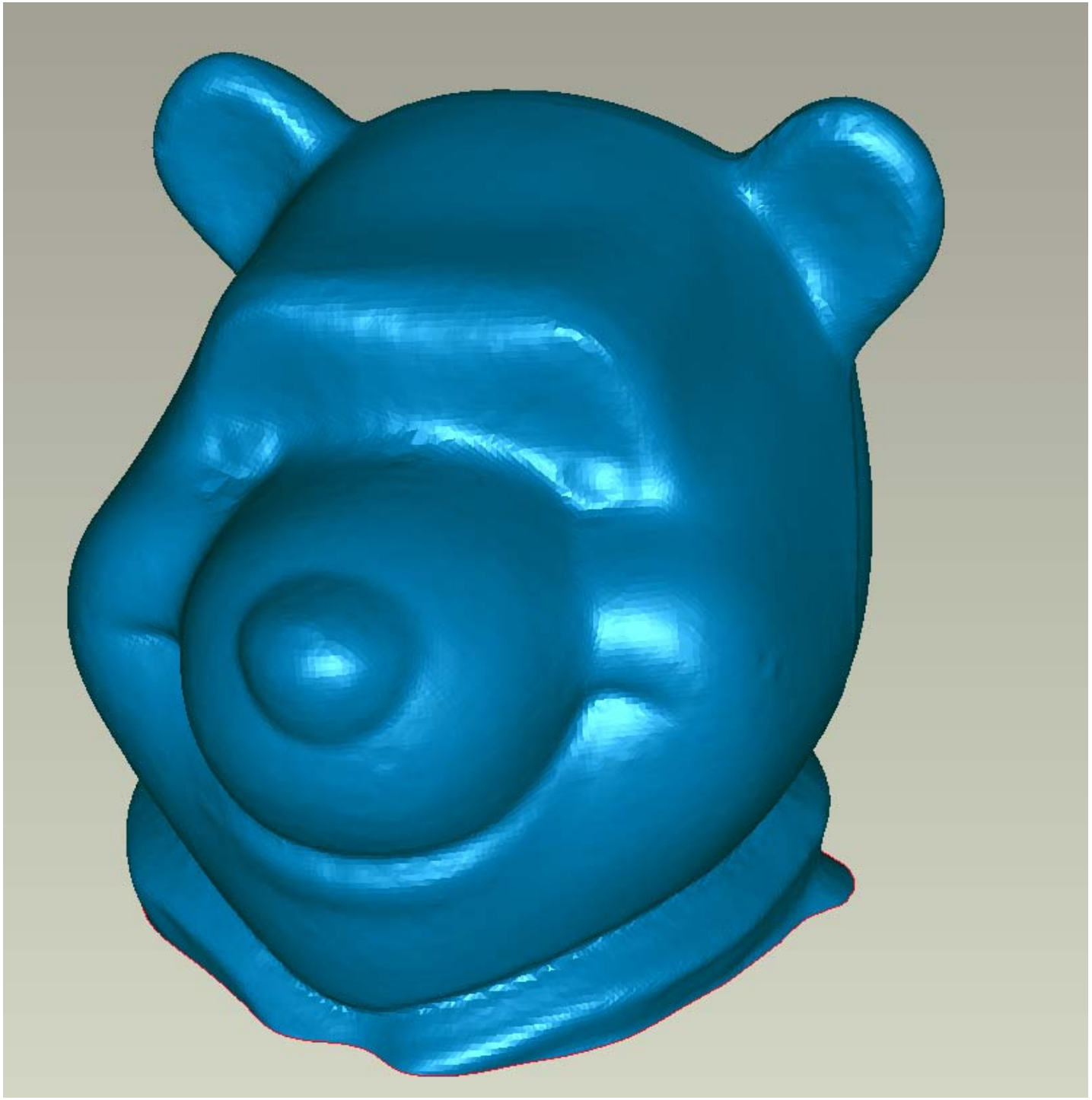

Y Axis
X Axis
Z Axis


Y-Axis
Z-Axis
X-Axis

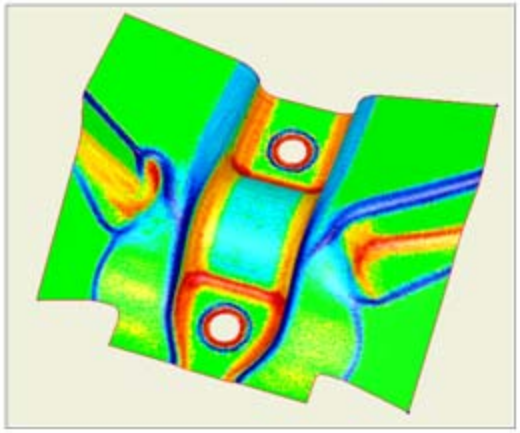
Noise Reduction

- Move each point to its local reference surface



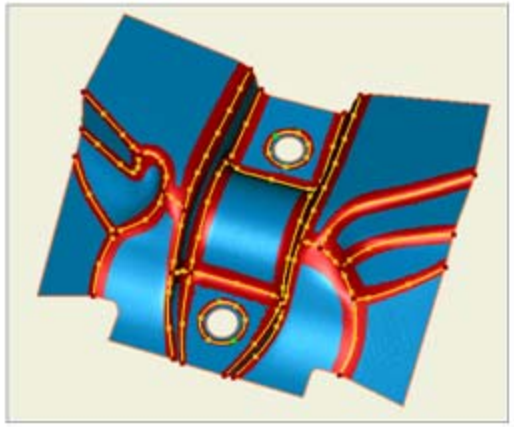


Polygons to CAD

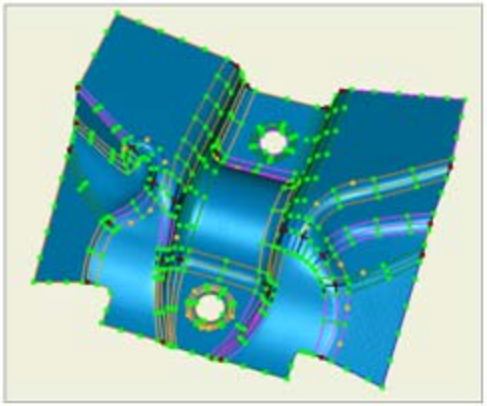


a. Numerical Curvature Map

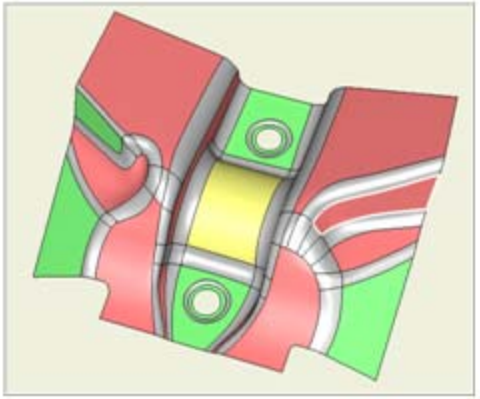
b. Regions and Separator Sets



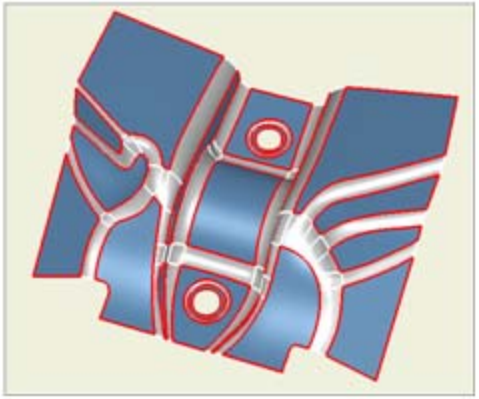
c. Contour extraction



d. Extending Contours



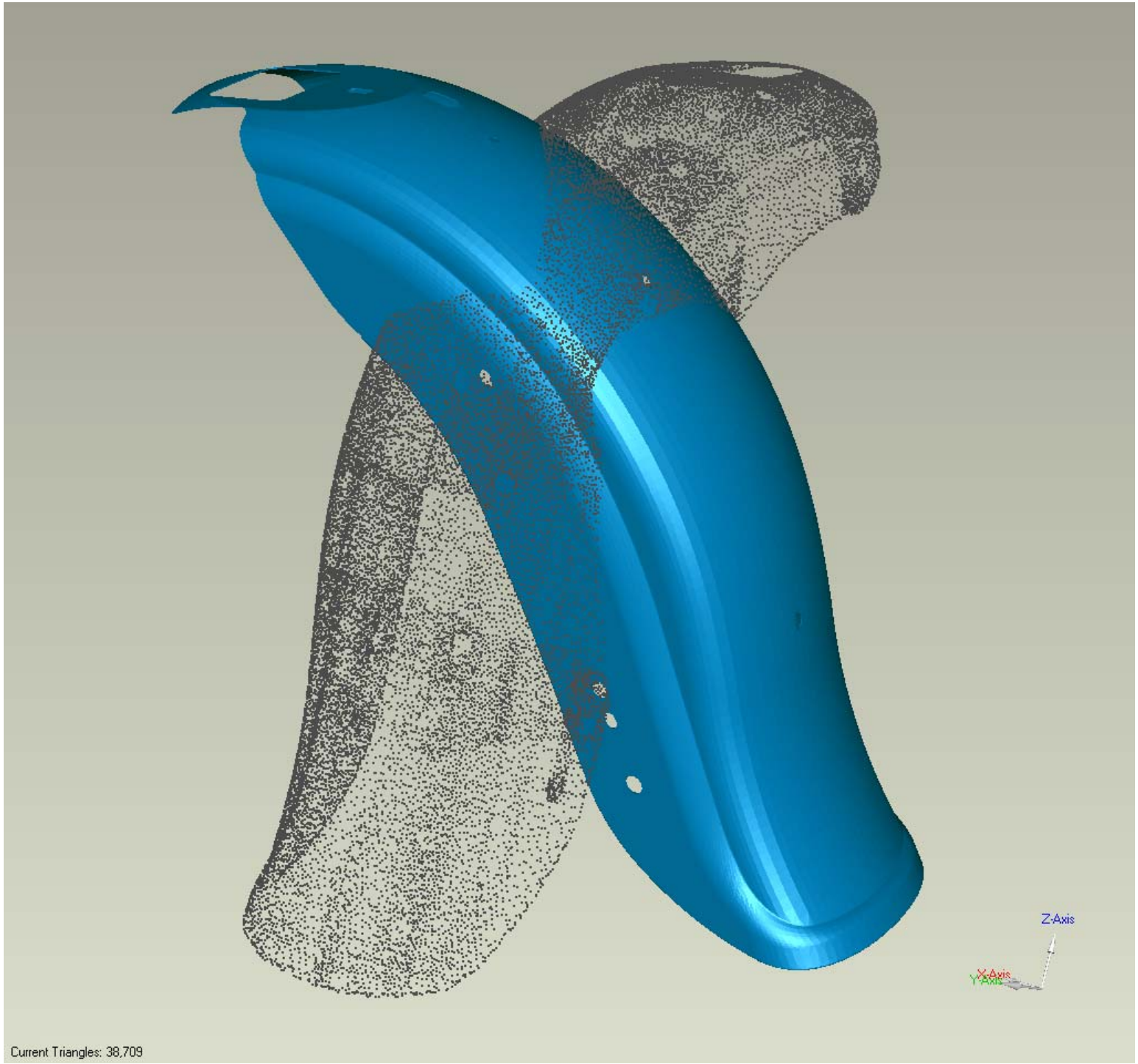
e. Classification

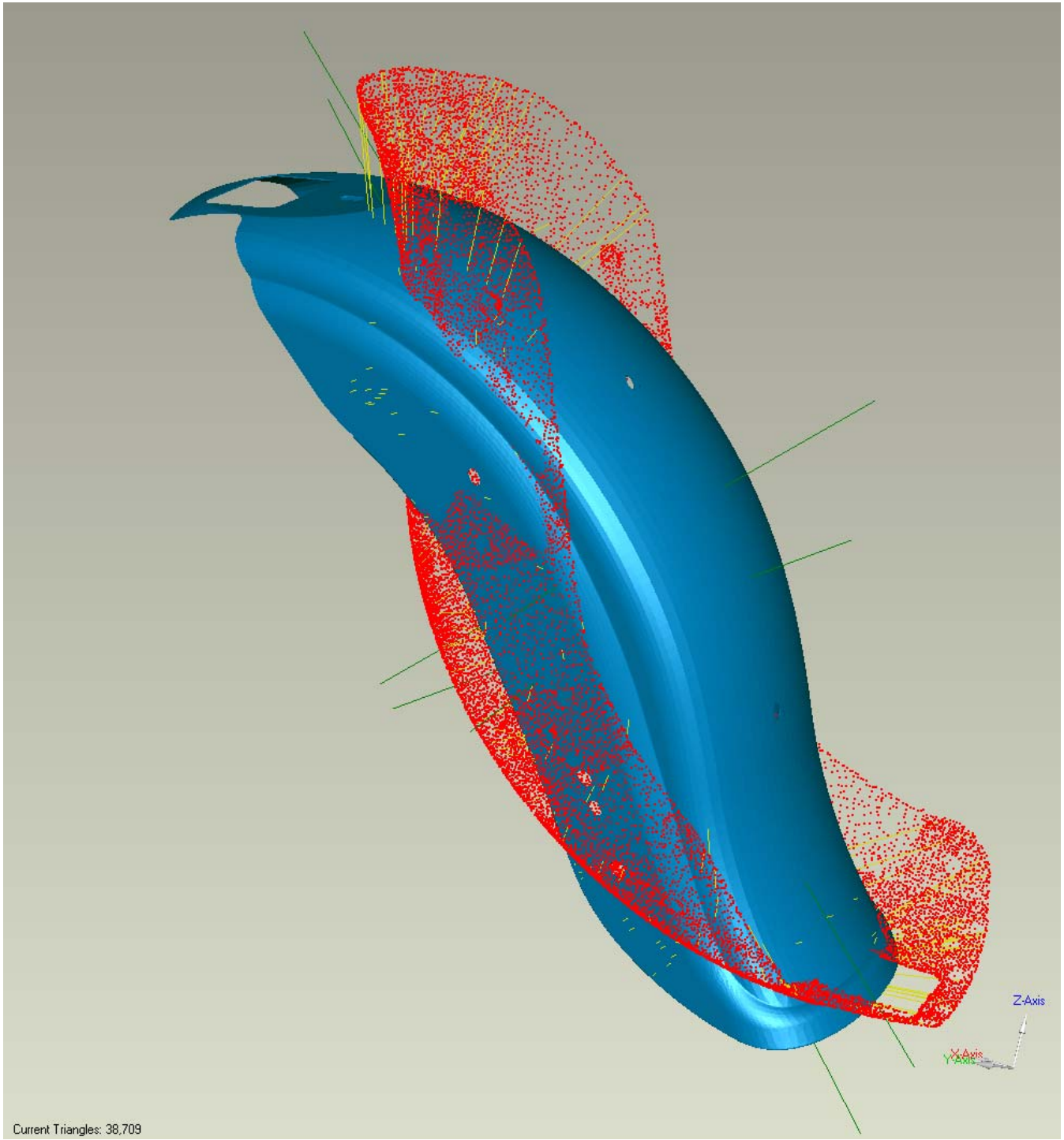


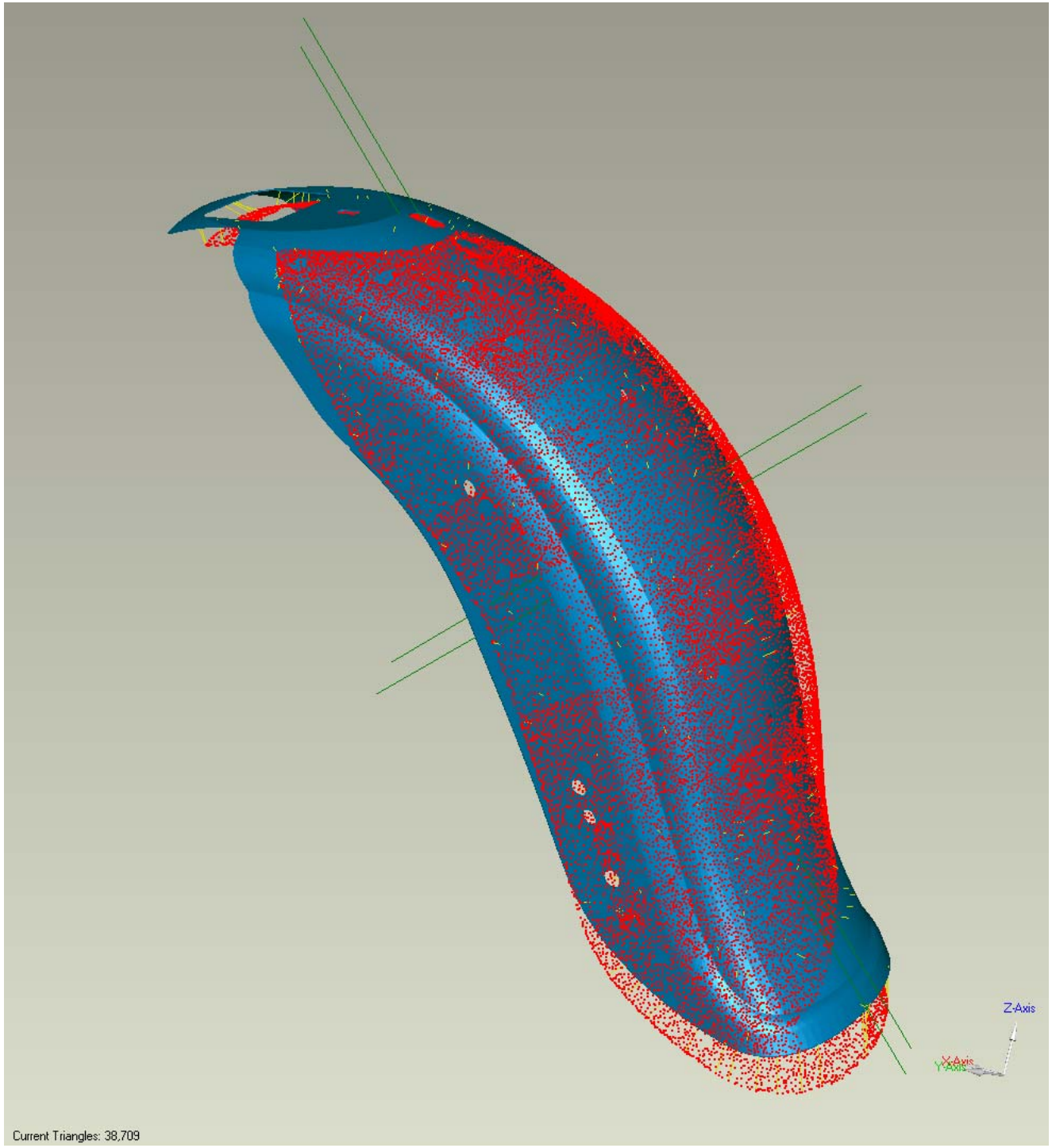
f. Trimmed Primaries

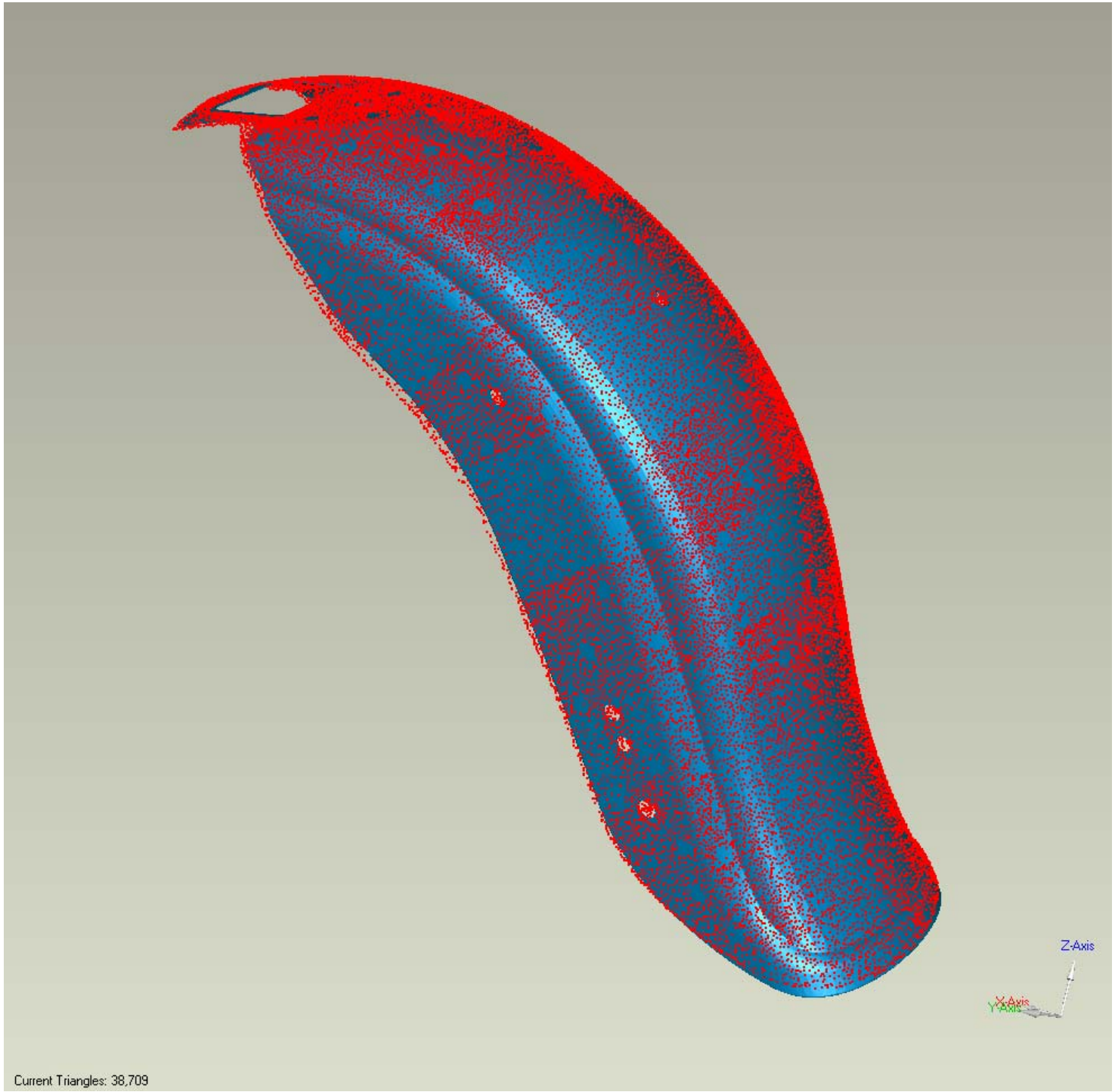
Alignment of congruent shapes

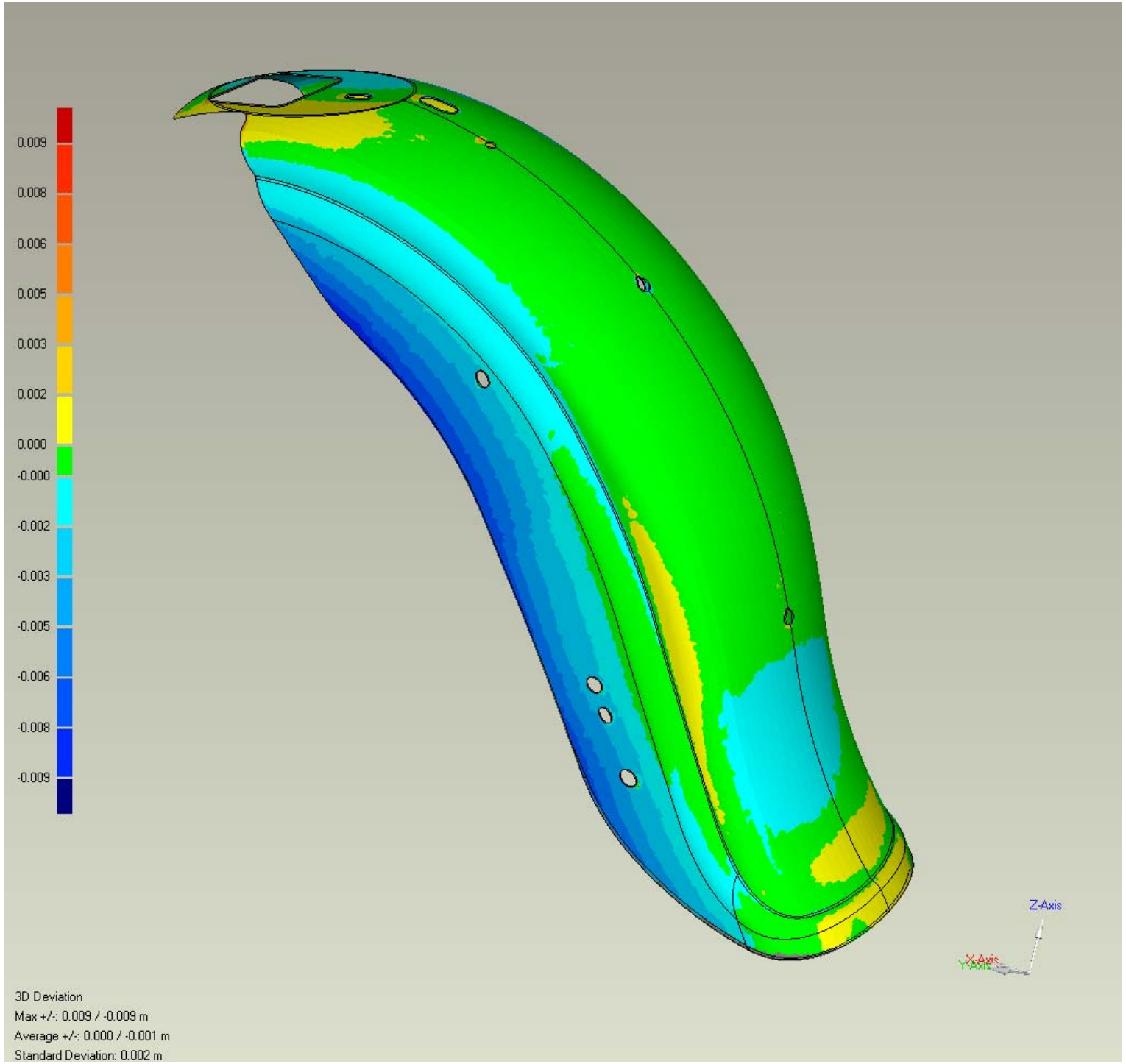
- Choose set of discrete samples S on first object
- Iterated Corresponding Point (ICP)
  - Find a correspondent c(x) (usually closest point) on second object for each sample x in S
  - Compute rigid motion M that minimizes the sum over all x in S of the square distance from M(c(x)) to x
  - Apply motion M to second object
  - Repeat until M is negligible


Z-Axis
X-Axis
Y-Axis
Current Triangles: 38,709


Z-Axis
X-Axis
Y-Axis
Current Triangles: 38,709

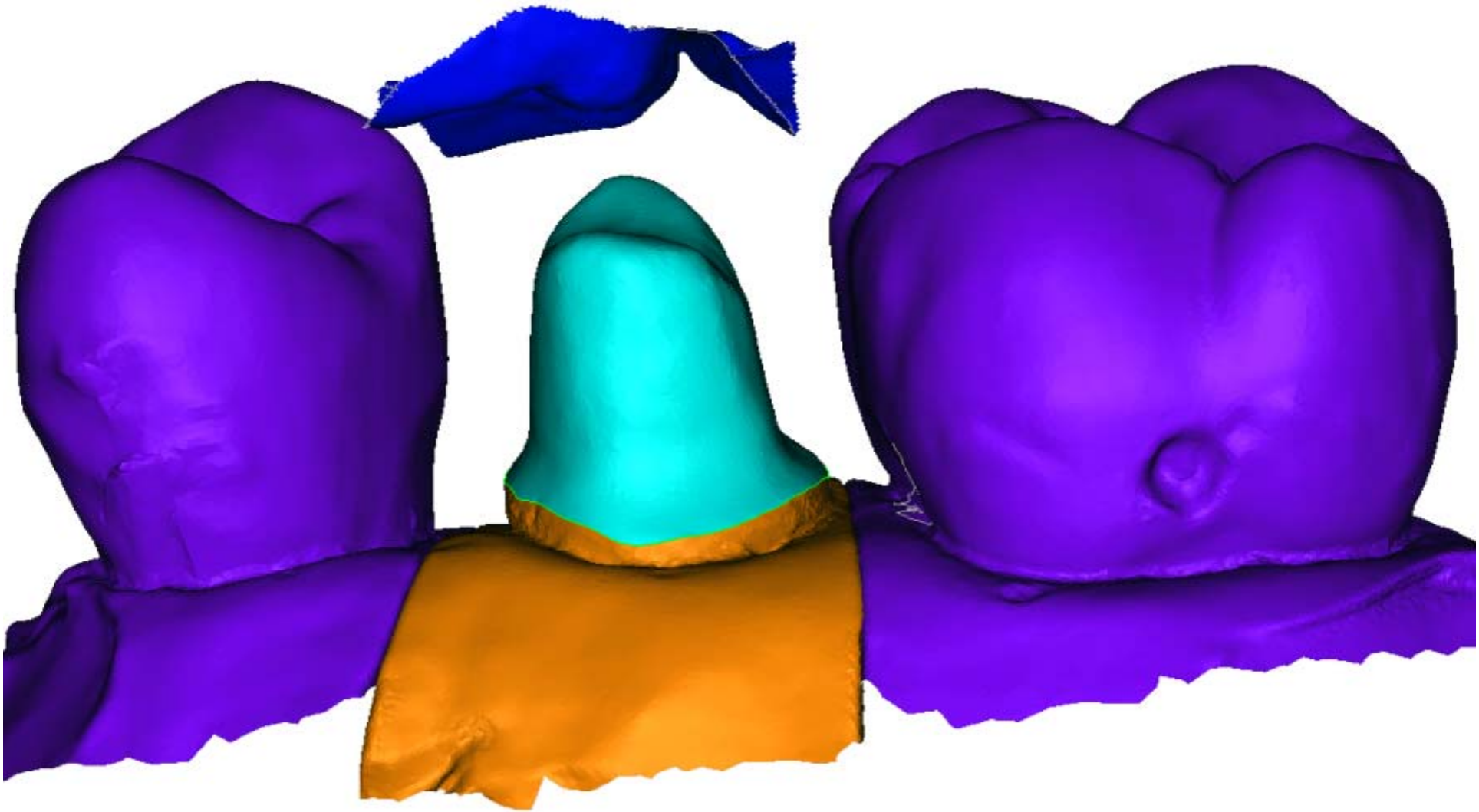

Z-Axis
X-Axis
Y-Axis
Current Triangles: 38,709

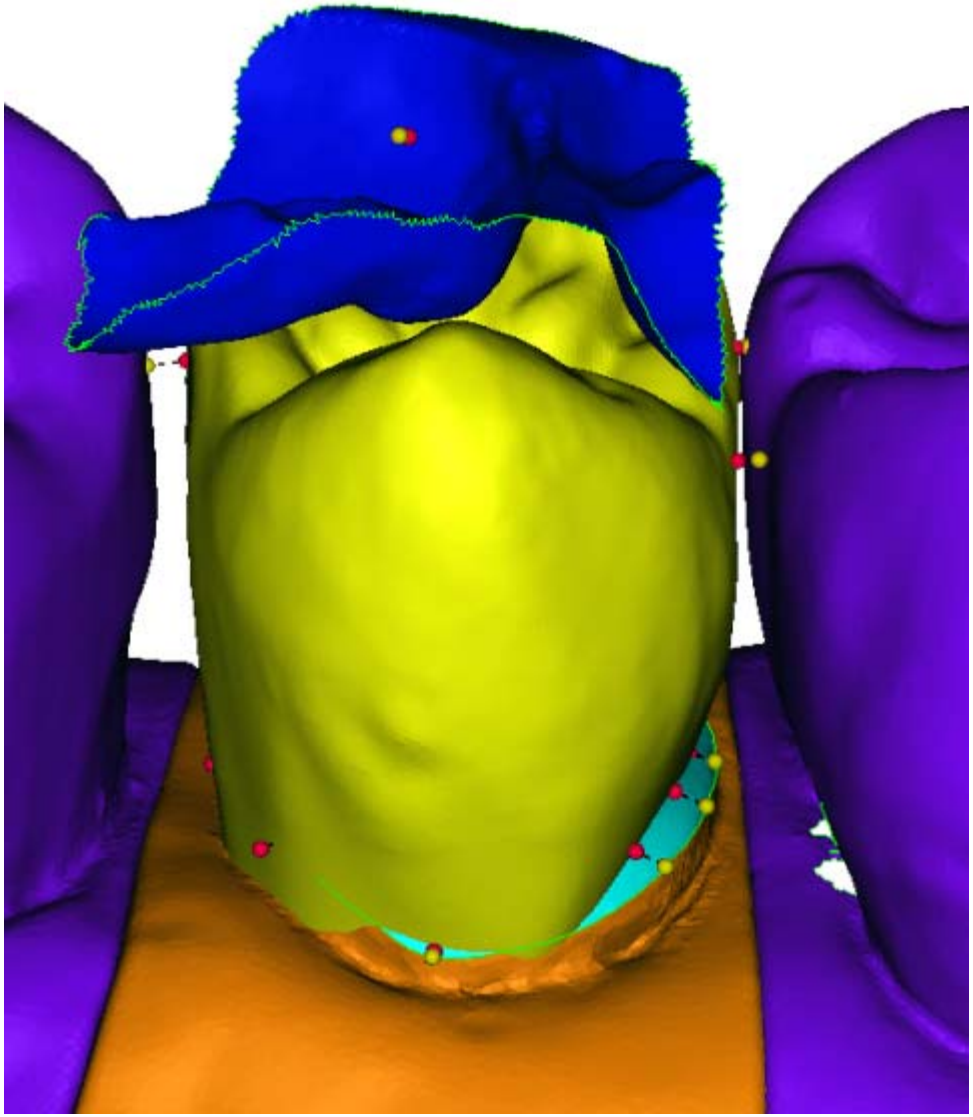

Z-Axis
X-Axis
Y-Axis
Current Triangles: 38,709


0.009
0.008
0.006
0.005
0.003
0.002
0.000
-0.000
-0.002
-0.003
-0.005
-0.006
-0.008
-0.009
Z-Axis
X-Axis
Y-Axis
3D Deviation
Max +/-: 0.009 / -0.009 m
Average +/-: 0.000 / -0.001 m
Standard Deviation: 0.002 m

Shape Warping with Tie Points

- Given a discrete set of samples P on a surface S and a corresponding set of (arbitrary) targets T, find a (smooth) mapping that moves each sample in P to its corresponding target in T in such a way that the distortion of S is minimized.

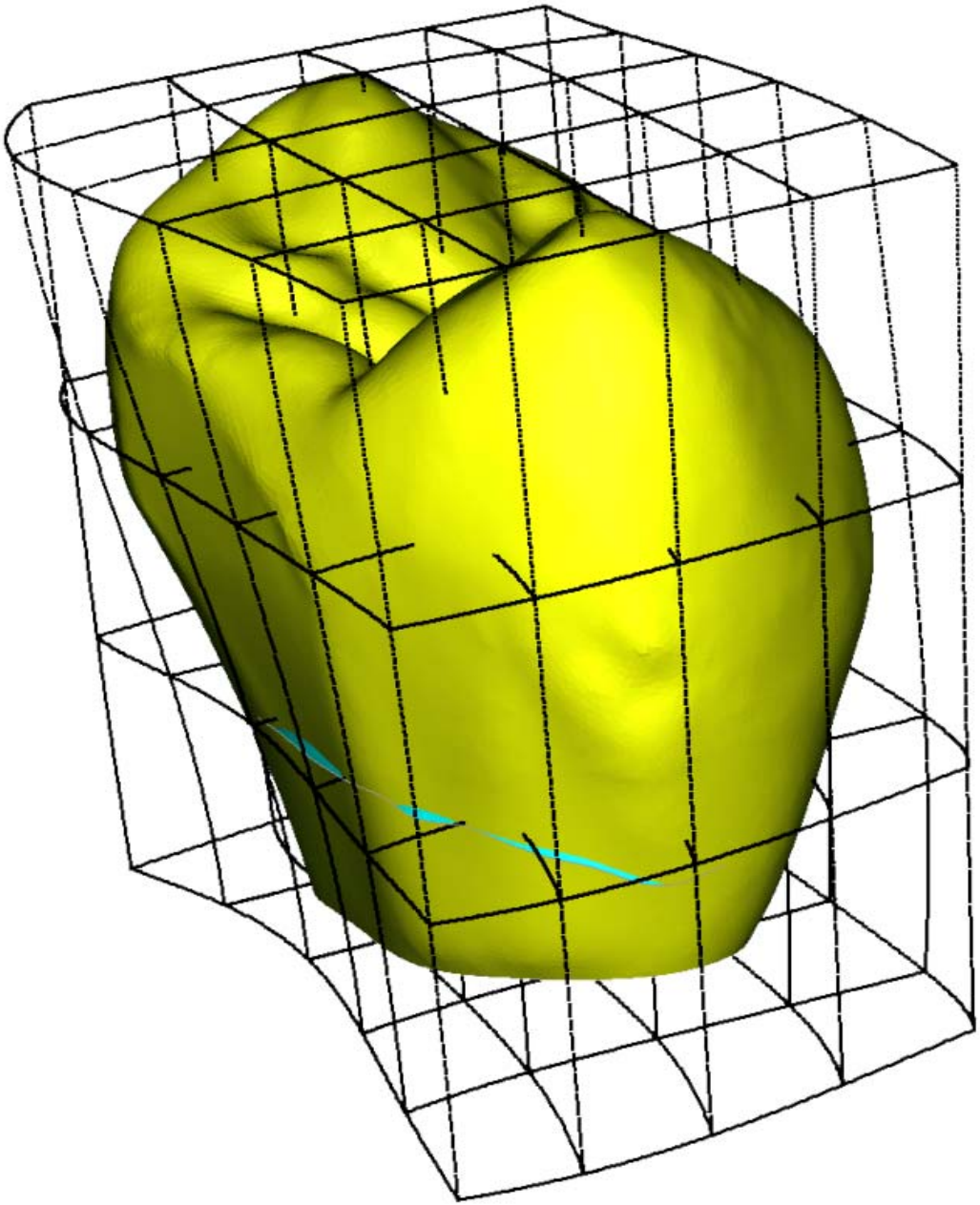




- For any regular rectilinear lattice *P*, B-spline theory provides a set of basis functions which allow the warp

$$W_P(\mathbf{x}) \quad = \quad \mathbf{x} + \sum_{\mathbf{p} \in P} w_{\mathbf{p}}(\mathbf{x}) \cdot \Delta \mathbf{p}$$

  to be written in terms of *perturbations* of the lattice points
- We can then solve for the lattice perturbation that minimizes the sum of square deviations of the warped tie-points from their targets.
- If this linear system is under-constrained then standard SVD solvers will produce the “minimum norm” perturbation defining a warp that exactly matches the tie-points.



Exactly constrained warps are usually bad

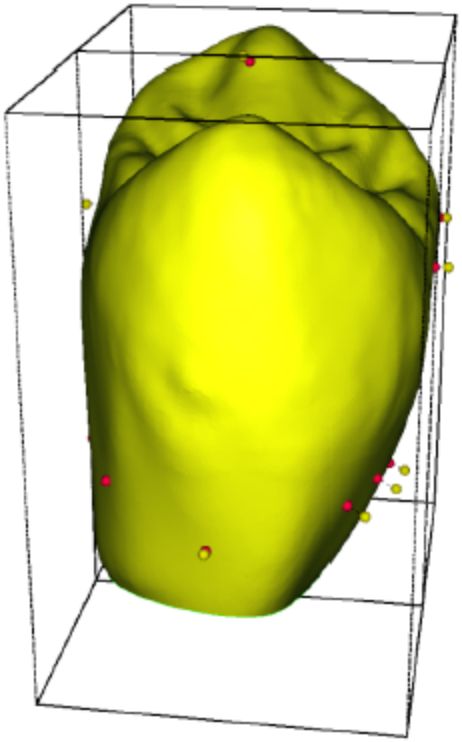


Figure 14: Control lattice whose size exactly matches the number of tie-points

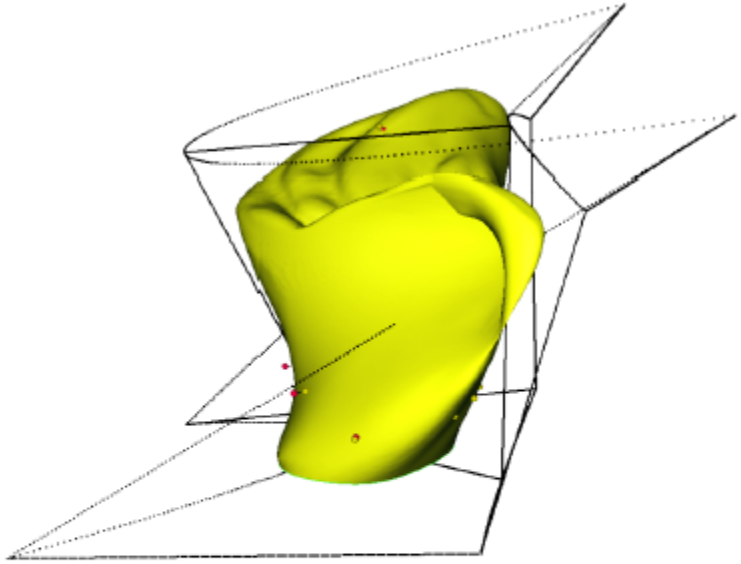
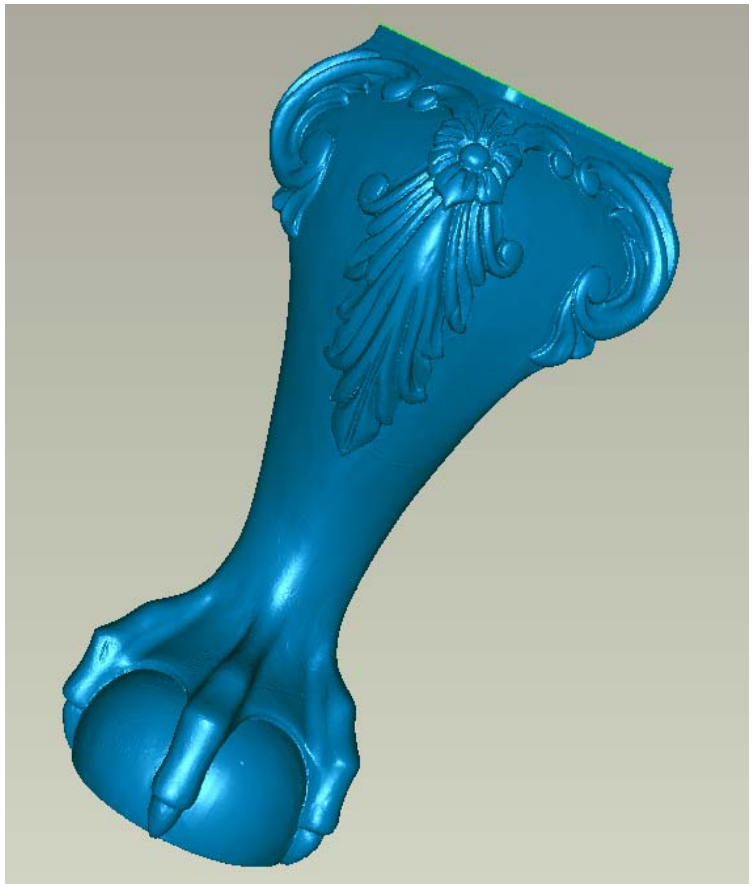


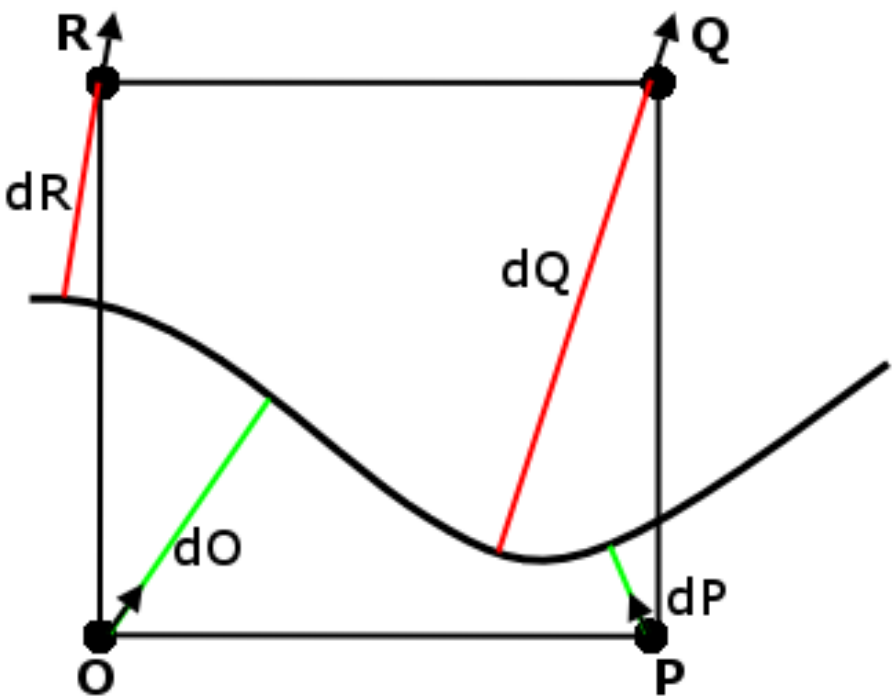
Figure 15: Ill-behaved warp

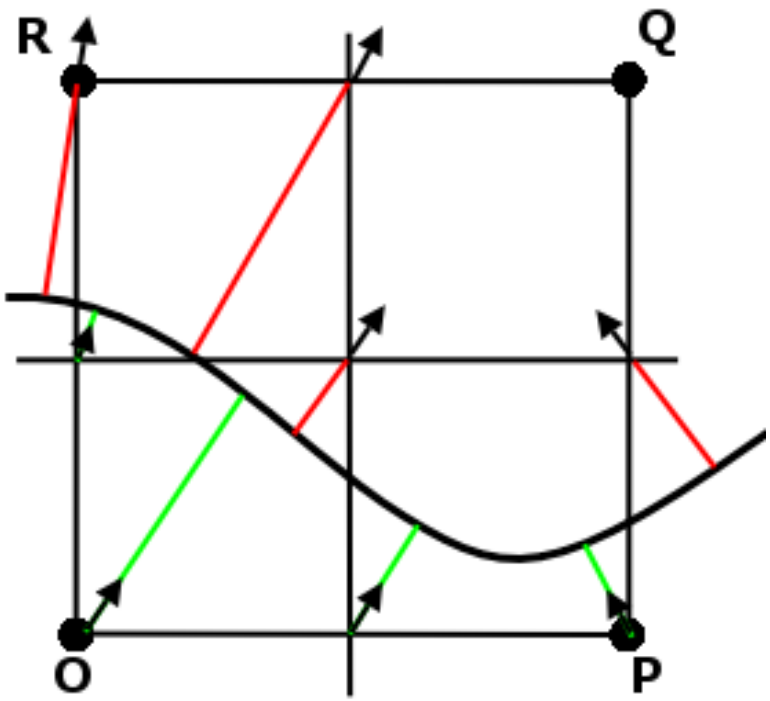
Shape Compression

- There is a difference between the data (polygons) and the shape information it represents.



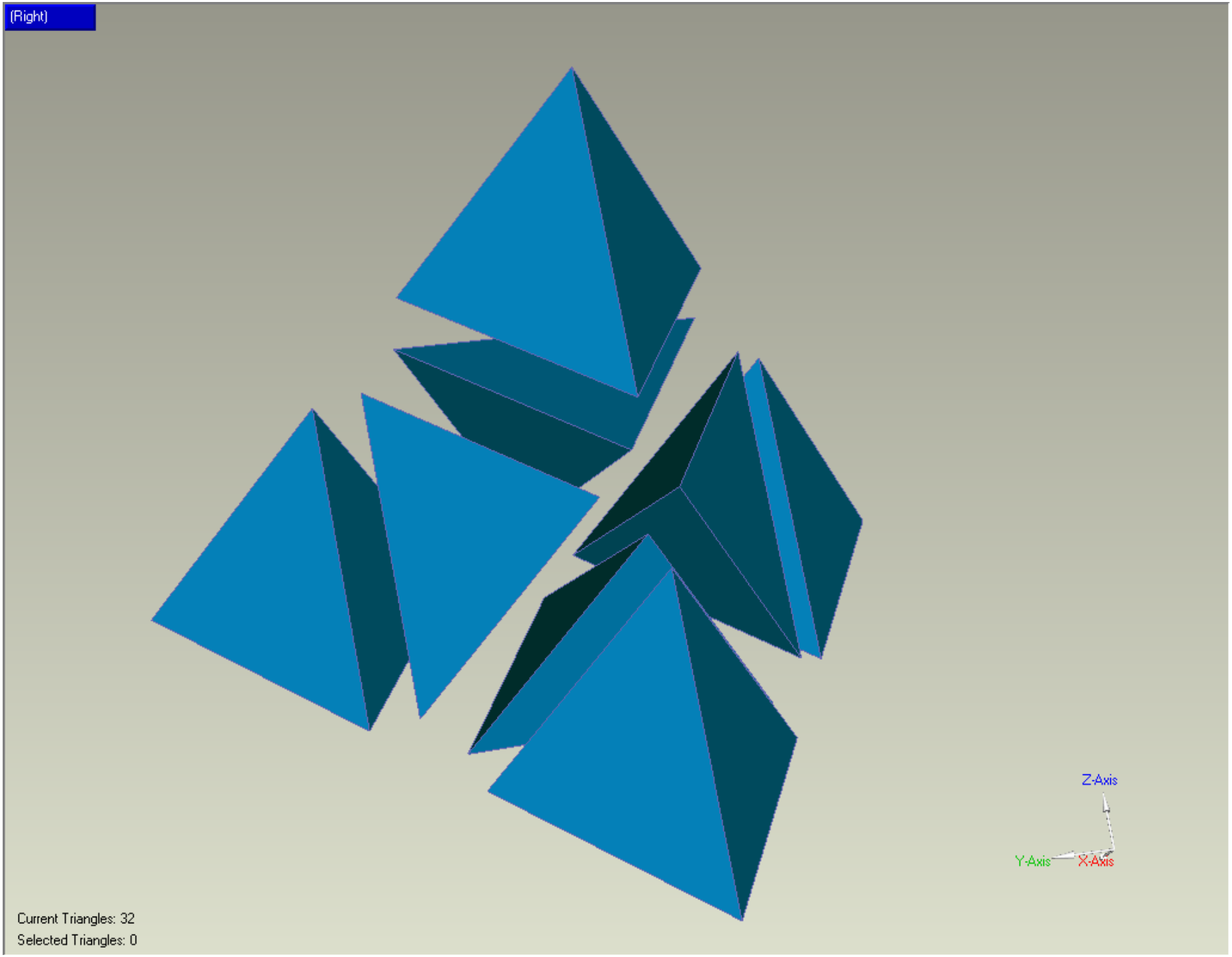
Surface representation by Adaptive Distance Function over a lattice





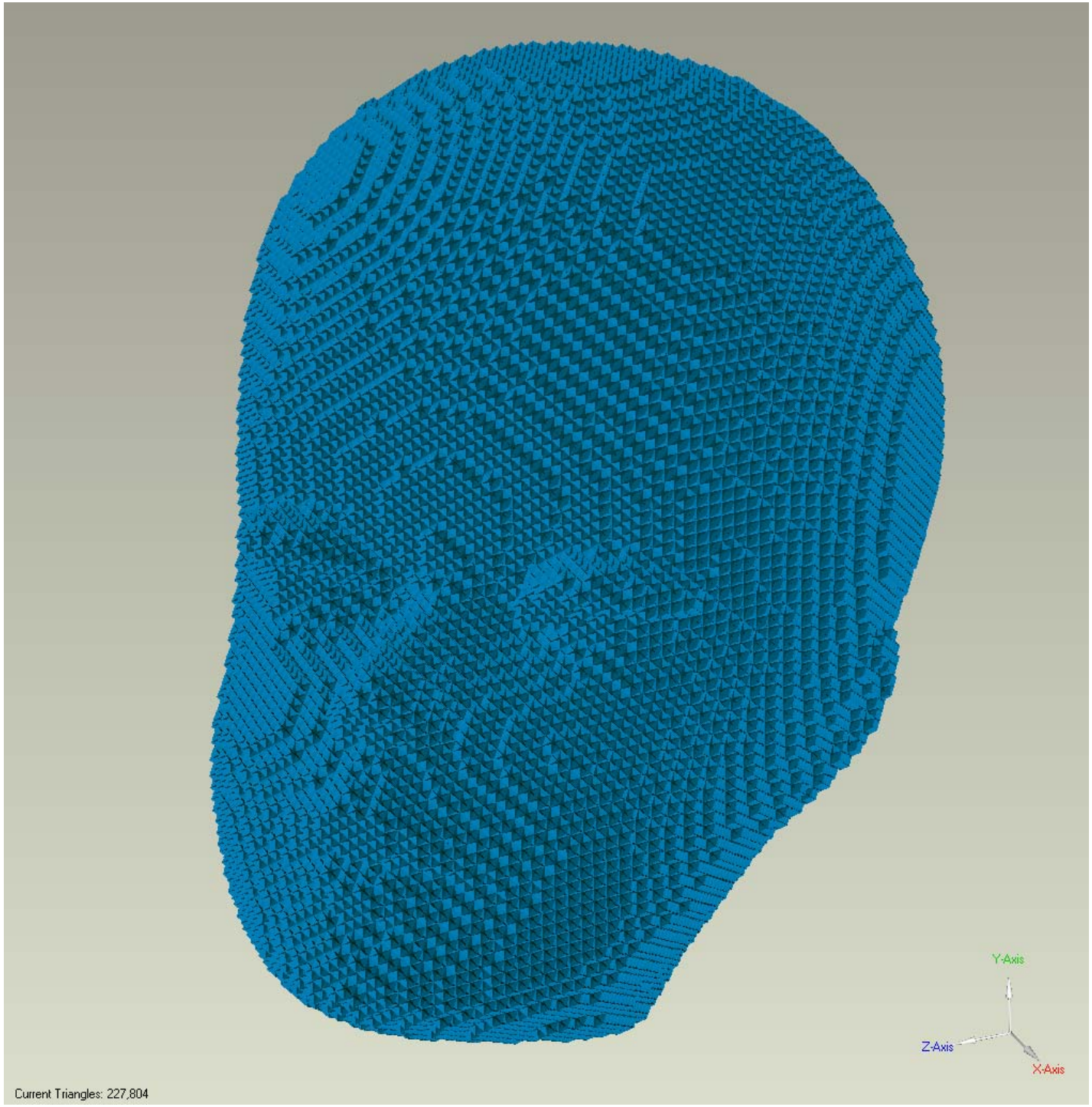
Compression

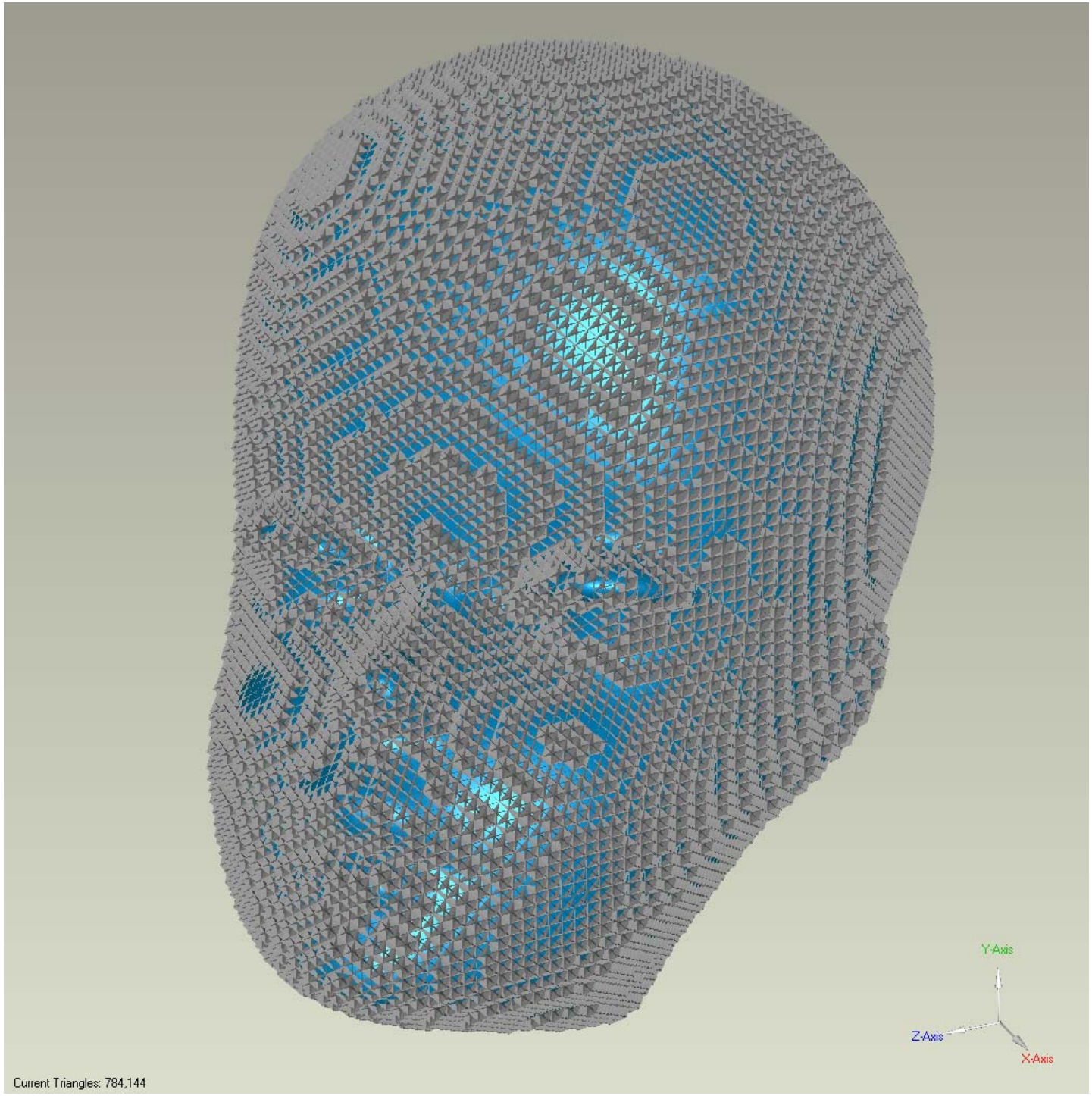
- Determine “shell” of lattice cells (at suitably fine level) containing surface
- Compute distance values for all shell vertices
- Assume that a consistent surface representation (reconstruction) on each cell is computable from the distance values at its vertices.
- Consolidate cells
- Compressed representation is a list of indices for the vertices of the consolidated shell together with their distance values

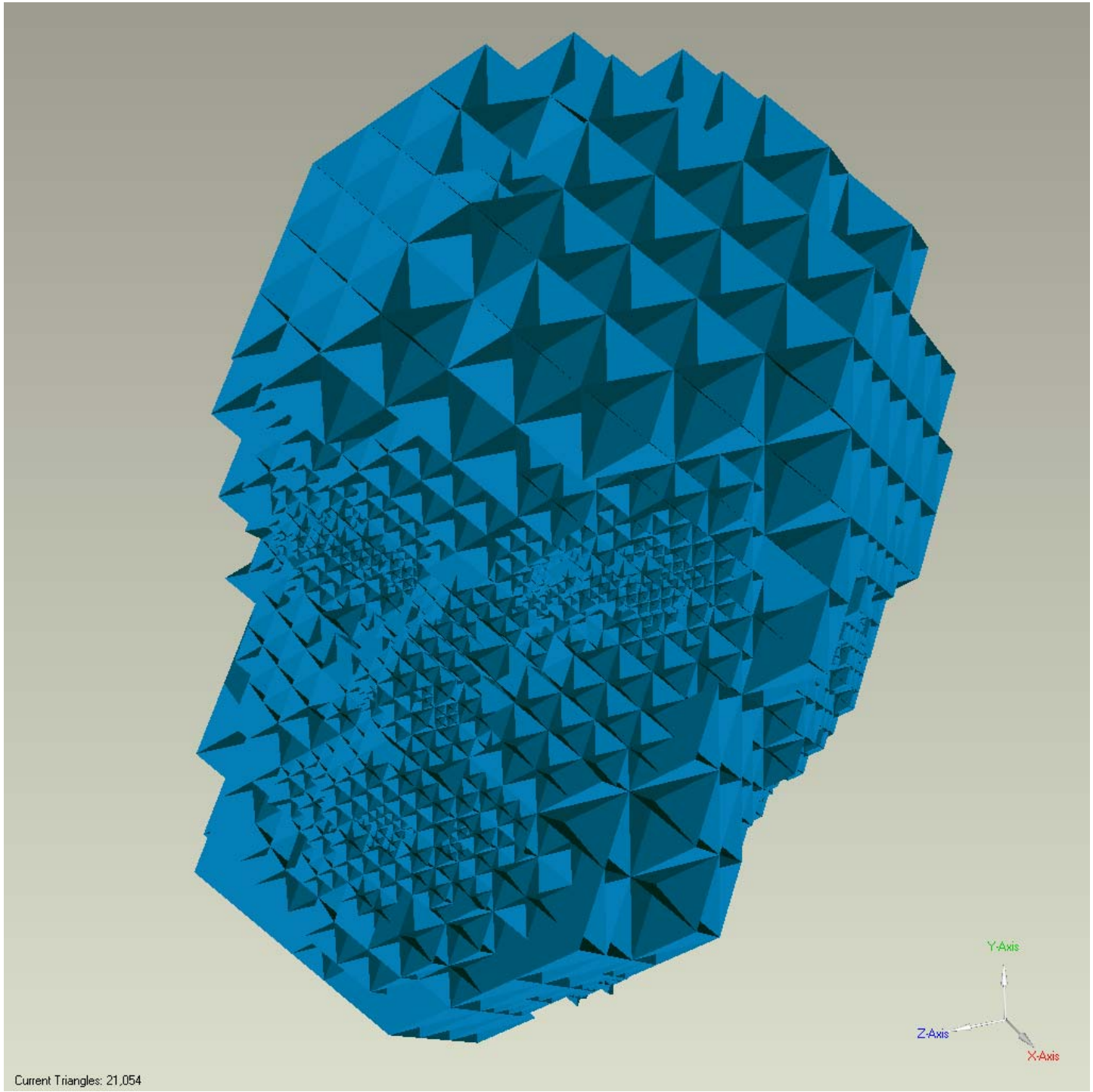

(Right)
Z-Axis
Y-Axis
X-Axis
Current Triangles: 32
Selected Triangles: 0

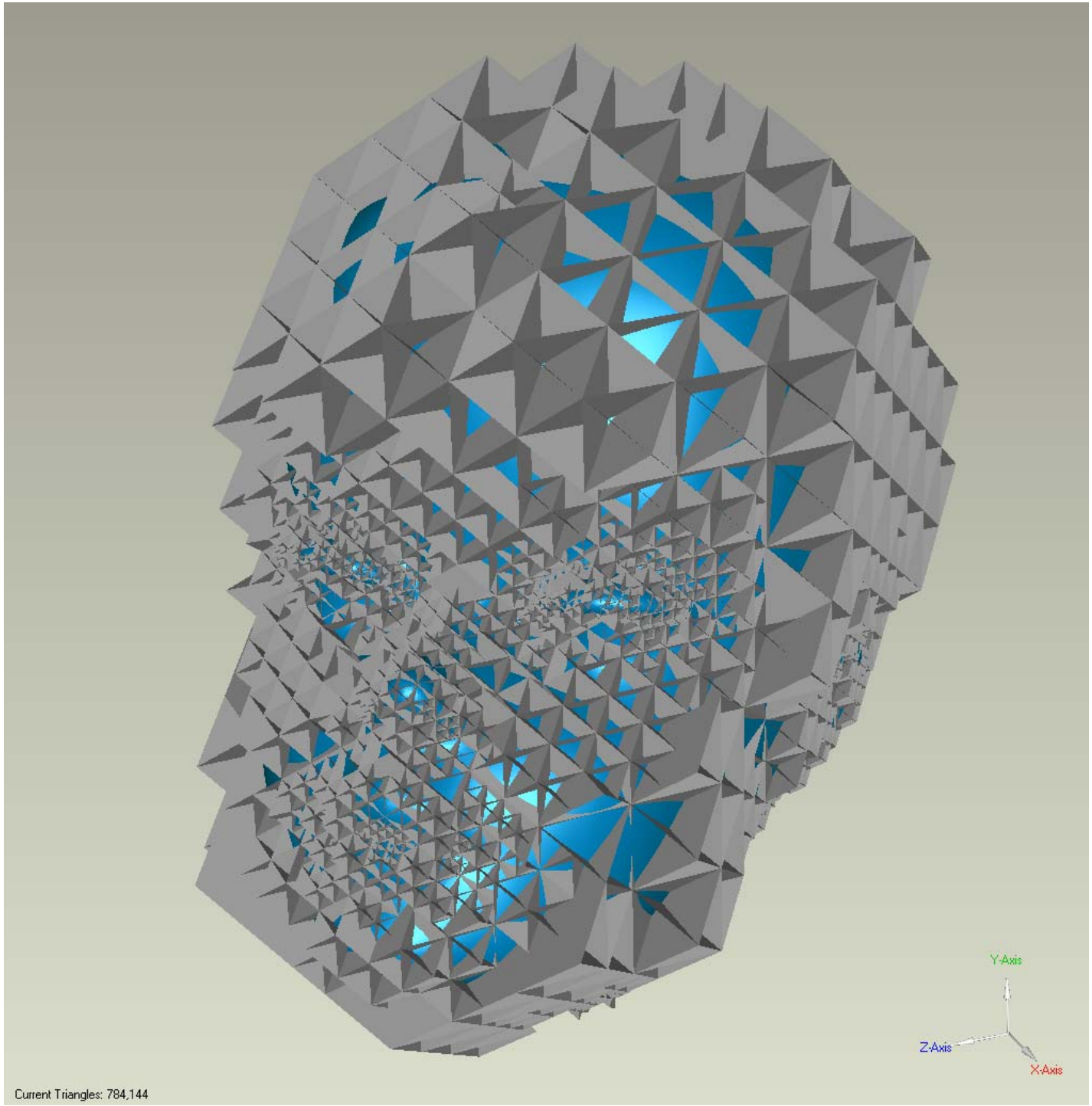
BCC Cell Representation

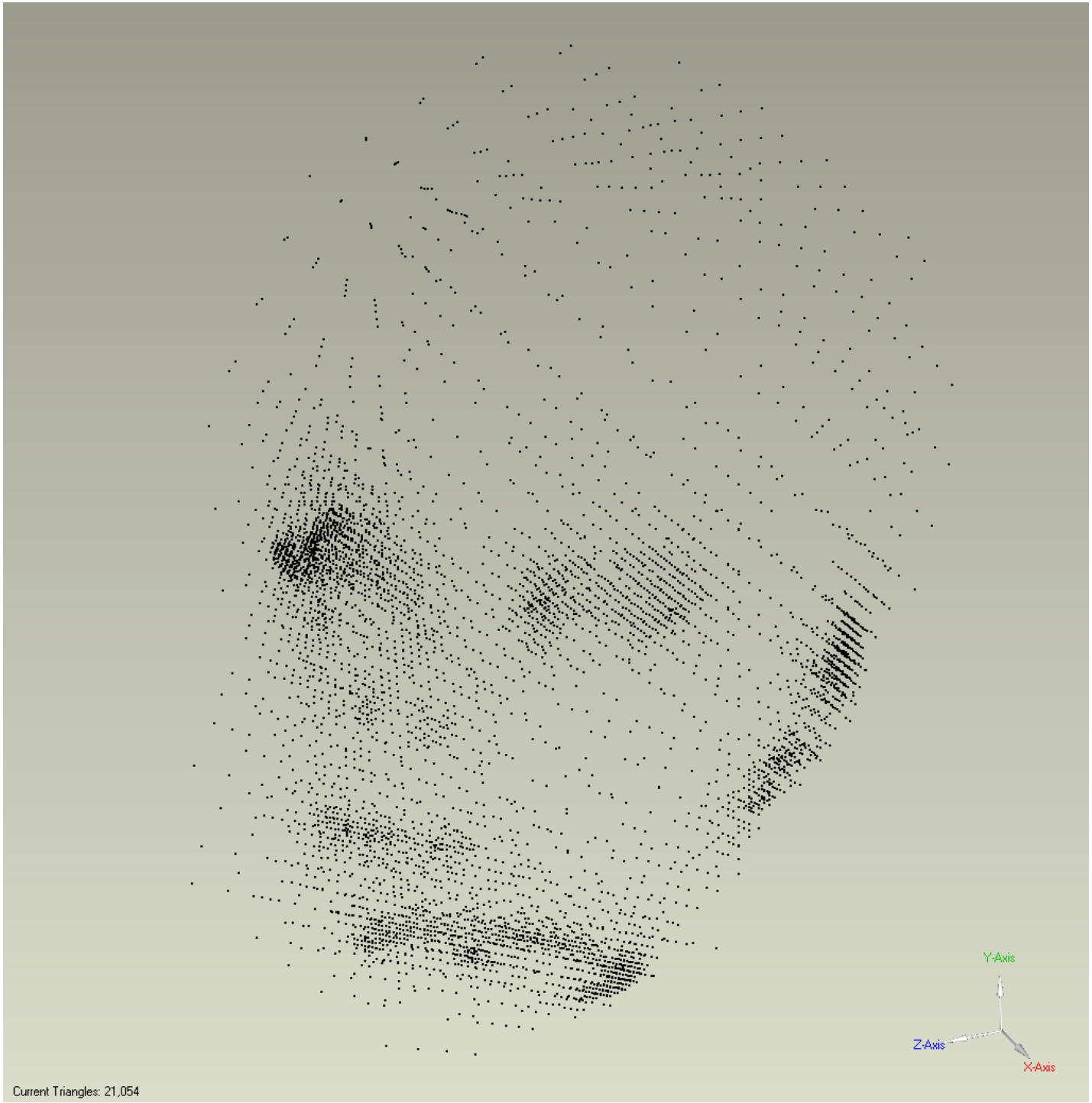
- The *unique* quadric surface whose distance from the four vertices and six edge midpoints matches the sampled values.


Y-Axis
Z-Axis
X-Axis
Current Triangles: 227,804


Y-Axis
Z-Axis
X-Axis
Current Triangles: 784,144

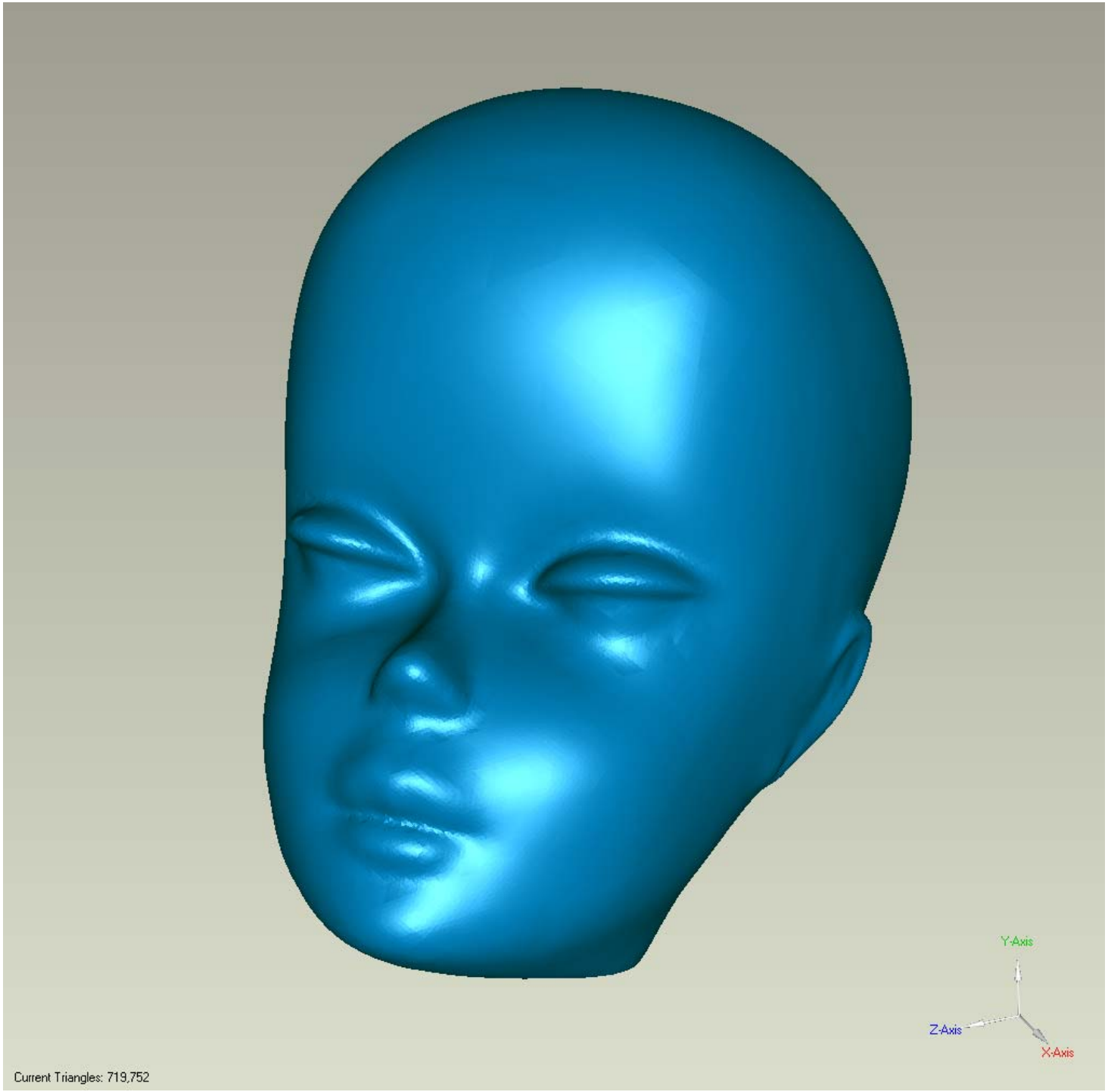

Y-Axis
Z-Axis
X-Axis
Current Triangles: 21,054

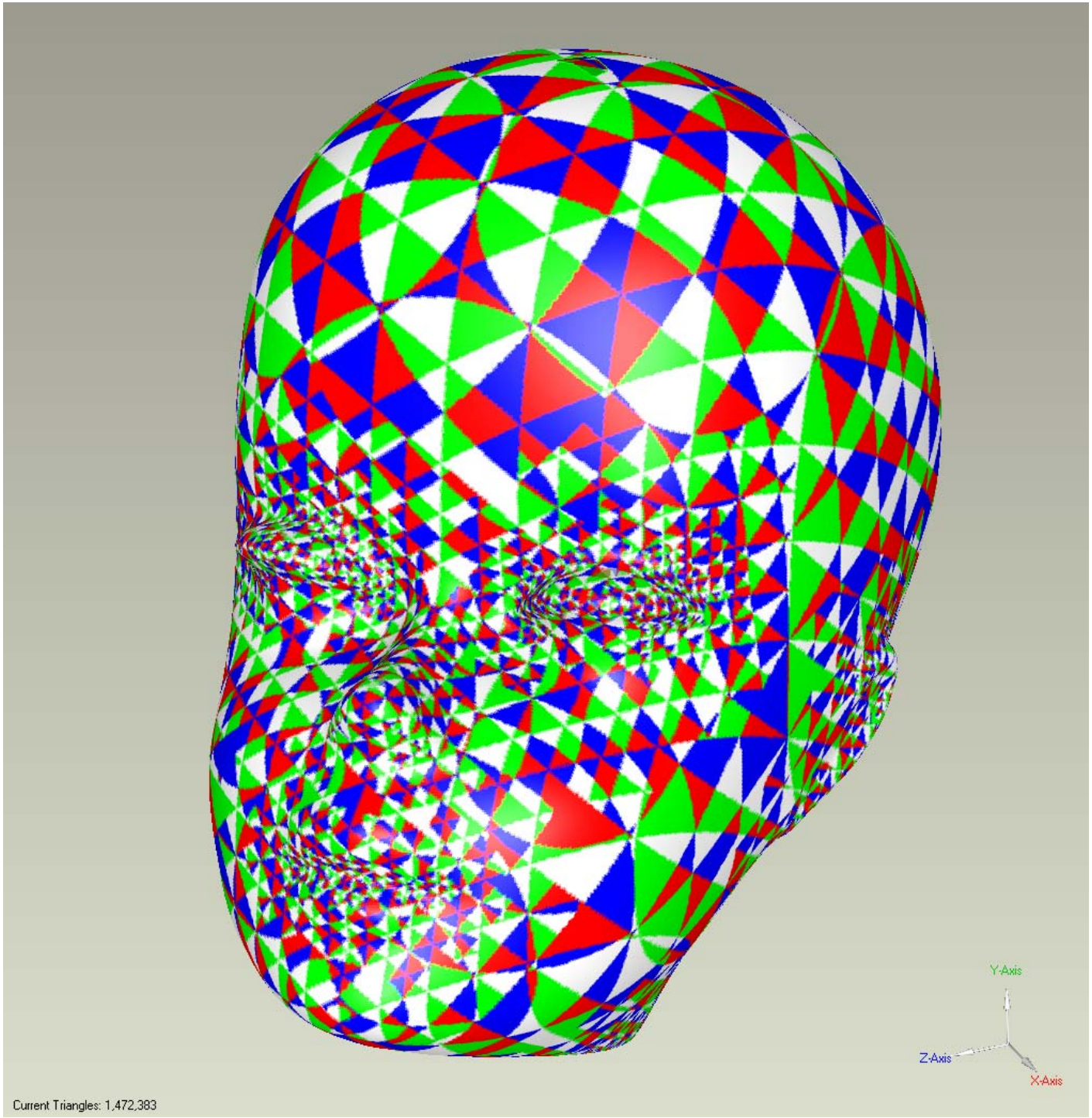

Y-Axis
Z-Axis
X-Axis
Current Triangles: 784,144

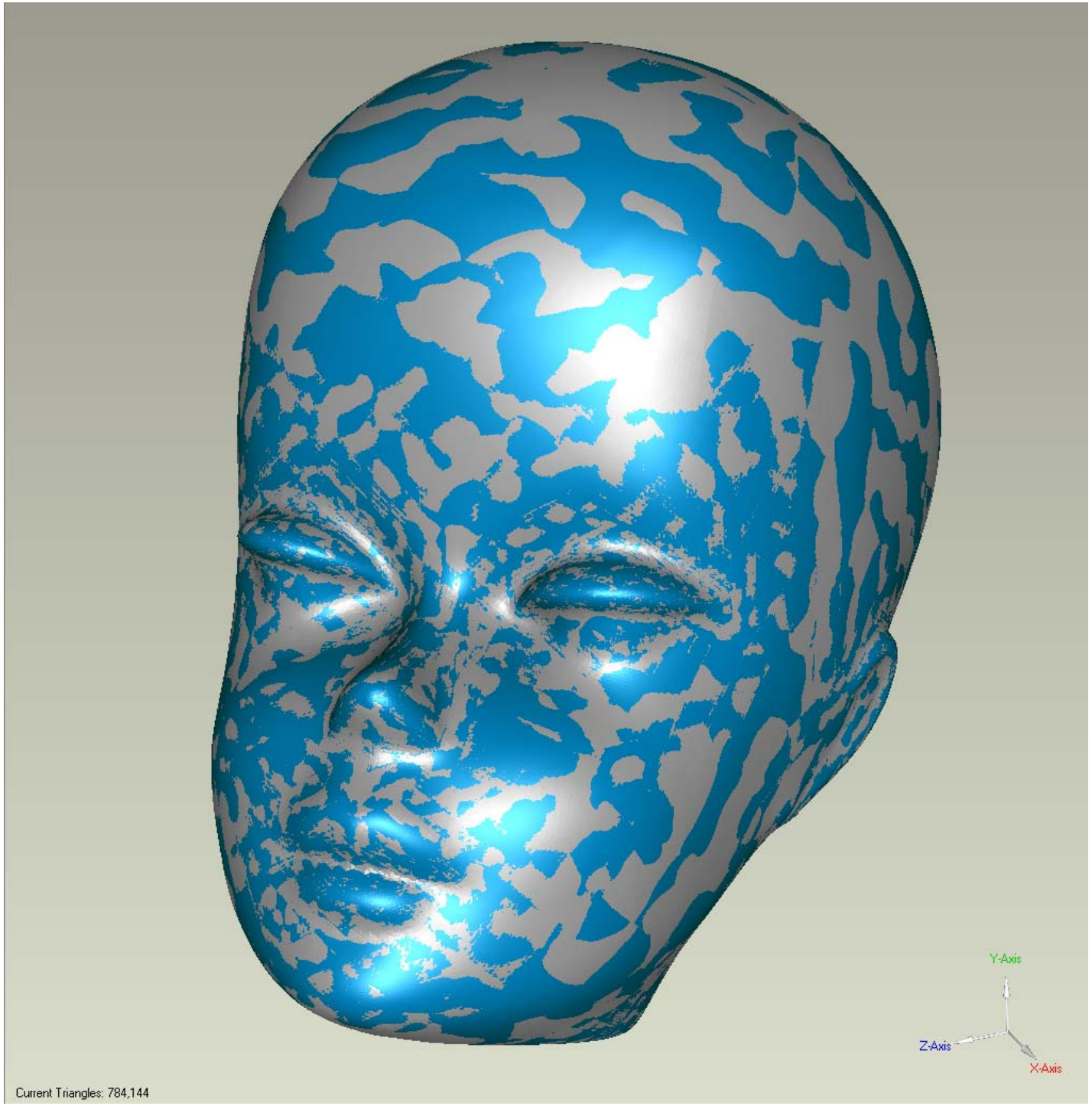

Y-Axis
Z-Axis
X-Axis
Current Triangles: 21,054

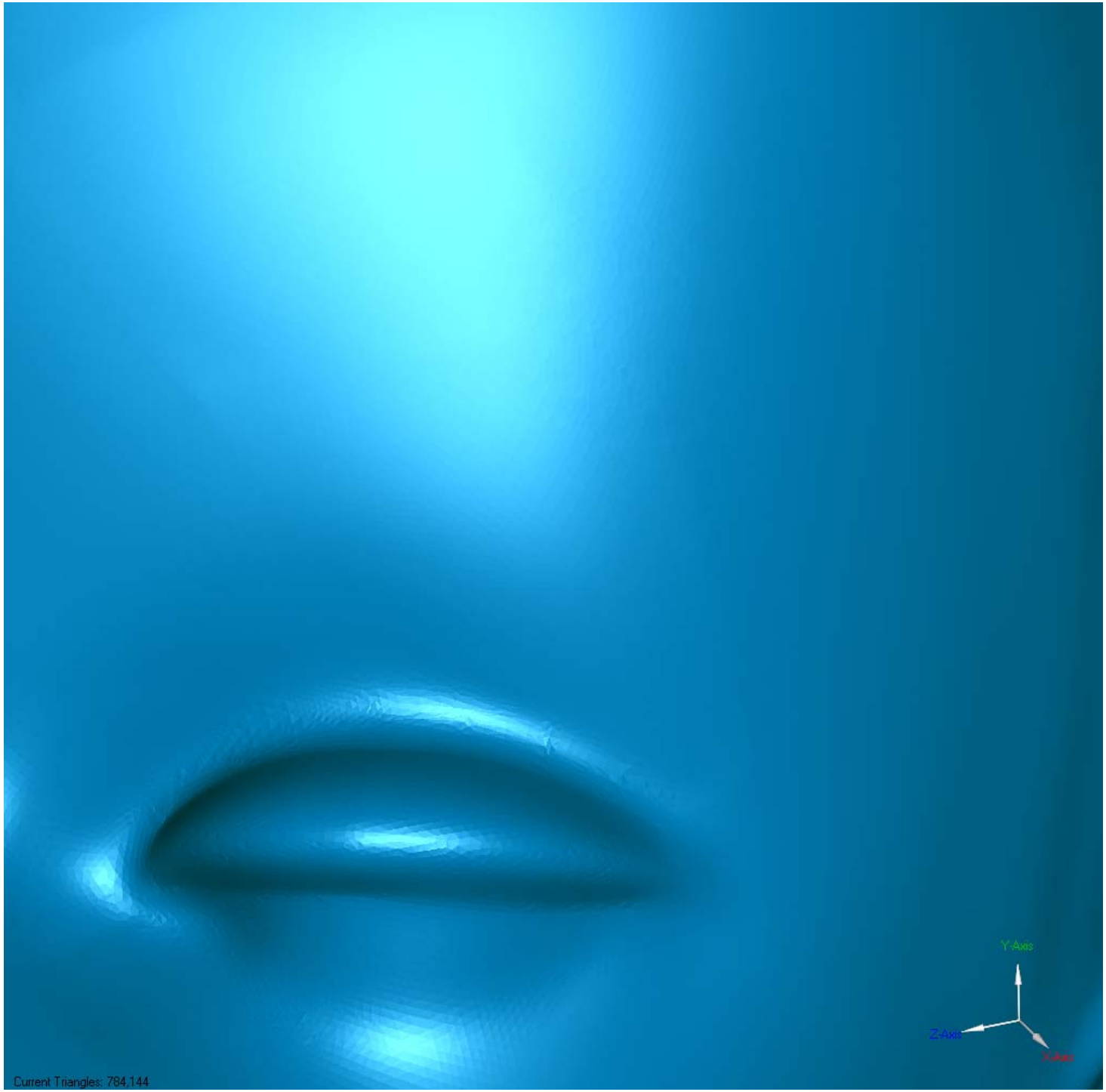
Reconstruction

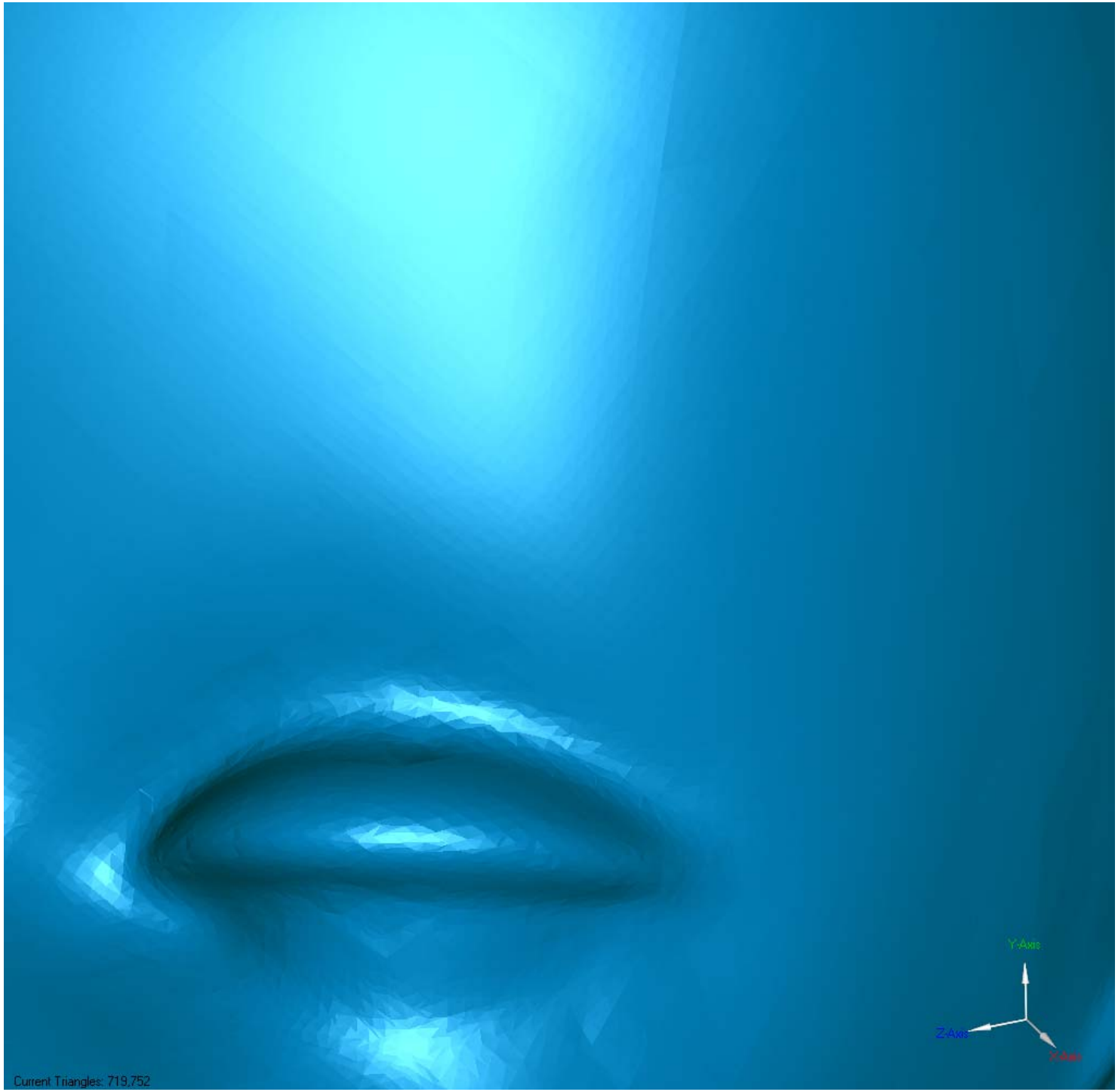
- Produce distance values at finest level shell vertices
  - If value is already present in consolidated shell, use it.
  - If not, reproduce it by interpolating the rep from the smallest cell of the consolidated shell that contains it.
- Produce surface (polygon) representation of each cell.


Y-Axis
Z-Axis
X-Axis
Current Triangles: 719,752


Y-Axis
Z-Axis
X-Axis
Current Triangles: 1,472,383

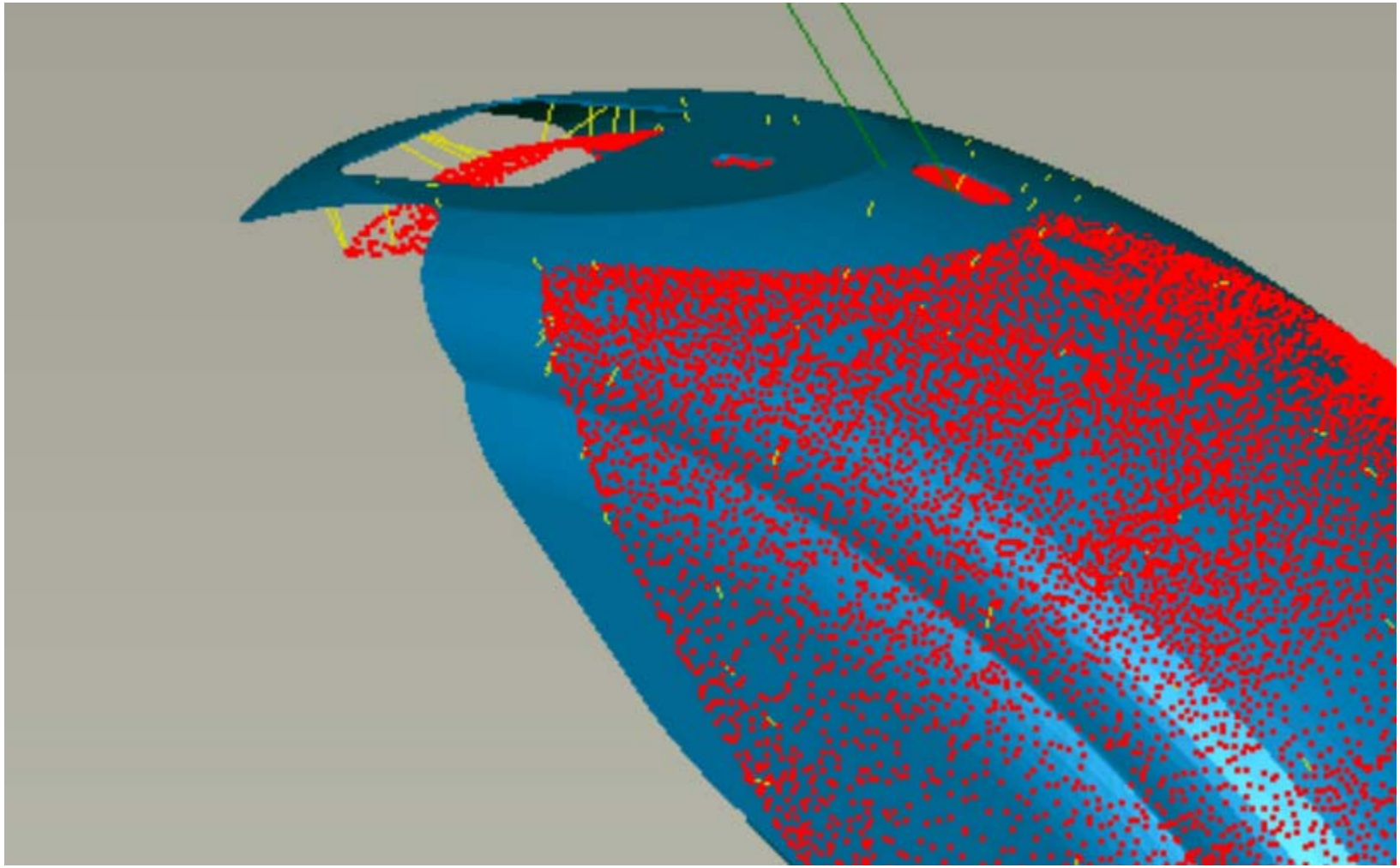

Y-Axis
Z-Axis
X-Axis
Current Triangles: 784,144


Y-Axis
Z-Axis
X-Axis
Current Triangles: 784,144


Y-Axis
Z-Axis
X-Axis
Current Triangles: 719,752

- In this example the data representation of the 780K polygon model is 25MB.
- The compressed representation of 25K indices and distance values is 200KB

A Closer Look at Registration



Recall that any rotation R on $\mathbb{R}^3$ can be written as $R = I + 2Q_{\mathrm{q}}$ where Q_{q} is defined in terms of a unit quaternion $\mathrm{q} = [q_0, \mathbf{q}] = [q_0, q_x, q_y, q_z]^T$ $(q_0^2 + q_x^2 + q_y^2 + q_z^2 = 1)$ by

$$Q_{\mathrm{q}} = \begin{bmatrix} -(q_y^2 + q_z^2) & q_x q_y - q_0 q_z & q_x q_z + q_0 q_y \\ q_x q_y + q_0 q_z & -(q_x^2 + q_z^2) & q_y q_z - q_0 q_x \\ q_x q_z - q_0 q_y & q_y q_z + q_0 q_x & -(q_x^2 + q_y^2) \end{bmatrix} \equiv q_0 \mathbf{q}^\times + (\mathbf{q}^\times)^2.$$

Which says that the rotation $R_{\mathbf{q}}$ corresponding to the quaternion $\mathbf{q}$ can be written in terms of cross products as

$$R_{\mathbf{q}}\mathbf{v} = \mathbf{v} + 2q_0 \mathbf{q} \times \mathbf{v} + 2\mathbf{q} \times (\mathbf{q} \times \mathbf{v}).$$

For small rotations, however, recall that $q_0 \approx 1$ and $\mathbf{q} \times (\mathbf{q} \times \mathbf{v}) \approx \mathbf{0}$ so that

$$R_{\mathbf{q}}\mathbf{v} \approx \mathbf{v} + 2\mathbf{q} \times \mathbf{v}.$$

Point to Plane Alignment. Let P be a discrete set of points on a smooth surface that have undergone a rigid motion with respect to their original positions R on a corresponding set of tangent planes $\mathcal{P}$ defined by unit normals N so that $\mathbf{n}_i^T\mathbf{x} = \mathbf{n}_i^T\mathbf{r}_i$ is the equation of the ith plane. If a subsequent arbitrary rigid motion M is applied to the points of P while the planes remain fixed, then the (directed) distance of the ith point $\mathbf{p}_i$ from its corresponding plane is

$$\begin{aligned}
d_i &= \mathbf{n}_i^T(M(\mathbf{p}_i) - \mathbf{r}_i) \\
&= \mathbf{n}_i^T(R_{\mathbf{q}}\mathbf{p}_i + \mathbf{t} - \mathbf{r}_i) \\
&\approx \mathbf{n}_i^T(\mathbf{p}_i + 2\mathbf{q} \times \mathbf{p}_i) + \mathbf{n}_i^T\mathbf{t} - \mathbf{n}_i^T\mathbf{r}_i \\
&= 2(\mathbf{p}_i \times \mathbf{n}_i)^T\mathbf{q} + \mathbf{n}_i^T\mathbf{t} - \mathbf{n}_i^T(\mathbf{r}_i - \mathbf{p}_i) \\
&= \mathbf{a}_i^T\mathbf{q} + \mathbf{n}_i^T\mathbf{t} - k_i
\end{aligned}$$

where $\mathbf{a}_i \equiv 2(\mathbf{p}_i \times \mathbf{n}_i)$, $k_i \equiv \mathbf{n}_i^T(\mathbf{r}_i - \mathbf{p}_i)$ is the *initial separation* between the ith point and its corresponding plane.

So when we consider the *total separation* $S \equiv \sum d_i^2$, we get

$$\begin{aligned}
\partial_{\mathbf{q}} S &= \sum_i 2(\partial_{\mathbf{q}} d_i) d_i \approx 2 \sum_i \mathbf{a}_i d_i \\
&\approx 2 \sum_i \left(\mathbf{a}_i \mathbf{a}_i^T + \mathbf{a}_i \mathbf{n}_i^T \mathbf{t} - k_i \mathbf{a}_i \right) \\
\partial_{\mathbf{t}} S &= \sum_i 2(\partial_{\mathbf{t}} d_i) d_i = 2 \sum_i \mathbf{n}_i d_i \\
&\approx 2 \sum_i \left(\mathbf{n}_i \mathbf{a}_i^T \mathbf{q} + \mathbf{n}_i \mathbf{n}_i^T \mathbf{t} - k_i \mathbf{n}_i \right).
\end{aligned}$$

S will be minimized only when $\partial_{\mathbf{q}} S = \partial_{\mathbf{t}} S = 0$, and as seen from the above this happens precisely when $[\mathbf{q}, \mathbf{t}]^T$ satisfies the symmetric linear system:

$$\begin{bmatrix} \sum \mathbf{a}_i \mathbf{a}_i^T & \sum \mathbf{a}_i \mathbf{n}_i^T \\ \sum \mathbf{n}_i \mathbf{a}_i^T & \sum \mathbf{n}_i \mathbf{n}_i^T \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \sum k_i \mathbf{a}_i \\ \sum k_i \mathbf{n}_i \end{bmatrix}. \tag{1}$$

Slip Theory

If the points $\mathbf{p}_i \in P$ are sampled from a surface $\mathcal{S}$ and the planes of $\mathcal{P}$ are the planes tangent to $\mathcal{S}$ at $\mathbf{p}_i$, then the points and normals effectively furnish a *shape sample* of $\mathcal{S}$ and we must have $\mathbf{r}_i \equiv \mathbf{p}_i \Rightarrow k_i \equiv 0$ so that (1) simplifies to

$$\begin{bmatrix} 4\sum \mathbf{a}_i\mathbf{a}_i^T & 2\sum \mathbf{a}_i\mathbf{n}_i^T \\ 2\sum \mathbf{n}_i\mathbf{a}_i^T & \sum \mathbf{n}_i\mathbf{n}_i^T \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{2}$$

and the classical theorems tell us that (2) will have non-trivial solutions if and only if its coefficient matrix is singular. In this case, the eigenvector(s) of zero describe rigid perturbing motions for which the separation remains 0. We refer to such motions as *slips* and note that they are intrinsic to the surface $\mathcal{S}$ as sampled by the point-normal pairs $(\mathbf{p}_i, \mathbf{n}_i)$. These slips (if they exist) describe axes of the rotational/translational symmetries of $\mathcal{S}$.