Homotopy methods for finding the limit cycle of oscillators

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1. Basic concepts

2. Homotopy techniques

3. Homotopy techniques for oscillators

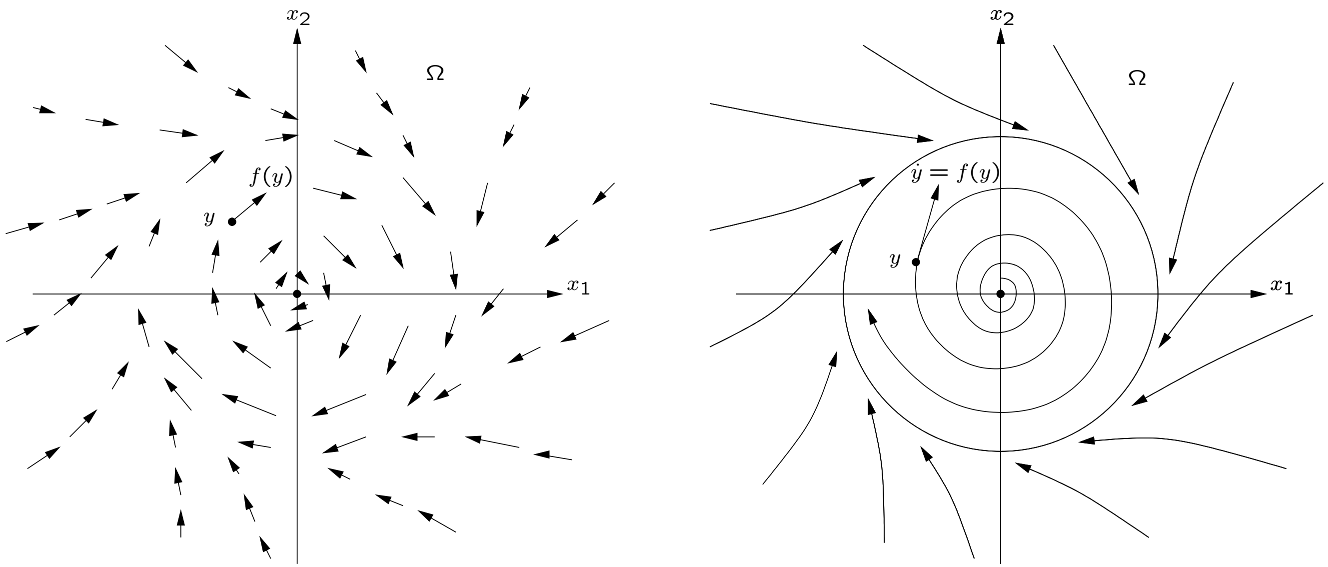
4. Simulation results

System of autonomous ODEs or DAEs of dimension N

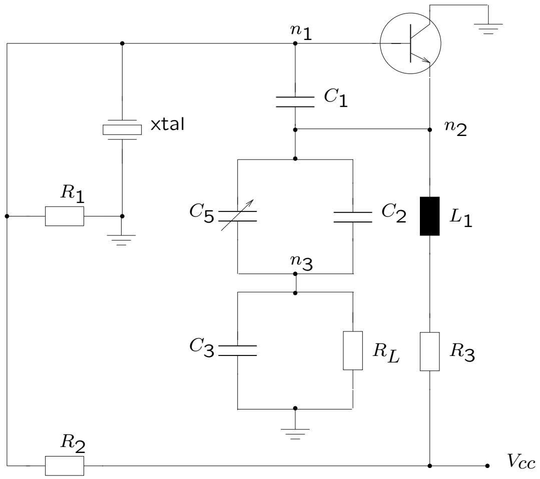
$$\dot{x} = f(x), \qquad \dot{q}(x) + f(x) = 0$$

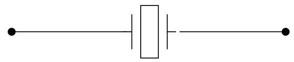
with periodic limit cycle $x_{ss}(t) = x_{ss}(t+T)$

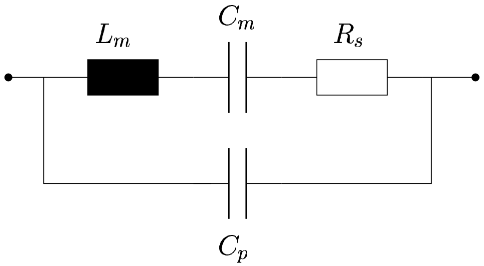
Representation in phase space



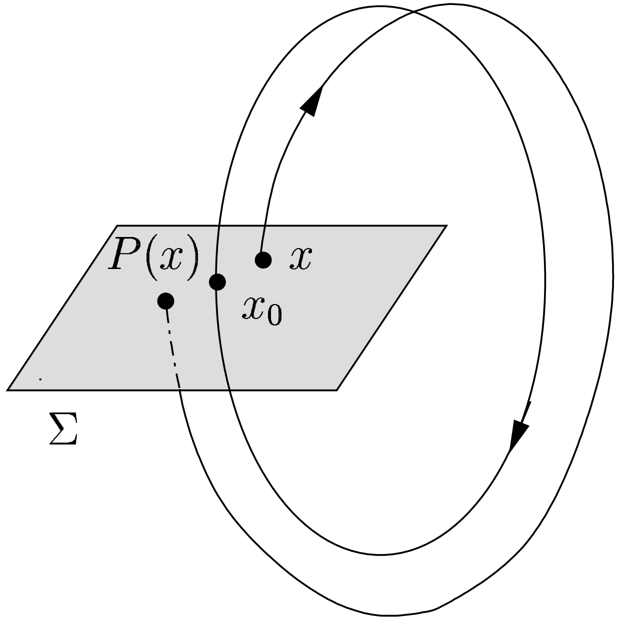
Colpitts oscillator with xtal







Uncountable set of solutions $x_{ss}(t + \Delta t)$, t, $\Delta t \in \mathbb{R}$
Poincaré hyperplane Σ

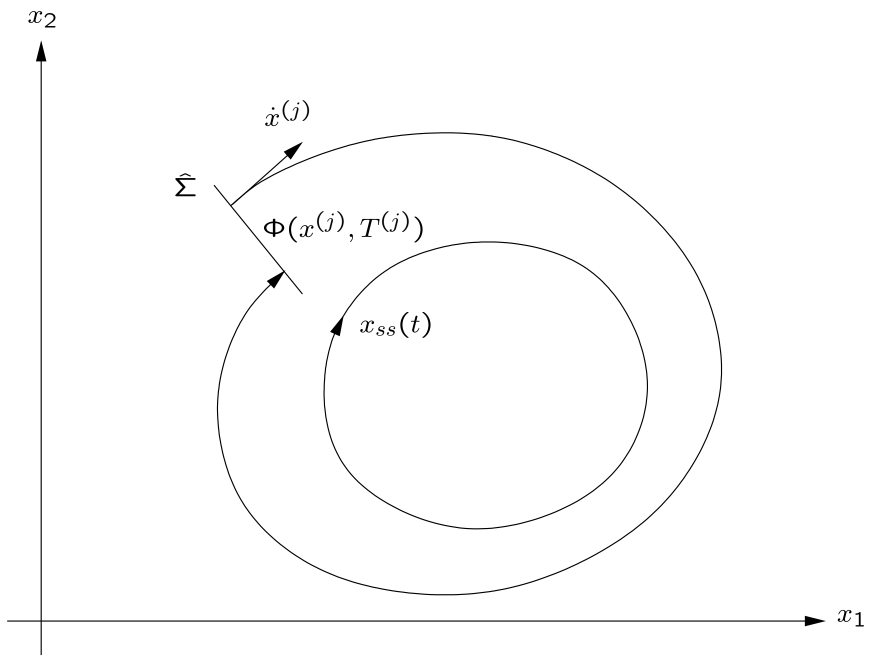


Assumption: State transition $\Phi(y, t)$ function unique for any initial condition

Shooting methods: reformulation as boundary value problem

$$\Phi(y, T) - y = 0$$

Additional equation by approximating the Poincaré hyperplane Σ



Finite-Difference or Harmonic Balance; K gridpoints

$$F(X,T) := I(X) + \nabla(T)\,Q(X) = 0$$

where

$$\nabla_{nn} = \mathrm{circ}\,(a_0,\, a_1,\, \ldots,\, a_{K-1}) =: T(c,\, r)$$

$$T(c,\, r) = \begin{bmatrix} c_1 & r_2 & \cdots & \cdots & r_p \\ c_2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & r_2 \\ c_p & \cdots & \cdots & c_2 & c_1 \end{bmatrix}$$

Underdetermined: $K \cdot N$ equations for $K \cdot N + 1$ unknowns

Selecting a specific solution: Orthogonality to fundamental waveform

$$\frac{1}{T}\int_0^T x_1(t)\cdot \cos\left(\frac{2\,\pi}{T}\,t\right)\mathrm{d}t = 0$$

However: Unstable operating point still a solution of the discretized equations

Often convergence to the trivial solution

Use the fact that operating point is unstable

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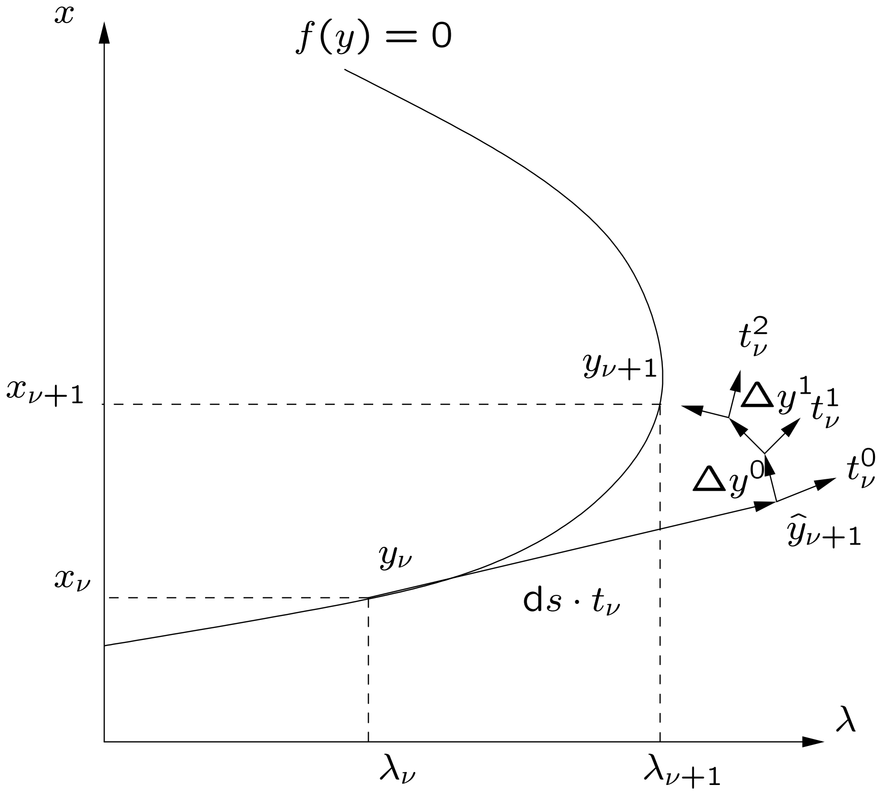
$$f(x(\lambda),\ \lambda) = 0, \qquad 0 \le \lambda \le 1$$
$$f:\ \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$$



Search direction is orthogonal to tangent vector
Tangent vector is kernel of the Jacobian

$$t_{\nu+1} \in \ker \left[f_x(y_{\nu+1}) \quad f_\lambda(y_{\nu+1}) \right]$$

Note: Nullspace can be calculated without SVD or QR decomposition

begin

$y \in \mathbb{R}^{n+1}$ such that $f(y) = 0$

$ds > 0$

$\hat{t} \in \mathbb{R}^{n+1}$

end

repeat

solve $\begin{bmatrix} f_y(y) \\ \hat{t}^T \end{bmatrix} t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for t

$t := \frac{t}{||t||_2}$

if $\hat{t}^T\, t < 0$ then $t := -t$ fi

calculate initial guess $\hat{y} := y + ds\, t$

repeat

solve $\begin{bmatrix} f_y(\hat{y}) \\ t^T \end{bmatrix} \Delta y = \begin{bmatrix} f(\hat{y}) \\ 0 \end{bmatrix}$ for Δy

$\hat{y} := \hat{y} - \Delta y$

until $||\Delta y||_2 < \epsilon$

$y := \hat{y}$

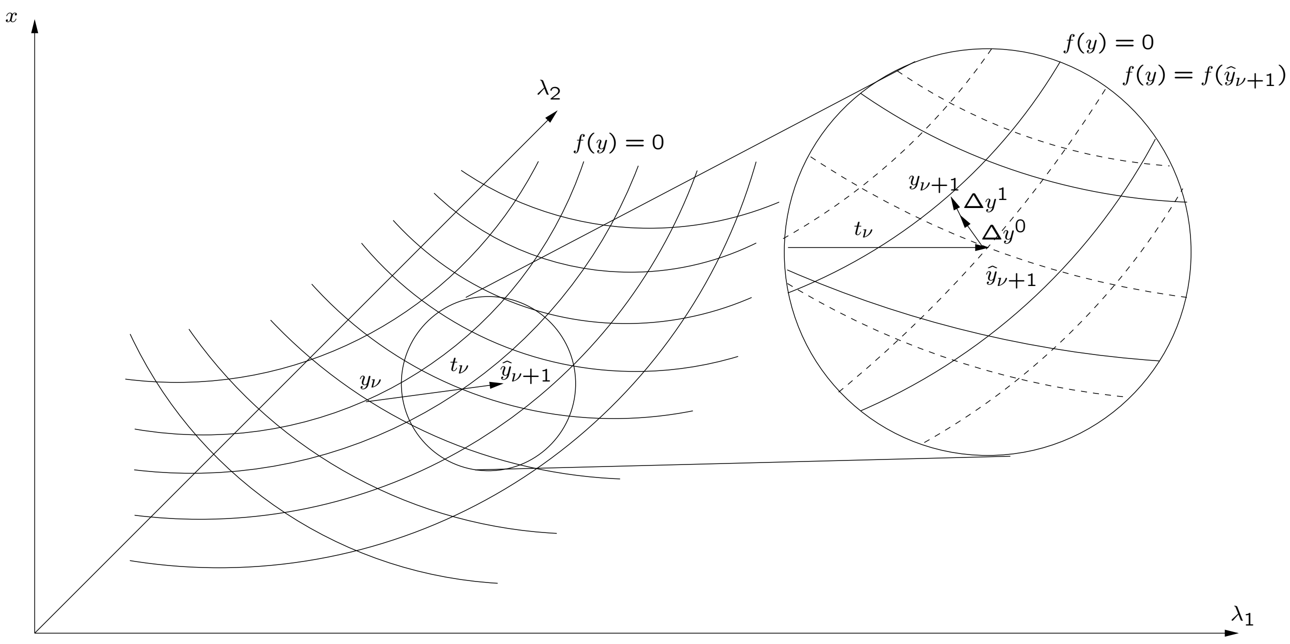
choose a new steplength $ds > 0$

$\hat{t} = t$

until traversing is stopped

Abbildung 1: Pseudo-Arclength Homotopieverfahren

$$f(x(\lambda_1, \lambda_2),\, \lambda_1, \lambda_2) = 0, \qquad f:\, \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n$$



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$$P(x^{(j)}) = \Phi\left(x^{(j)}, T^{(j)}\right),\ x^{(0)} = x_0$$

Assumption: fixed point iteration converges asymptotically towards limit cycle

Boundary value problem (T_0 initial state) is underdetermined

$$x - \Phi\left(x, \lambda_2 = T/T_0\right) = 0$$

Continuation from initial value problem $\lambda_1 = 0$ to steady state condition $\lambda_1 = 1$

$$x - \Phi\left(\lambda_1\, x + (1-\lambda_1)\, x_0, \lambda_2 = T/T_0\right) = 0$$

Solve for x and λ_1, λ_2 in a least squares sense
$\Rightarrow$ Two-dimensional homotopy

Results for crystal oscillators · Initials by eigenvalue analysis at operating point

	Clapp	Colpitts	Pierce	grounded-base
f_0/MHz	20	20	2	50
$\frac{f}{f_0}$	0.999978	0.99996	0.999976	0.999996
steps	16	16	16	27

Note: The higher the Q of an oscillator the burdensomer the limit cycle, the easier the estimate of the oscillation frequency

Simulation results for oscillators of medium quality factor

	RF-VCO	Pierce	VC0-C1000	VHF	Symm-Osc
f_0/MHz	941	1.60	1025	138	96.46
steps	8	18	9	10	9