Initial Condition Strategies for Multiple-time Partial Differential Equations

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Collaborators

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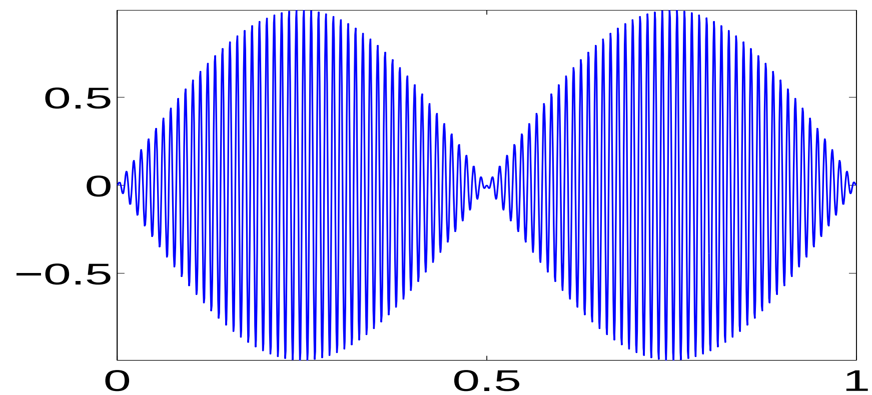
What are MPDEs?

- Multi-time Partial Differential Equations
- An approach for faster simulation of oscillatory DAEs
- Convert an ODE/DAE into a multi-time PDE/PDAE
- Introduce new time variable as a spatial variable
- Savings come from periodicity of new time variable
- Can recover ODE/DAE solution from MPDE solution
- Motivation: Speedups of 100x to 1000x!

Representation of multi-rate signals

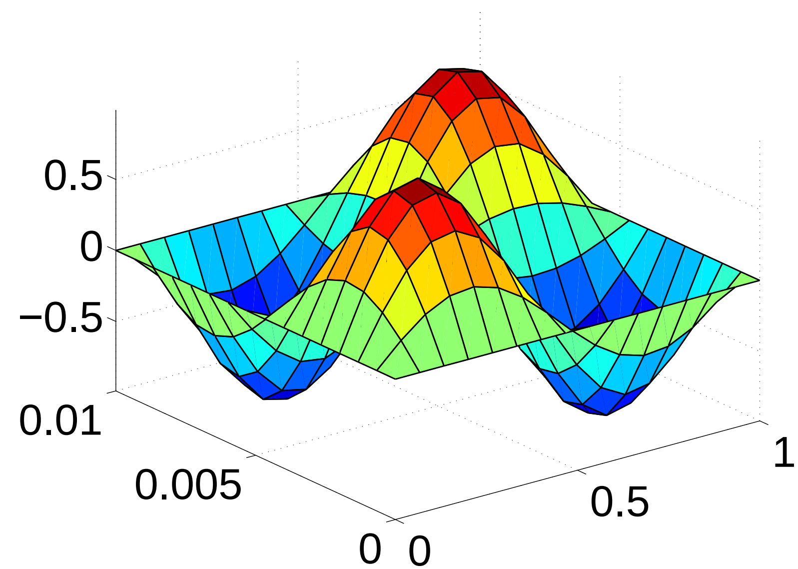
- $b(t) = \sin(2\pi t)\sin(2\pi 10^2 t)$

arclength $\approx$ 254.7
15 points / 4.2 arclength
$\approx$ 910 samples



- $\hat{b}(t_1, t_2) = \sin(2\pi t_1)\sin(2\pi 10^2 t_2), \quad b(t) = \hat{b}(t, t)$

t_1 arclength $\approx$ 4.2
t_2 arclength $\approx$ 4.2
$\approx$ 225 samples



Research Context

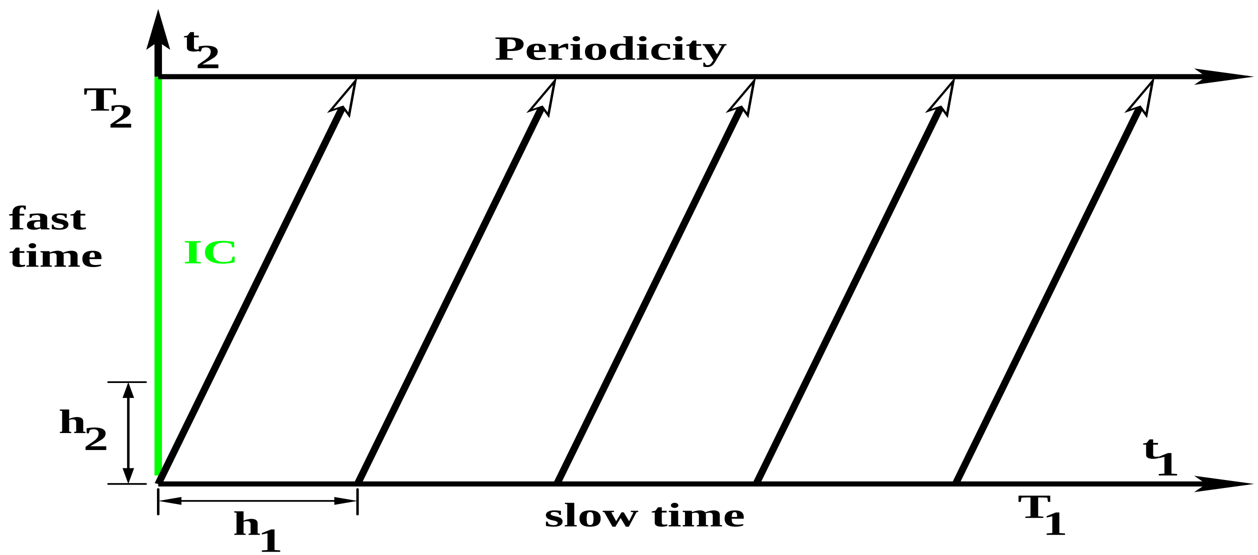
- Time-domain envelope methods:
  - L. Petzold 1981
  - K. Kundert et al. 1988
- Frequency-domain envelope methods:
  - E. Ngoya / R. Larchevéque 1996
  - D. Sharrit 1996
  - P. Feldmann / J. Roychowdhury 1996
- Time and frequency domain MPDE methods:
  - H-G. Brachtendorf et al. 1996
  - J. Roychowdhury 2001
  - Fourier envelope robustness: J. Roychowdhury 2002
  - Discretization: Coffey, Roychowdhury, et al 2004

Circuit transformation

- Re-write circuit DAE using multi-time variables
  - DAE: $\frac{d\mathbf{q}(\mathbf{x})}{dt} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t) = 0$
  - MPDE: $\frac{\partial\mathbf{q}(\widehat{\mathbf{x}})}{\partial t_1} + \frac{\partial\mathbf{q}(\widehat{\mathbf{x}})}{\partial t_2} + \mathbf{f}(\widehat{\mathbf{x}}) + \widehat{\mathbf{b}}(t_1, t_2) = 0$
- $\widehat{\mathbf{x}}(t_1, t_2) =$ vector of multi-time unknowns
- $\widehat{\mathbf{b}}(t_1, t_2) =$ multi-time input
- Solution of MPDE $\Longrightarrow$ solution of DAE
  - $\widehat{\mathbf{b}}(t, t) = \mathbf{b}(\mathbf{t}) \Longrightarrow \mathbf{x}(\mathbf{t}) = \widehat{\mathbf{x}}(\mathbf{t}, \mathbf{t})$
- Solve multi-time formulation directly

Efficiency

- We've turned a DAE into a 2-D PDAE
- Quasi-periodicity of fast oscillation

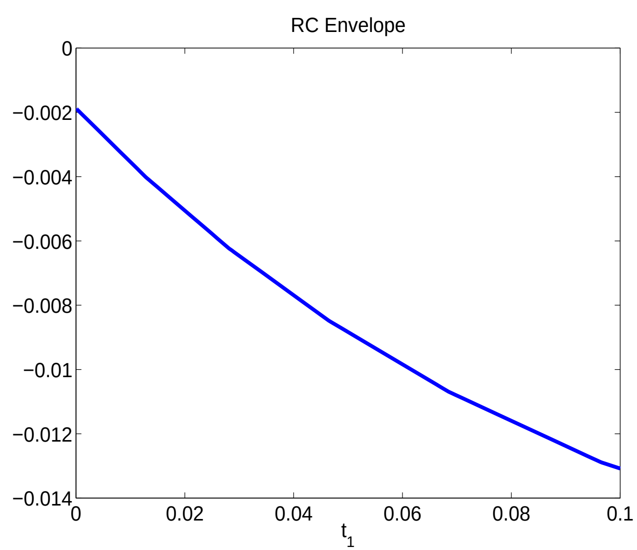


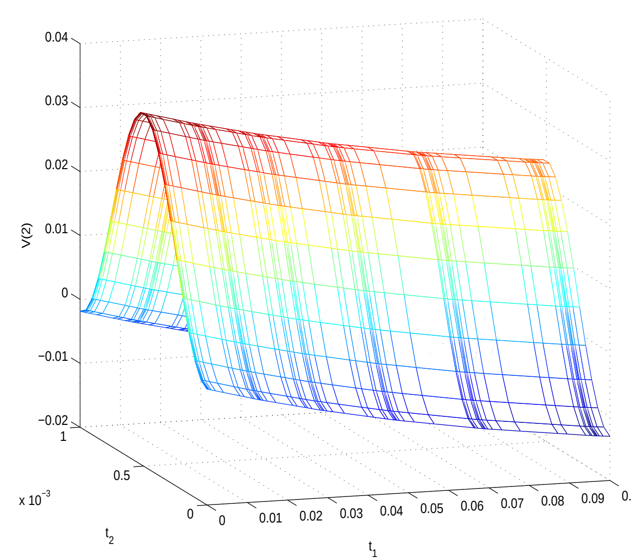
- Frequency gap between rates of change: Need at least 100x difference for 2-D MPDE

Initial Condition Strategies

- Five strategies
- Basic explanation
- Pro & Con

1. DCOP
2. Integrate
3. Integrate & Interpolate
4. Steady state with slope
5. Optimization





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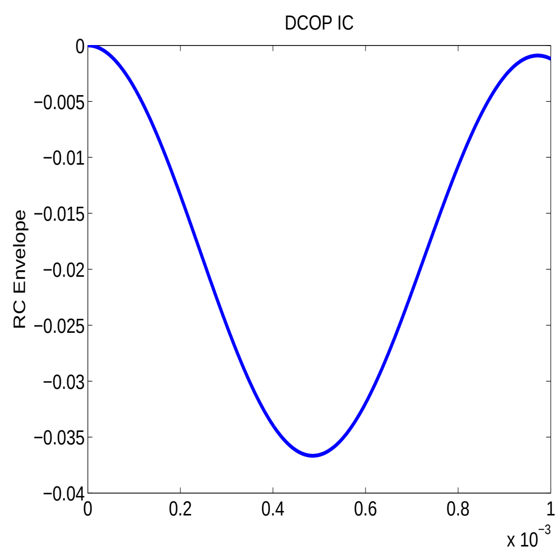
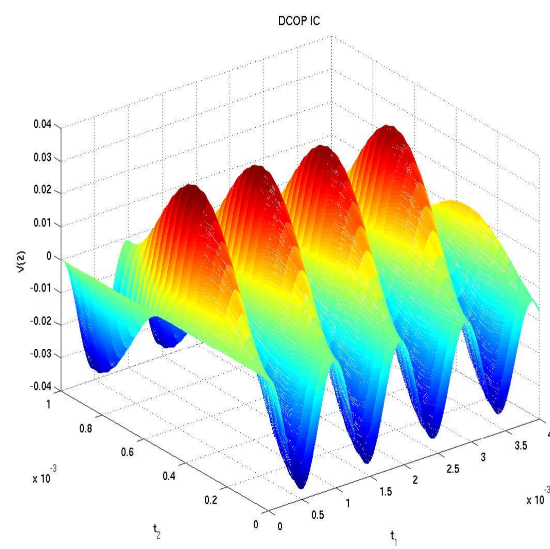
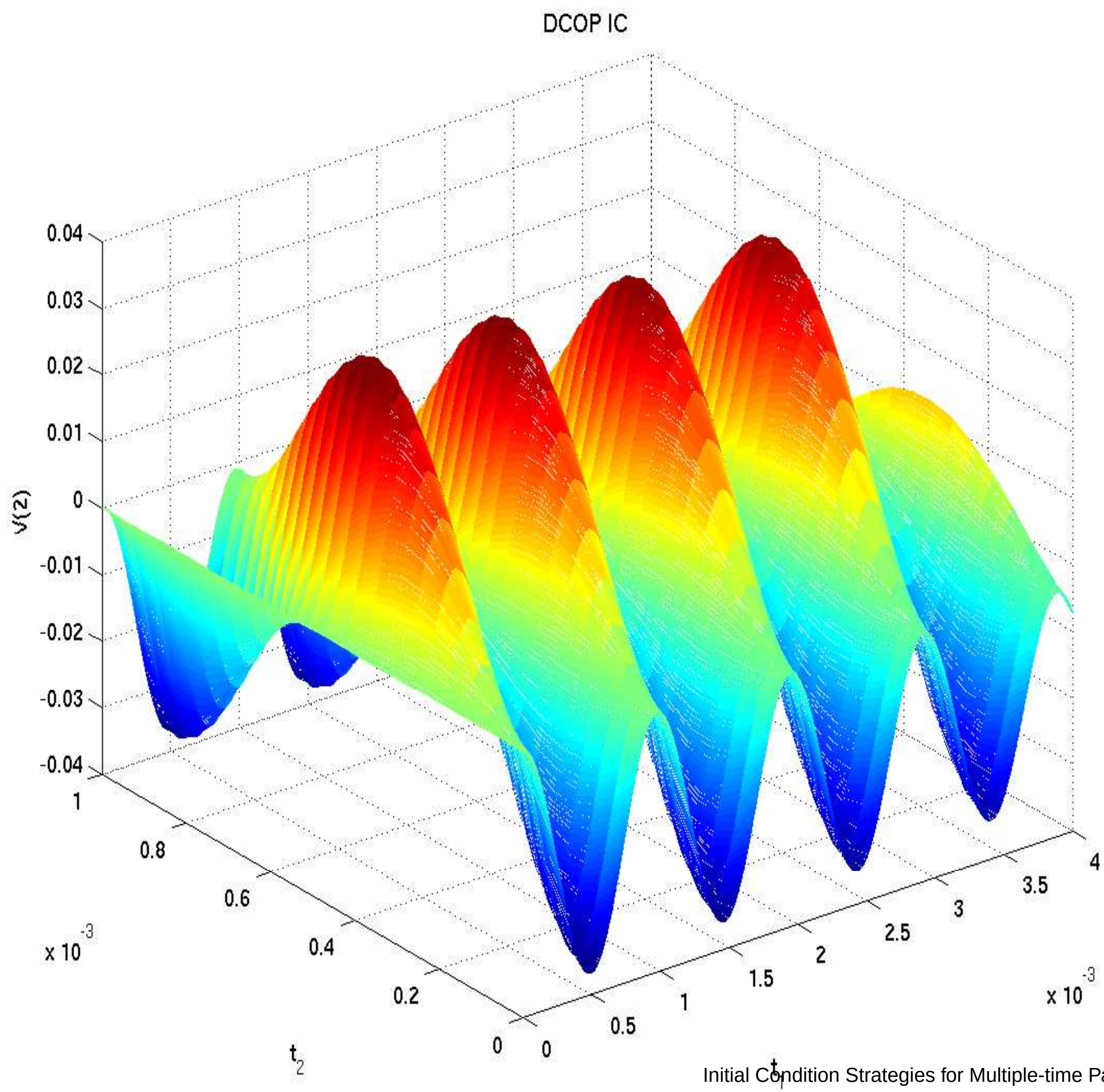
IC 1: DCOP

- DCOP: DC Operating Point
  Consistent initial condition for DAE problem
- Pro: Free, periodic
- Con: Does not represent oscillations





IC 1: DCOP IC MPDE output



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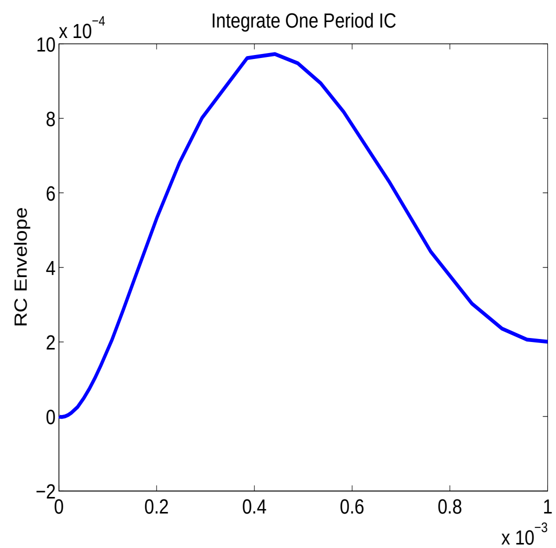
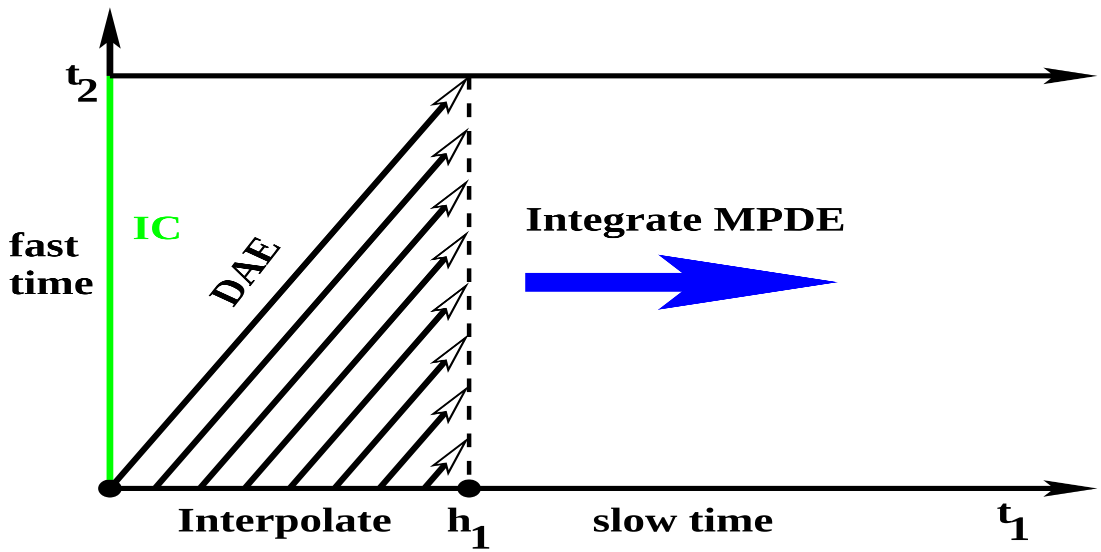
IC 2: Integrate one fast period

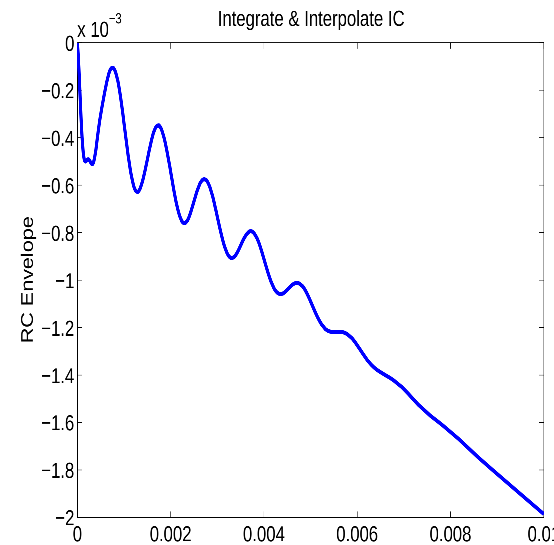
- Integrate one fast period as MPDE IC
- Pro: low cost: 1 period
- Con: not periodic



IC 3: Integrate & Interpolate

- Integrate DAE one fast period then interpolate to form new DAE IC for integrating to MPDE IC

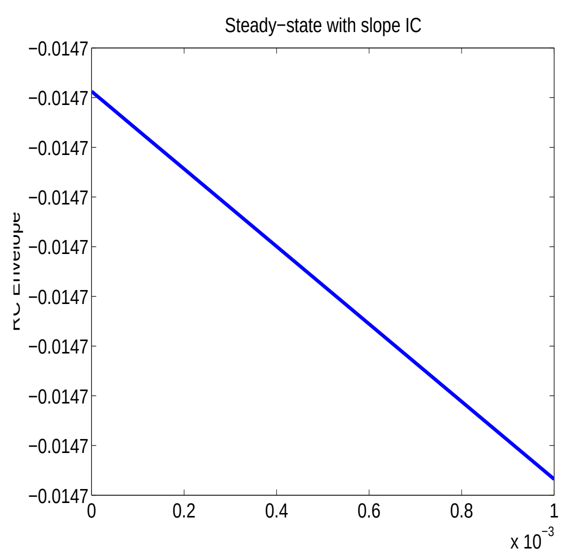




- Pro: periodic
- Con: cost: $(n_2 + 1)/2$ periods

IC 4: Steady-state with Slope

- Integrate one period to estimate $\frac{\partial \mathbf{q}(\widehat{\mathbf{x}})}{\partial t_1} = \gamma$
- Solve for $\widehat{\mathbf{x}}(0, t_2)$: $\gamma + \frac{\partial \mathbf{q}(\widehat{\mathbf{x}})}{\partial t_2} + \mathbf{f}(\widehat{\mathbf{x}}) + \widehat{\mathbf{b}}(0, t_2) = 0$
- Pro: fairly cheap: 1 period + nonlinear solve, fast source oscillations well represented
- Con: IC based only at $t_1 = 0$



IC 5: Optimization

- Consider the simplified MPDE problem:

$$\frac{\partial \widehat{\mathbf{x}}}{\partial t_1} + \frac{\partial \widehat{\mathbf{x}}}{\partial t_2} + \mathbf{f}(\widehat{\mathbf{x}}) + \widehat{\mathbf{b}}(t_1, t_2) = 0$$
$$\widehat{\mathbf{x}}(0, t_2) = \mathbf{v}(t_2), \qquad \mathbf{v}(0) = x_0$$

- Formulate unconstrained minimization problem

$$\min_{v \in \mathsf{U}} |J(\mathbf{v}) - \gamma|, \qquad J(\mathbf{v}) = \frac{1}{2}\int_0^T \int_0^T \left| \frac{\partial \widehat{\mathbf{x}}}{\partial t_1} \right|^2 dt_1 dt_2$$

- Pro: periodic, increase step-sizes
- Con: high cost: $2n_2$ periods/iteration

Optim: Related ODE problem

- Assume $\widehat{\mathbf{b}}(t_1, t_2) = \mathbf{b}(t_2)$
  The sources produce only fast oscillations
- This produces the following related problem:

$$\dot{\mathbf{x}} + \mathbf{f}(\mathbf{x}) + \mathbf{b}(t+s) = 0, \qquad \mathbf{x}(0; s) = \mathbf{v}(s)$$

  - $\widehat{\mathbf{x}}(t_1, t_2) = \begin{cases} \mathbf{x}(t_1, t_2 - t_1), & t_2 > t_1 \\ \mathbf{x}(t_1, t_2 - t_1 + T), & t_2 < t_1 \end{cases}$
  - We can compute MPDE solutions by solving a perturbed ODE problem
- Transform our objective function:

$$J(\mathbf{v}) = \frac{1}{2} \int_0^T \int_0^T |\dot{\mathbf{x}}(t; s) - \mathbf{x}_s(t; s)|^2 \, dt ds$$

Optim: Gradient & adjoint equations

- The gradient is a function $s \mapsto \mathbf{g}$ given by:

$$\mathbf{g} = \mathbf{p}(0;s) - \dot{\mathbf{x}}(0;s) + \mathbf{x}_s(0;s), \quad s \in (0,T)$$

- where $\mathbf{p} = \mathbf{p}(t;s)$ is computed by solving the adjoint equations:

$$-\dot{\mathbf{p}} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\mathsf{T}} \mathbf{p} + \left(\frac{d}{dt} - \frac{d}{ds}\right) \mathbf{x}(t;s) = 0$$

$$\mathbf{p}(T;s) = \dot{\mathbf{x}}(T;s) - \mathbf{x}_s(T;s)$$

- We have continuous & discrete analysis for fully nonlinear DAE case

Optim: Steps for gradient

Recall gradient and adjoint equations:

$$\mathbf{g}(s) = \mathbf{p}(0;s) - \dot{\mathbf{x}}(0;s) + \mathbf{x}_s(0;s) \quad \text{for} \quad s \in (0,T)$$

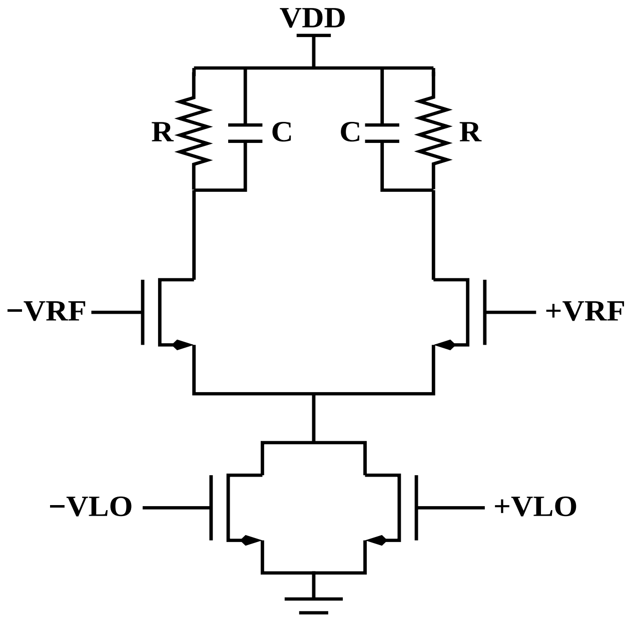
$$-\dot{\mathbf{p}} + \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)^{\mathsf{T}} \mathbf{p} + \left(\frac{d}{dt} - \frac{d}{ds}\right)\mathbf{x}(t;s) = 0$$

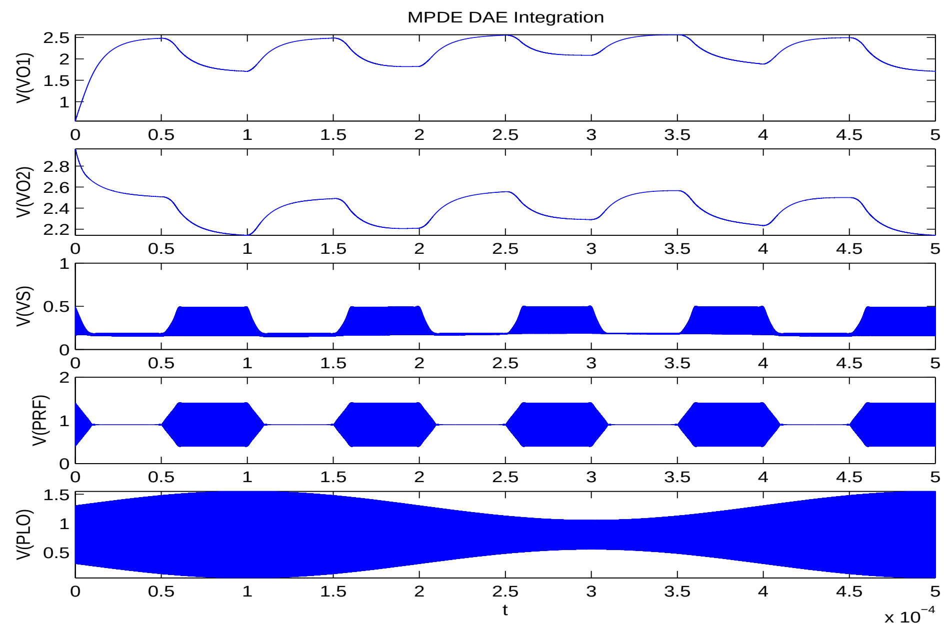
$$\mathbf{p}(T;s) = \dot{\mathbf{x}}(T;s) - \mathbf{x}_s(T;s)$$

- Specify $\mathbf{v}$ as input
- Solve forward problem for $\mathbf{x}(t;s), \quad \forall s \in (0,T)$
- Form $J(\mathbf{v})$
- Solve adjoint equations for $\mathbf{p}(0;s), \quad \forall s \in (0,T)$
- Form gradient $\mathbf{g}(s)$

Communication Speed-up Example

- Balanced CMOS direct-downconversion mixer





- Sources:
  - +VLO = 450Mhz sinusoid modulated by 2.4khz sinusoid
  - +VRF = 900Mhz carrier modulated by 10kbps bitstream
- DAE simulation time: $\approx$ 5 days

IC Performance

Communication Example Circuit (time in min)

	IC Time	MPDE Time	Total
DCOP	N/A	N/A	N/A
Integrate & Interpolate	5.2	18.3	23.5
Integrate	0.08	12.3	12.4
Steady state w/slope	0.12	13.1	13.2
Optimization	14.0	11.4	25.4
DAE			$\approx$ 7600

Note $\approx$ 600x speed-up!

Conclusions

- MPDE methods offer considerable speed-up
- Choice of IC is very important:
  Optimization may be helpful, but it needs to be cheap.
  - Interesting application of adjoint equation theory
  - Improves accuracy & speed-up
  - Captures some start-up behavior
- Fast time axis must represent oscillation well to get speedup

Thank you!