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Lecture Notes on Continuum Mechanics on Arbitrary Moving Domains

Guglielmo Scovazzi, Thomas J.R. Hughes

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Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

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Guglielmo Scovazzi
1431 Computational Shock- and Multi-physics Department
Sandia National Laboratories
P.O. Box 5800, MS 1319
Albuquerque, NM 87185-1319, USA
gscovaz@sandia.gov

Thomas J. R. Hughes
Institute for Computational Sciences and Engineering
The University of Texas at Austin
1 University Station C0200
Austin, TX 78712-0027
hughes@ices.utexas.edu

Abstract

These notes have been prepared on occasion of a short course taught by the first of the authors at Sandia National Laboratories in the Fall 2007. They are an attempt to present a unified approach to the topic of continuum mechanics on arbitrarily moving domains. The motivation of this work stands on the fact that in the literature up to date such a unifying view is missing. A space-time approach is also developed, which is shown to incorporate the more traditional thinking of time and space as separate entities. As applications of the proposed framework, the compressible and incompressible Navier-Stokes equations are derived. Additional considerations on the role of so-called geometric conservation laws are presented, for their implication in numerical computations.

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Lecture Notes on Continuum Mechanics on Arbitrary Moving Domains

Between the late 60's and the late 80's, arbitrary Lagrangian-Eulerian (ALE) methods were developed in academic institutions and National Laboratories inside and outside the United States. The numerical algorithms that have been developed since that time are derived using the methods of continuum mechanics on arbitrarily moving domains. Classical publications which contain information about ALE methods are [11, 4, 15, 7, 2]. A large number of publications appeared in the technical literature on ALE methods for various applications, among which an *non-exhaustive* list is [5, 23, 22, 27, 26, 28, 6, 1, 19, 20, 17, 16, 31, 18]. The ability to move computational domains in an arbitrary fashion is particularly effective in the treatment of fluid/structure interaction (FSI) problems, where a structure, typically meshed on a computational grid that follows the material is mechanically coupled to a fluid, which typically requires an Eulerian-type (fixed) mesh. The idea in this case is to relax the requirement for the fluid mesh to be Eulerian and accommodate for the movement of the interface fluid/solid by means of a computational grid that does not follow the motion of the material, nor is fixed in space at all times.

One specific aspect of the technical references published to date, is the lack of a general view on the derivation of continuum mechanics equations on arbitrary moving domains. This is the aim and scope of the present work. Our approach is to present a comprehensive view on the kinematic problem of expressing the rate of change of the integral of a conserved quantity, when the domain moves with an arbitrary motion. This task can be accomplished in many ways, by the use of a number of integral transport theorems, each of which corresponds to a particular reference frame in which the time rate of an integral quantity can be measured (or *observed*). For the sake of generality, we develop a space-time approach to ALE formulations, which has the advantage of unifying in an even more decisive way the representation of a certain conservation law in different reference frames. The tool to move from a space-time reference frame to another is a space-time version of the Piola identity, which is stated, and, for the sake of completeness, proved.

Finally, in order to provide examples of application of the framework developed, the compressible and incompressible Navier-Stokes equations and the equations of compressible, isothermal, non-linear elasticity are derived. Additional considerations on the role of so-called geometric conservation laws [18] are presented, for their implication in numerical computations.

The rest of the exposition proceeds as follows: Section 1 presents a number of theorems on mappings which are fundamental in continuum mechanics, Section 2 is devoted to kinematics and its space-time interpretation, Section 3 details the proof of transport theorems, which are then used in Section 4 to write the various ALE *incarnations* of the master balance law. Section 5 presents a generalized space-time view on master balance laws. In Section 6 the master balance law is used to derive conservation laws, but no specific reference to the consitutive law of the materials is made. This aspect is postponed to Section 7, where the case of compressible and incompressible Navier-Stokes equations is discussed. A number of additional considerations on the role of geometric conservation laws and their impact on numerical computations is the scope of Section 8.

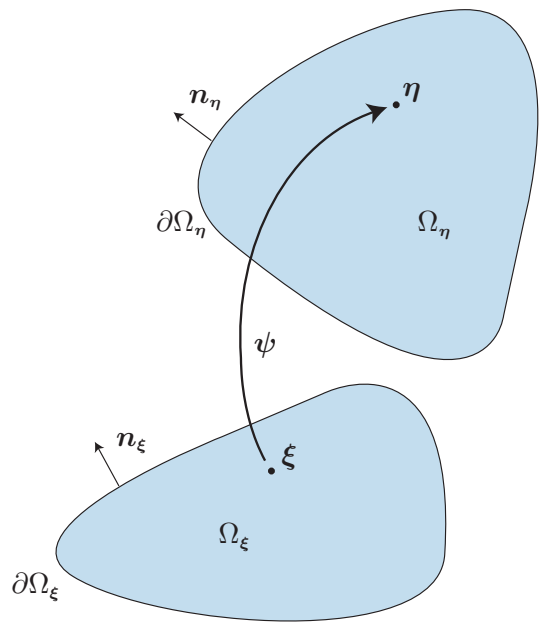

Figure 1. Sketch of a general map $\boldsymbol{\psi}$ between the reference frame associated with the coordinate $\boldsymbol{\xi}$ and the reference frame associated with the coordinate $\boldsymbol{\eta}$.

1 Some fundamental results of vector calculus

A number of fundamental results of vector calculus, of pervasive application in continuum mechanics, are recalled. In what follows, n_d indicates the number of space dimensions ($n_d = 2, 3$), $\Omega_{\boldsymbol{\xi}}$ and $\Omega_{\boldsymbol{\eta}}$ are open sets in $\mathbb{R}^{n_d}$, with boundaries $\partial\Omega_{\boldsymbol{\xi}}$ and $\partial\Omega_{\boldsymbol{\eta}}$, and outward normals $\boldsymbol{n}_{\boldsymbol{\xi}}$ and $\boldsymbol{n}_{\boldsymbol{\eta}}$, respectively. Let $\boldsymbol{\psi}$ be a *diffeomorphism* (which, without dwelling on a detailed technical discussion, can be thought of as a smooth invertible map between smooth manifolds with smooth inverse, as depicted in Fig. 1):

$$\boldsymbol{\psi} : \Omega_{\boldsymbol{\xi}} \quad \rightarrow \quad \Omega_{\boldsymbol{\eta}} = \boldsymbol{\psi}(\Omega_{\boldsymbol{\xi}}) \ , \tag{1}$$

$$\boldsymbol{\xi} \quad \mapsto \quad \boldsymbol{\eta} = \boldsymbol{\psi}(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \in \Omega_{\boldsymbol{\xi}}, \boldsymbol{\eta} \in \Omega_{\boldsymbol{\eta}} \ , \tag{2}$$

Although the following results apply in a more general sense, for the purpose of the present work, the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ may be thought of as two different sets of coordinates, associated with two reference frames. The *Jacobian matrix* of the map $\boldsymbol{\psi}$, namely,

$$\boldsymbol{F}_{\boldsymbol{\psi}} = \nabla_{\boldsymbol{\xi}} \boldsymbol{\psi} = \nabla_{\boldsymbol{\xi}} \boldsymbol{\eta} = \frac{\partial \boldsymbol{\eta}}{\partial \boldsymbol{\xi}}, \quad \text{or,} \quad (F_{\boldsymbol{\psi}})_{ij} = \frac{\partial \eta_i}{\partial \xi_j}, \quad \forall i, j \in \{1, \ldots, n_d\} \ , \tag{3}$$

is *non-singular*, since $\boldsymbol{\psi}$ is invertible, and its *Jacobian determinant* is defined as

$$J_{\boldsymbol{\psi}} = \det \boldsymbol{F}_{\boldsymbol{\psi}} \ . \tag{4}$$

The meaning of $J_{\boldsymbol{\psi}}$ as the change in volume due to the change of coordinates $\boldsymbol{\psi}$ can be easily understood from the following

Theorem 1 (Interpretation of J_ψ) In dimension $n_d = 3$, let $\mathrm{d}\Omega_{\boldsymbol{\xi}}$ be an infinitesimal domain, given by the triple product $\mathrm{d}\Omega_{\boldsymbol{\xi}} = \mathrm{d}\boldsymbol{\xi}^{(1)} \cdot \left(\mathrm{d}\boldsymbol{\xi}^{(2)} \wedge \mathrm{d}\boldsymbol{\xi}^{(3)}\right)$. Also let $\mathrm{d}\Omega_{\boldsymbol{\eta}} = \mathrm{d}\boldsymbol{\eta}^{(1)} \cdot \left(\mathrm{d}\boldsymbol{\eta}^{(2)} \wedge \mathrm{d}\boldsymbol{\eta}^{(3)}\right)$ be the push forward of $\mathrm{d}\Omega_{\boldsymbol{\xi}}$, that is $\mathrm{d}\boldsymbol{\eta}^{(i)} = \boldsymbol{F}_\psi \mathrm{d}\boldsymbol{\xi}^{(i)}$, for $i \in \{1, 2, 3\}$. Then:

$$\mathrm{d}\Omega_{\boldsymbol{\eta}} = J_\psi \mathrm{d}\Omega_{\boldsymbol{\xi}} \ . \tag{5}$$

Proof:
Recalling that the triple product $\mathrm{d}\boldsymbol{\eta}^{(1)} \cdot \left(\mathrm{d}\boldsymbol{\eta}^{(2)} \wedge \mathrm{d}\boldsymbol{\eta}^{(3)}\right)$ can be computed using the determinant of the matrix

$$\left[\ \mathrm{d}\boldsymbol{\eta}^{(1)} \ \middle|\ \mathrm{d}\boldsymbol{\eta}^{(2)} \ \middle|\ \mathrm{d}\boldsymbol{\eta}^{(3)}\ \right] \ , \tag{6}$$

one obtains

$$\begin{aligned}
\mathrm{d}\Omega_{\boldsymbol{\eta}} &= \det\left(\left[\ \mathrm{d}\boldsymbol{\eta}^{(1)} \ \middle|\ \mathrm{d}\boldsymbol{\eta}^{(2)} \ \middle|\ \mathrm{d}\boldsymbol{\eta}^{(3)}\ \right]\right) \\
&= \det\left(\left[\ \boldsymbol{F}_\psi \mathrm{d}\boldsymbol{\xi}^{(1)} \ \middle|\ \boldsymbol{F}_\psi \mathrm{d}\boldsymbol{\xi}^{(2)} \ \middle|\ \boldsymbol{F}_\psi \mathrm{d}\boldsymbol{\xi}^{(3)}\ \right]\right) \\
&= \det\left(\boldsymbol{F}_\psi \left[\ \mathrm{d}\boldsymbol{\xi}^{(1)} \ \middle|\ \mathrm{d}\boldsymbol{\xi}^{(2)} \ \middle|\ \mathrm{d}\boldsymbol{\xi}^{(3)}\ \right]\right) \\
&= \det\left(\boldsymbol{F}_\psi\right) \det\left(\left[\ \mathrm{d}\boldsymbol{\xi}^{(1)} \ \middle|\ \mathrm{d}\boldsymbol{\xi}^{(2)} \ \middle|\ \mathrm{d}\boldsymbol{\xi}^{(3)}\ \right]\right) \\
&= J_\psi \mathrm{d}\Omega_{\boldsymbol{\xi}} \ .
\end{aligned} \tag{7}$$

□

Remark 1 Notice that the previous proof can be easily reformulated in dimension $n_d = 2$, by extruding from a two-dimensional domain a three-dimensional prism. It can also be extended to $n_d > 3$, by observing that in the general case, the determinant of the matrix $[\mathrm{d}\boldsymbol{\xi}^{(1)}|\mathrm{d}\boldsymbol{\xi}^{(2)}|\ldots|\mathrm{d}\boldsymbol{\xi}^{(n_d)}]$ represents the volume measure of the hyper-parallelepiped spanned by the vectors $\mathrm{d}\boldsymbol{\xi}^{(1)}$, $\mathrm{d}\boldsymbol{\xi}^{(2)}$, $\ldots \mathrm{d}\boldsymbol{\xi}^{(n_d)}$.

Remark 2 The result in (5) can be generalized to infinitesimal domains of any shape.

Remark 3 The result in (5) is important, for example, whenever a change of coordinate in volume integrals needs to be performed.

In what follows, the map $\boldsymbol{\psi}$ is assumed not to invert domains, a condition implying $J_\psi > 0$. Recalling the definition of the *cofactor matrix* of an invertible matrix $\boldsymbol{A}$,

$$\mathbf{cof}\,\boldsymbol{A} = (\det \boldsymbol{A})\boldsymbol{A}^{-T} \ , \tag{8}$$

it is possible to state Nanson's theorems on normals:

Theorem 2 (Nanson's formula) With the previous definitions,

$$\boldsymbol{n}_{\boldsymbol{\eta}}\ \mathrm{d}(\partial\Omega_{\boldsymbol{\eta}}) = \mathbf{cof}\,\boldsymbol{F}_\psi \boldsymbol{n}_{\boldsymbol{\xi}}\ \mathrm{d}(\partial\Omega_{\boldsymbol{\xi}}) \ . \tag{9}$$

Proof:
A proof using vector calculus tools can be found in Odgen [24], p. 88, or Malvern [21], p. 169. Only a sketch of the proof is presented, since for a consistent derivation of Nanson's formula, more advanced differential geometry tools may be needed. The derivations loosely follow the appraoch of Odgen [24]. Let us consider the infinitesimal oriented surface $\boldsymbol{n}_{\boldsymbol{\xi}}\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}}) = \mathrm{d}\tilde{\boldsymbol{\xi}}^{(a)} \wedge \mathrm{d}\tilde{\boldsymbol{\xi}}^{(b)}$ ($\mathrm{d}\tilde{\boldsymbol{\xi}}^{(a)}$ and $\mathrm{d}\tilde{\boldsymbol{\xi}}^{(b)}$ lie on the tangent plane to the surface $\partial\Omega_{\boldsymbol{\xi}}$, with outward-pointing normal $\boldsymbol{n}_{\boldsymbol{\xi}}$). The inner product $\mathrm{d}\boldsymbol{\xi} \cdot \boldsymbol{n}_{\boldsymbol{\xi}}\ \mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})$, represents

the measure of the volume extruded from $\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})$ along the infinitesimal increment $\mathrm{d}\boldsymbol{\xi}$. The *push forward* through the map $\boldsymbol{\psi}$ of such volume transforms according to (5):

$$\mathrm{d}\boldsymbol{\eta}\cdot\boldsymbol{n}_{\boldsymbol{\eta}}\mathrm{d}(\partial\Omega_{\boldsymbol{\eta}}) \;=\; J_{\boldsymbol{\psi}}\;\left(\mathrm{d}\boldsymbol{\xi}\cdot\boldsymbol{n}_{\boldsymbol{\xi}}\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})\right)\;, \tag{10}$$

where $\boldsymbol{n}_{\boldsymbol{\eta}}\mathrm{d}(\partial\Omega_{\boldsymbol{\eta}}) = \mathrm{d}\tilde{\boldsymbol{\eta}}^{(a)}\wedge\mathrm{d}\tilde{\boldsymbol{\eta}}^{(b)}$, $\mathrm{d}\tilde{\boldsymbol{\eta}}^{(a)} = \boldsymbol{F}_{\boldsymbol{\psi}}\mathrm{d}\tilde{\boldsymbol{\xi}}^{(a)}$, $\mathrm{d}\tilde{\boldsymbol{\eta}}^{(b)} = \boldsymbol{F}_{\boldsymbol{\psi}}\mathrm{d}\tilde{\boldsymbol{\xi}}^{(b)}$, and $\mathrm{d}\boldsymbol{\eta} = \boldsymbol{F}_{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{\xi}$. Hence

$$\begin{aligned}
\mathrm{d}\boldsymbol{\eta}\cdot\boldsymbol{n}_{\boldsymbol{\eta}}\mathrm{d}(\partial\Omega_{\boldsymbol{\eta}}) &= J_{\boldsymbol{\psi}}\;\left(\mathrm{d}\boldsymbol{\xi}\cdot\boldsymbol{n}_{\boldsymbol{\xi}}\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})\right)\\
&= J_{\boldsymbol{\psi}}\;\boldsymbol{F}_{\boldsymbol{\psi}}^{-1}\mathrm{d}\boldsymbol{\eta}\cdot\boldsymbol{n}_{\boldsymbol{\xi}}\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})\\
&= \mathrm{d}\boldsymbol{\eta}\cdot\left(J_{\boldsymbol{\psi}}\;\boldsymbol{F}_{\boldsymbol{\psi}}^{-T}\boldsymbol{n}_{\boldsymbol{\xi}}\right)\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})\\
&= \mathrm{d}\boldsymbol{\eta}\cdot\left(\mathbf{cof}\,\boldsymbol{F}_{\boldsymbol{\psi}}\boldsymbol{n}_{\boldsymbol{\xi}}\right)\mathrm{d}(\partial\Omega_{\boldsymbol{\xi}})\;.
\end{aligned} \tag{11}$$

The result in (11) must hold for any vector $\mathrm{d}\boldsymbol{\eta}$. Since the inner product is a linear operator on a finite-dimensional vector space, (11) yields (9). □

Remark 4 The result just proved is very useful, for example, when changing coordinates in boundary integral involving normal fluxes.

Let $\nabla_{\boldsymbol{\xi}}$ be the gradient operator in the reference frame associated with the coordinate $\boldsymbol{\xi}$, and let us recall that the divergence operator acts on the last index of tensors, that is, if $\boldsymbol{A}(\boldsymbol{\xi})$ is a n-rank tensor (repeated index sum is implied throughout),

$$\nabla_{\boldsymbol{\xi}}\cdot\boldsymbol{A} = \partial_{\xi_{i_n}}A_{i_1 i_2\cdots i_n}\;. \tag{12}$$

Theorem 3 (Piola identity) With the previous definitions,

$$\nabla_{\boldsymbol{\xi}}\cdot\left(\mathbf{cof}\;\boldsymbol{F}_{\boldsymbol{\psi}}\right) = \mathbf{0}\;. \tag{13}$$

Proof:
The Piola identity can be proved in several ways, all very instructive on the type of manipulations that are commonly used in continuum mechanics. One possibility, presented next, is to apply the Gauss divergence theorem to an arbitrary *constant* vector field $\boldsymbol{f}$ and use Nanson's formula (9). Let $\omega_{\boldsymbol{\eta}}$ be an open subset of $\Omega_{\boldsymbol{\eta}}$, thus

$$\begin{aligned}
0 &= \int_{\omega_{\boldsymbol{\eta}}\subset\Omega_{\boldsymbol{\eta}}}\nabla_{\boldsymbol{\eta}}\cdot\boldsymbol{f}\\
&= \int_{\partial\omega_{\boldsymbol{\eta}}}\boldsymbol{f}\cdot\boldsymbol{n}_{\boldsymbol{\eta}}\\
&= \int_{\partial\omega_{\boldsymbol{\xi}}=\boldsymbol{\psi}^{-1}(\partial\omega_{\boldsymbol{\eta}})}\boldsymbol{f}\cdot\left(\mathbf{cof}\,\boldsymbol{F}_{\boldsymbol{\psi}}\boldsymbol{n}_{\boldsymbol{\xi}}\right)\\
&= \int_{\partial\omega_{\boldsymbol{\xi}}}\left(J_{\boldsymbol{\psi}}\boldsymbol{F}_{\boldsymbol{\psi}}^{-1}\boldsymbol{f}\right)\cdot\boldsymbol{n}_{\boldsymbol{\xi}}\\
&= \int_{\omega_{\boldsymbol{\xi}}}\nabla_{\boldsymbol{\xi}}\cdot\left(J_{\boldsymbol{\psi}}\boldsymbol{F}_{\boldsymbol{\psi}}^{-1}\boldsymbol{f}\right)\;.
\end{aligned} \tag{14}$$

Using index notation, it is possible to further manipulate the integrand in (14),

$$\begin{aligned}
\nabla_{\boldsymbol{\xi}} \cdot \left(J_{\boldsymbol{\psi}} \boldsymbol{F}_{\boldsymbol{\psi}}^{-1} \boldsymbol{f}\right) &= \partial_{\xi_i} \left(J_{\boldsymbol{\psi}} (F_{\boldsymbol{\psi}}^{-1})_{ij} f_j\right) \\
&= f_j \partial_{\xi_i} \left(J_{\boldsymbol{\psi}} (F_{\boldsymbol{\psi}}^{-1})_{ij}\right) \\
&= f_j \partial_{\xi_i} \left(J_{\boldsymbol{\psi}} (F_{\boldsymbol{\psi}}^{-T})_{ji}\right) \\
&= \boldsymbol{f} \cdot \left(\nabla_{\boldsymbol{\xi}} \cdot \left(J_{\boldsymbol{\psi}} \boldsymbol{F}_{\boldsymbol{\psi}}^{-T}\right)\right) \\
&= \boldsymbol{f} \cdot \left(\nabla_{\boldsymbol{\xi}} \cdot (\mathbf{cof} \boldsymbol{F}_{\boldsymbol{\psi}})\right) \ , \qquad (15)
\end{aligned}$$

where we recall that $\boldsymbol{f}$ is constant. Hence, (14) can be recast as

$$0 = \int_{\omega_{\boldsymbol{\xi}}} \nabla_{\boldsymbol{\xi}} \cdot \left(J_{\boldsymbol{\psi}} \boldsymbol{F}_{\boldsymbol{\psi}}^{-1} \boldsymbol{f}\right) = \boldsymbol{f} \cdot \left(\int_{\omega_{\boldsymbol{\xi}}} \nabla_{\boldsymbol{\xi}} \cdot (\mathbf{cof} \boldsymbol{F}_{\boldsymbol{\psi}})\right) \ . \qquad (16)$$

Recalling that $\boldsymbol{f}$ is arbitrary and $\boldsymbol{\psi}$ is smooth, and using the localization theorem in the limit of a domain $\omega_{\boldsymbol{\xi}}$ of vanishing measure, (16) implies

$$\nabla_{\boldsymbol{\xi}} \cdot (\mathbf{cof} \boldsymbol{F}_{\boldsymbol{\psi}}) = \mathbf{0} \ , \qquad (17)$$

which concludes the proof. □

Remark 5 The Piola identity is very useful when changing coordinates of integral formulations involving the divergence of a tensor or vector quantity.

Recalling the definition of the trace operator of a matrix $\boldsymbol{A}$, $\mathrm{tr}\boldsymbol{A} = A_{jj}$, we have now the following

Theorem 4 (Differentiation of a matrix determinant)

$$\mathrm{d}(\det \boldsymbol{A}) = \det \boldsymbol{A} \ \mathrm{tr}(\mathrm{d}\boldsymbol{A} \ \boldsymbol{A}^{-1}) \ . \qquad (18)$$

Proof:
There are a few ways of proving (18). In what follows, an approach using the properties of the cofactor matrix is proposed. For an alternative proof, one can reference to Gurtin [10], p. 23. Let us first recall the Laplace formula for the determinant of a matrix:

$$\det \boldsymbol{A} = \sum_k A_{kl} \ \mathrm{cof} A_{kl} = \sum_l A_{kl} \ \mathrm{cof} A_{kl} \qquad (19)$$

An alternative definition of the cofactor matrix is $\mathrm{cof} A_{ij} = (-1)^{i+j} \det(\mathrm{minor} A_{ij})$, where $\mathrm{minor} A_{ij}$ is the sub-matrix obtained by eliminating the entries of $\boldsymbol{A}$ on the row and column containing A_{ij}. Then, the differential of a matrix $\boldsymbol{A}$ can be expressed using the chain rule in terms of the differentials of its entries, namely

$$\begin{aligned}
\mathrm{d}(\det \boldsymbol{A}) &= \frac{\partial(\det \boldsymbol{A})}{\partial A_{ij}} \ \mathrm{d}A_{ij} \\
&= \frac{\partial}{\partial A_{ij}} \left(\sum_l A_{il} \ \mathrm{cof} A_{il}\right) \mathrm{d}A_{ij} \\
&= \mathrm{cof} A_{ij} \ \mathrm{d}A_{ij} \ , \qquad (20)
\end{aligned}$$

where, in the last step of the previous derivation, the fact that all entries of the type $\mathrm{cof} A_{il}$ do not contain A_{ij} has been used. Hence:

$$\begin{aligned}
\mathrm{d}(\det \boldsymbol{A}) &= \mathrm{cof} A_{ij} \mathrm{d} A_{ij} \\
&= \det \boldsymbol{A} \; A_{ij}^{-T} \mathrm{d} A_{ij} \\
&= \det \boldsymbol{A} \; A_{ji}^{-1} \mathrm{d} A_{ij} \\
&= \det \boldsymbol{A} \; \mathrm{tr}(\mathrm{d}\boldsymbol{A} \; \boldsymbol{A}^{-1}) \; , \qquad (21)
\end{aligned}$$

which concludes the proof. □

Remark 6 The previous result is very useful when computing the rate of change of volume due to a mapping, when replacing $\boldsymbol{A}$ with $\boldsymbol{F}_{\psi}$.

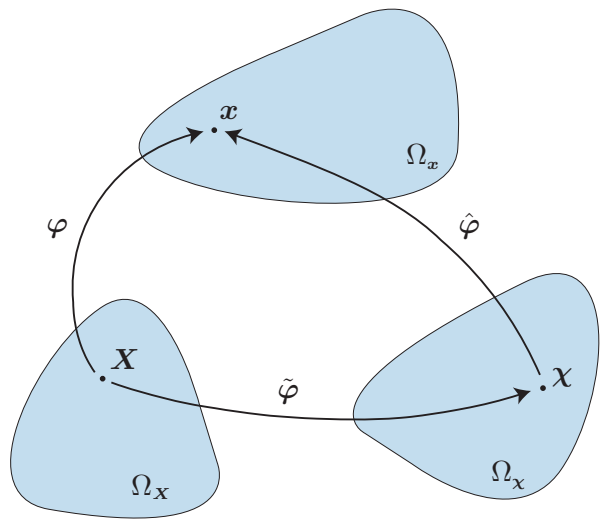

Figure 2. Sketch of the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\tilde{\boldsymbol{\varphi}}$.

2 Kinematics

A point of departure in the analysis of the kinematics of deformable bodies is to define the *material* (or *Lagrangian*), *referential*, and *current* (or *Eulerian*) reference frames, and the relative transformation maps. Then, expressions for the *Lagrangian* or *material* time derivative in such frames are derived, together with a number of very useful kinematic identities. The notation we adopt is inspired by [15, 2, 4, 7, 6], but significant differences are present. In what follows, $\Omega_{\boldsymbol{X}}$, $\Omega_{\boldsymbol{\chi}}$, and $\Omega_{\boldsymbol{x}}$ are open sets in $\mathbb{R}^{n_d}$ (see Fig. 2).

2.1 Lagrangian-to-Eulerian map $\boldsymbol{\varphi}$

The *motion* of a deformable body can be tracked in terms of the *deformation*, a *family* of diffeomorphic maps (see Fig. 2), parametrized by the time coordinate $t \in \mathbb{R}^+$:

$$\boldsymbol{\varphi}(\cdot, t) : \Omega_{\boldsymbol{X}} \quad \rightarrow \quad \Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}}, t) \ , \quad \forall t \geq 0 \ , \tag{22}$$

$$\boldsymbol{X} \quad \mapsto \quad \boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t), \quad \forall \boldsymbol{X} \in \Omega_{\boldsymbol{X}} \ . \tag{23}$$

Here $\boldsymbol{x}$ represent the current position of an infinitesimal material particle, *originally* at $\boldsymbol{X}$, which implies

$$\boldsymbol{\varphi}(\cdot, 0) = \boldsymbol{id}(\cdot) \ , \tag{24}$$

the identity map. $\Omega_{\boldsymbol{x}}(t)$ and $\Omega_{\boldsymbol{X}} = \Omega_{\boldsymbol{x}}(t = 0)$ are usually referred to as the *current* and the *original* configuration of the body, respectively.

Remark 7 Because any point $\boldsymbol{x} \in \Omega_{\boldsymbol{x}}(t)$, can be traced back to its original position $\boldsymbol{X} = \boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t) \in \Omega_{\boldsymbol{X}}$, $\boldsymbol{X}$ represents the material (or Lagrangian) coordinate of the point identified at time t by the Eulerian coordinate vector $\boldsymbol{x}$. Consequently, the motion of a deformable body can be observed and described from either the current configuration coordinate frame $\boldsymbol{x}$, or the original configuration coordinate frame $\boldsymbol{X}$ (see Fig. 2). The fact that $\boldsymbol{\varphi}$ is an omeomorphism makes these representations perfectly equivalent.

The displacement undergone by a material particle can be expressed, in view of (24), as

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{\varphi}(\boldsymbol{X},t) - \boldsymbol{\varphi}(\boldsymbol{X},0) \\ &= \boldsymbol{\varphi}(\boldsymbol{X},t) - \boldsymbol{X} \ , \end{aligned} \tag{25}$$

The velocity of a material particle identified by the Lagrangian coordinate $\boldsymbol{X}$, or *material velocity*, is defined as the increment in position per unit time, namely,

$$\boldsymbol{v} = \left.\frac{\partial \boldsymbol{\varphi}}{\partial t}\right|_{\boldsymbol{X}} = \dot{\boldsymbol{\varphi}} \ . \tag{26}$$

In view of (25), the material velocity can also be defined as the incremental displacement per unit time:

$$\boldsymbol{v} = \left.\frac{\partial \boldsymbol{u}}{\partial t}\right|_{\boldsymbol{X}} = \dot{\boldsymbol{u}} \ . \tag{27}$$

The *deformation gradient*, and the *Jacobian determinant* are defined as

$$\begin{aligned} \boldsymbol{F} &= \nabla_{\boldsymbol{X}} \boldsymbol{\varphi} = \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} \\ &= \nabla_{\boldsymbol{X}} \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \ , \end{aligned} \tag{28}$$

$$J = \det(\boldsymbol{F}) \ , \tag{29}$$

where, in index notation, $F_{ij} = \frac{\partial \varphi_i}{\partial X_j} = \frac{\partial x_i}{\partial X_j}$.

Remark 8 $J(\boldsymbol{X},t)$ represents the (local) ratio between the current and original volumes occupied by an infinitesimal particle of material, initially located at $\boldsymbol{X}$.

Using (26) in combination with the chain rule of differentiation, an expression for the Lagrangian time derivative of a scalar function α can be derived in terms of the Eulerian reference frame,

$$\begin{aligned} \dot{\alpha}(\boldsymbol{x},t) &= \left.\frac{\partial \alpha(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{X}} \\ &= \left.\frac{\partial \alpha(\boldsymbol{\varphi}(\boldsymbol{X},t),t)}{\partial t}\right|_{\boldsymbol{X}} \\ &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \alpha \cdot \left.\frac{\partial \boldsymbol{\varphi}}{\partial t}\right|_{\boldsymbol{X}} \\ &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \alpha \ . \end{aligned} \tag{30}$$

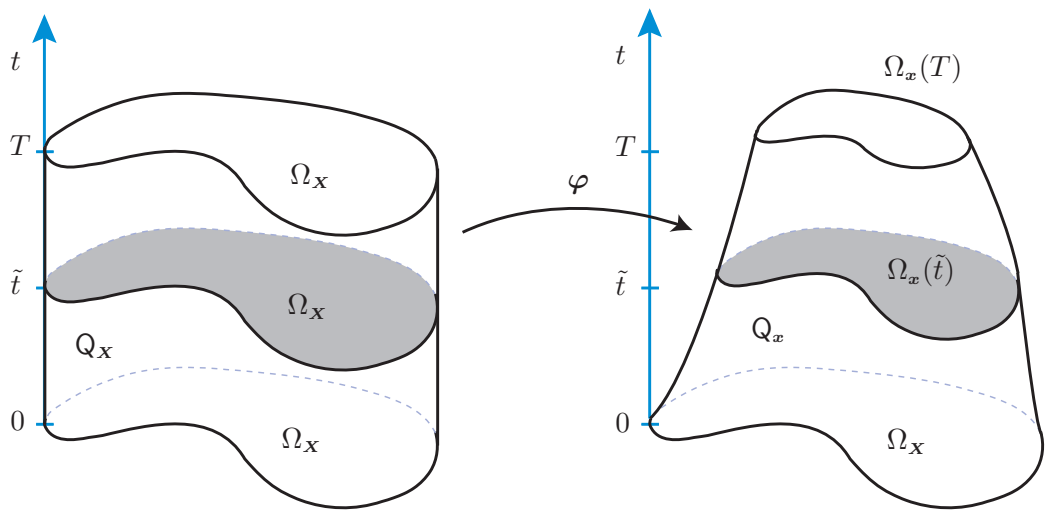

Figure 3. Space-time interpretation of the map $\boldsymbol{\varphi}$, with non-relativistic synchronization of times.

By means of (18), and recalling that, by the inverse mapping theorem, $\boldsymbol{F}^{-1} = \nabla_{\boldsymbol{x}} \boldsymbol{X}$, the Lagrangian time derivative of J can be expressed in terms of the current configuration velocity divergence, namely:

$$
\begin{aligned}
\dot{J} &= \left.\frac{\partial(\det \boldsymbol{F})}{\partial t}\right|_{\boldsymbol{X}} \\
&= J \operatorname{tr}\left(\left.\frac{\partial \boldsymbol{F}}{\partial t}\right|_{\boldsymbol{X}} \boldsymbol{F}^{-1} \right) \\
&= J \operatorname{tr}\left(\left(\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}} \right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \right) \\
&= J \operatorname{tr}\left(\left(\frac{\partial}{\partial \boldsymbol{X}} \left.\frac{\partial \boldsymbol{\varphi}}{\partial t}\right|_{\boldsymbol{X}} \right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \right) \\
&= J \operatorname{tr}\left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}} \right) \\
&= J \, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} \, .
\end{aligned} \tag{31}
$$

2.1.1 A generalized space-time perspective on the map $\boldsymbol{\varphi}$

More generally, the deformation $\boldsymbol{\varphi}$ can be interpreted as a transformation in space-time

$$
\boldsymbol{\varphi} : \quad \mathsf{Q}_{\boldsymbol{X}} \;\rightarrow\; \mathsf{Q}_{\boldsymbol{x}} = \boldsymbol{\varphi}(\mathsf{Q}_{\boldsymbol{X}}) \, , \text{ with } \mathsf{Q}_{\boldsymbol{X}} = \Omega_{\boldsymbol{X}} \times [0, T] \, , \tag{32}
$$

$$
(\boldsymbol{X}, \tau) \;\mapsto\; (\boldsymbol{x}, t) = \boldsymbol{\varphi}((\boldsymbol{X}, \tau)), \quad \forall (\boldsymbol{X}, \tau) \in \mathsf{Q}_{\boldsymbol{X}} \, . \tag{33}
$$

In what follows we will assume that the mechanical system under consideration is *non-relativistic*, so that *synchronization of times* ($t = \tau$) is always implied. Hence, the chain rule $\nabla_{\boldsymbol{X}}(\cdot) = \boldsymbol{F}^T \nabla_{\boldsymbol{x}}(\cdot)$ and (30) can

be used to define a generalized *space-time deformation gradient*,

$$\mathsf{F} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{v} & \boldsymbol{F} \end{bmatrix} . \tag{34}$$

Defining

$$\nabla_{[t,\boldsymbol{X}]} = \begin{Bmatrix} \partial_t|_{\boldsymbol{X}} \\ \nabla_{\boldsymbol{X}} \end{Bmatrix} , \tag{35}$$

$$\nabla_{[t,\boldsymbol{x}]} = \begin{Bmatrix} \partial_t|_{\boldsymbol{x}} \\ \nabla_{\boldsymbol{x}} \end{Bmatrix} , \tag{36}$$

it is straightforward to observe that (34) yields a transformation rule for space-time gradients:

$$\nabla_{[t,\boldsymbol{X}]} = \mathsf{F}^T \nabla_{[t,\boldsymbol{x}]} . \tag{37}$$

Using the Shur complement formula (see [12], p. 21),

$$\mathsf{F}^{-1} = \begin{bmatrix} 1 & \mathbf{0}^T \\ -\boldsymbol{F}^{-1}\boldsymbol{v} & \boldsymbol{F}^{-1} \end{bmatrix} . \tag{38}$$

Finally, applying Laplace formula (19) along the the first row of F to compute J,

$$\mathsf{J} = \det(\mathsf{F}) = J , \tag{39}$$

due to the mentioned synchronization of times, as there is neither stretch nor contraction along the time axis. We conclude this section with the proof of the space-time version of the Piola identity, of great importance in the development that follows.

Homework 1 Use the chain rule to show that (37) holds. Also invert by blocks the linear system $\mathsf{F}\mathsf{x} = \mathsf{b}$, to show that (38) is true.

Theorem 5 (Space-time Piola identity) The following identity holds:

$$\nabla_{[t,\boldsymbol{X}]} \cdot (\mathbf{cof}\ \mathsf{F}) = \mathbf{0} . \tag{40}$$

Proof:
Applying (13) to the map $\boldsymbol{\varphi}$, we obtain:

$$\nabla_{\boldsymbol{X}} \cdot (\mathbf{cof}\ \boldsymbol{F}) = \mathbf{0} , \tag{41}$$

with $\mathbf{cof}\ \boldsymbol{F} = J\boldsymbol{F}^{-T}$. Using (38), (39) and (41),

$$\begin{aligned} \nabla_{[t,\boldsymbol{X}]} \cdot (\mathbf{cof}\ \mathsf{F}) &= \nabla_{[t,\boldsymbol{X}]} \cdot (\mathsf{J}\mathsf{F}^{-T}) \\ &= \nabla_{[t,\boldsymbol{X}]} \cdot \left(J \begin{bmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{v} & \boldsymbol{F} \end{bmatrix}^{-T} \right) \\ &= \nabla_{[t,\boldsymbol{X}]} \cdot \left(\begin{bmatrix} J & -\boldsymbol{v}^T J\boldsymbol{F}^{-T} \\ \mathbf{0} & J\boldsymbol{F}^{-T} \end{bmatrix} \right) \\ &= \begin{bmatrix} \dot{J} - \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{v}^T J\boldsymbol{F}^{-T}) \\ \nabla_{\boldsymbol{X}} \cdot (\mathbf{cof}\ \boldsymbol{F}) \end{bmatrix} \\ &= \begin{bmatrix} \dot{J} - \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{v}^T J\boldsymbol{F}^{-T}) \\ \mathbf{0} \end{bmatrix} . \end{aligned} \tag{42}$$

It remains to verify that $\dot{J} - \nabla_{\boldsymbol{X}} \cdot (\boldsymbol{v}^T J \boldsymbol{F}^{-T}) = 0$. Using the index notation (sum on repeated indices is assumed), applying (41), and recalling (31),

$$
\begin{aligned}
\nabla_{\boldsymbol{X}} \cdot (\boldsymbol{v}^T J \boldsymbol{F}^{-T}) &= \frac{\partial}{\partial X_j}(J v_i F_{ij}^{-T}) \\
&= \frac{\partial v_i}{\partial X_j} J F_{ij}^{-T} + v_i \, \frac{\partial (J F_{ij}^{-T})}{\partial X_j} \\
&= J \, \frac{\partial v_i}{\partial X_j} \, F_{ji}^{-1} + \boldsymbol{v}^T \underbrace{(\nabla_{\boldsymbol{X}} \cdot (J \boldsymbol{F}^{-T}))}_{= 0 \text{ by } (41)} \\
&= J \frac{\partial v_i}{\partial X_j} \frac{\partial X_j}{\partial x_i} \\
&= J \, \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} \\
&= \dot{J} \, . \qquad (43)
\end{aligned}
$$

This concludes the proof. □

With arguments similar to the ones developed in Theorem 2, it is possible to derive a space-time Nanson's formula. Derivations are left to the reader.

2.2 Referential-to-Eulerian map $\hat{\boldsymbol{\varphi}}$

Computational methods on moving domains make use of a third reference frame, which is neither fixed in space nor attached to the material, but moves in time with an arbitrary motion. Usually, a notion of a computational grid or *mesh* is associated to this frame of reference. In an abstract sense, one can think of the mesh as a "virtual material" which deforms according to a law different from the "real material". Therefore, in close analogy with the Lagrangian-to-Eulerian case, one can define a a parametrized family of diffeomorphisms *referential map* $\hat{\boldsymbol{\varphi}}$ (see Fig. 2) such that:

$$
\hat{\boldsymbol{\varphi}}(\cdot, t) : \Omega_{\boldsymbol{\chi}} \;\rightarrow\; \Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}}, t) \, , \quad \forall t \geq 0 \, , \qquad (44)
$$

$$
\boldsymbol{\chi} \;\mapsto\; \boldsymbol{x} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t), \quad \forall \boldsymbol{\chi} \in \Omega_{\boldsymbol{\chi}} \, , \qquad (45)
$$

where $\boldsymbol{\chi}$ is the point vector in the referential frame. $\Omega_{\boldsymbol{\chi}}$, the domain occupied by the body in the referential frame, is mapped to $\Omega_{\boldsymbol{x}}$ by $\hat{\boldsymbol{\varphi}}$, and, again, this implies that $\hat{\boldsymbol{\varphi}}(\cdot, 0) = \boldsymbol{id}(\cdot)$. *Mesh displacements* and *velocities* can be expressed as

$$
\begin{aligned}
\hat{\boldsymbol{u}} &= \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, 0) \\
&= \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t) - \boldsymbol{\chi} \, , \qquad (46)
\end{aligned}
$$

$$
\hat{\boldsymbol{v}} = \left. \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial t} \right|_{\boldsymbol{\chi}} = \left. \frac{\partial \hat{\boldsymbol{u}}}{\partial t} \right|_{\boldsymbol{\chi}} \, . \qquad (47)
$$

In addition, the *mesh deformation gradient* and the *mesh Jacobian determinant* are defined as:

$$\begin{aligned}
\hat{\boldsymbol{F}} &= \nabla_{\boldsymbol{\chi}} \hat{\boldsymbol{\varphi}} = \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \boldsymbol{\chi}} \\
&= \nabla_{\boldsymbol{\chi}} \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\chi}} \,, && (48) \\
\hat{J} &= \det(\hat{\boldsymbol{F}}) \,. && (49)
\end{aligned}$$

The referential time derivative of a scalar function α reads

$$\begin{aligned}
\left.\frac{\partial \alpha(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{\chi}} &= \left.\frac{\partial \alpha(\hat{\boldsymbol{\varphi}}(\boldsymbol{\chi},t),t)}{\partial t}\right|_{\boldsymbol{\chi}} \\
&= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \alpha \cdot \left.\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{\chi}} \\
&= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}} \alpha \,, && (50)
\end{aligned}$$

and, using again (18), the referential time derivative of $\hat{J}$ is given by

$$\begin{aligned}
\left.\frac{\partial \hat{J}}{\partial t}\right|_{\boldsymbol{\chi}} &= \left.\frac{\partial (\det \hat{\boldsymbol{F}})}{\partial t}\right|_{\boldsymbol{\chi}} \\
&= \hat{J} \ \mathrm{tr}\left(\left.\frac{\partial \hat{\boldsymbol{F}}}{\partial t}\right|_{\boldsymbol{\chi}} \hat{\boldsymbol{F}}^{-1} \right) \\
&= \hat{J} \ \mathrm{tr}\left(\left(\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{\chi}} \frac{\partial \hat{\boldsymbol{\varphi}}}{\partial \boldsymbol{\chi}} \right) \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{x}} \right) \\
&= \hat{J} \ \mathrm{tr}\left(\left(\frac{\partial}{\partial \boldsymbol{\chi}} \left.\frac{\partial \hat{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{\chi}} \right) \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{x}} \right) \\
&= \hat{J} \ \mathrm{tr}\left(\frac{\partial \hat{\boldsymbol{v}}}{\partial \boldsymbol{\chi}} \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{x}} \right) \\
&= \hat{J} \ \nabla_{\boldsymbol{x}} \cdot \hat{\boldsymbol{v}} \,. && (51)
\end{aligned}$$

2.2.1 A generalized space-time perspective on the map $\hat{\boldsymbol{\varphi}}$

Assuming synchronization of times, $\hat{\boldsymbol{\varphi}}$ can also be interpreted as a space-time map:

$$\begin{aligned}
\hat{\boldsymbol{\varphi}} : \quad \mathsf{Q}_{\boldsymbol{\chi}} &\rightarrow \mathsf{Q}_{\boldsymbol{x}} = \boldsymbol{\varphi}(\mathsf{Q}_{\boldsymbol{\chi}}), \quad \text{with } \mathsf{Q}_{\boldsymbol{\chi}} = \Omega_{\boldsymbol{\chi}} \times [0,T] \,, && (52) \\
(\boldsymbol{\chi},t) &\mapsto (\boldsymbol{x},t) = \hat{\boldsymbol{\varphi}}((\boldsymbol{\chi},t)), \quad \forall (\boldsymbol{\chi},t) \in \mathsf{Q}_{\boldsymbol{\chi}} \,. && (53)
\end{aligned}$$

A *space-time mesh deformation gradient* for the map $\hat{\boldsymbol{\varphi}}$ can be defined as

$$\hat{\mathsf{F}} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \hat{\boldsymbol{v}} & \hat{\boldsymbol{F}} \end{bmatrix} \,, \tag{54}$$

with the following transformation rule for space-time referential gradients

$$\nabla_{[t,\boldsymbol{\chi}]} = \hat{\mathsf{F}}^T \nabla_{[t,\boldsymbol{x}]} , \tag{55}$$

where

$$\nabla_{[t,\boldsymbol{\chi}]} = \left\{ \begin{array}{c} \partial_t|_{\boldsymbol{\chi}} \\ \nabla_{\boldsymbol{\chi}} \end{array} \right\} . \tag{56}$$

In perfect analogy with the Lagrangian-to-Eulerian case,

$$\hat{\mathsf{F}}^{-1} = \left[\begin{array}{cc} 1 & \mathbf{0}^T \\ -\hat{\boldsymbol{F}}^{-1}\hat{\boldsymbol{v}} & \hat{\boldsymbol{F}}^{-1} \end{array} \right] , \tag{57}$$

$$\hat{\mathsf{J}} = \det(\hat{\mathsf{F}}) = \hat{J} . \tag{58}$$

Homework 2 Prove (55) and (57).

A space-time version of the referential Piola identity also holds:

Theorem 6 (Referential space-time Piola identity) The following identity holds:

$$\nabla_{[t,\boldsymbol{\chi}]} \cdot (\mathbf{cof}\ \hat{\mathsf{F}}) = \mathbf{0} . \tag{59}$$

Proof:
The proof is virtually identical to the case of the Lagrangian-to-Eulerian map $\boldsymbol{\varphi}$, once F, J, $\boldsymbol{F}$, J, $\boldsymbol{v}$, $\nabla_{[t,\boldsymbol{X}]}$, $\partial_t|_{\boldsymbol{X}}$, $\nabla_{\boldsymbol{X}}$ are replaced with $\hat{\mathsf{F}}$, $\hat{\mathsf{J}}$, $\hat{\boldsymbol{F}}$, $\hat{J}$, $\hat{\boldsymbol{v}}$, $\nabla_{[t,\boldsymbol{\chi}]}$, $\partial_t|_{\boldsymbol{\chi}}$, $\nabla_{\boldsymbol{\chi}}$, respectively. The version of the Piola identity in space to be used for the proof reads $\nabla_{\boldsymbol{\chi}} \cdot (\mathbf{cof}\ \hat{\boldsymbol{F}}) = \mathbf{0}$. □

2.3 Lagrangian-to-Referential map $\tilde{\boldsymbol{\varphi}}$

Finally, it is also important to consider the *Lagrangian-to-referential* transformation, which tracks the motion of the referential frame, *observed* from the Lagrangian reference frame,

$$\tilde{\boldsymbol{\varphi}}(\cdot,t) : \Omega_{\boldsymbol{X}} \rightarrow \Omega_{\boldsymbol{\chi}} = \tilde{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{X}},t), \quad \forall t \geq 0 , \tag{60}$$

$$\boldsymbol{X} \mapsto \boldsymbol{\chi} = \tilde{\boldsymbol{\varphi}}(\boldsymbol{X},t), \quad \forall \boldsymbol{X} \in \Omega_{\boldsymbol{X}} . \tag{61}$$

The previous map has to be interpreted as the following composition:

$$\tilde{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}^{-1} \circ \boldsymbol{\varphi} . \tag{62}$$

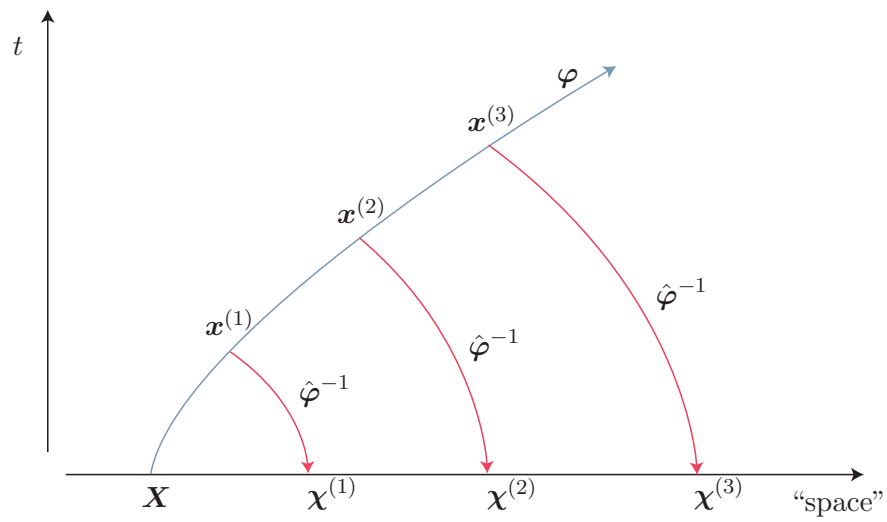

Figure 4. Interpretation of the displacement $\tilde{\boldsymbol{u}}$. Respectively, $\boldsymbol{\chi}^{(i)}$ and $\boldsymbol{X}$ represent the initial locations of a material particle and a "mesh particle" (e.g., a node, the baricenter of a cell, etc.), overlapping at time t in location $\boldsymbol{x}^{(i)}$. Hence, the nature of $\tilde{\boldsymbol{u}}^{(i)} = \boldsymbol{\chi}^{(i)} - \boldsymbol{X}$, $i = 1, 2, 3$, is closer to a distance vector than a true displacement. The kinematic interpretation of $\tilde{\boldsymbol{v}} = \partial_t \boldsymbol{\chi}|_{\boldsymbol{X}}$ is analogous.

Assuming $\tilde{\boldsymbol{\varphi}}(\cdot, 0) = \boldsymbol{id}(\cdot)$, $\tilde{\boldsymbol{u}}$ is the displacement undergone by the mesh, *observed* from the Lagrangian reference frame (see Fig. 4),

$$\begin{aligned} \tilde{\boldsymbol{u}} &= \tilde{\boldsymbol{\varphi}}(\boldsymbol{X}, t) - \tilde{\boldsymbol{\varphi}}(\boldsymbol{X}, 0) \\ &= \tilde{\boldsymbol{\varphi}}(\boldsymbol{X}, t) - \boldsymbol{X} \ . \end{aligned} \tag{63}$$

The velocity of the referential frame observed from the Lagrangian frame is

$$\tilde{\boldsymbol{v}} = \left.\frac{\partial \tilde{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{X}} = \left.\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}\right|_{\boldsymbol{X}} \ . \tag{64}$$

The Lagrangian-to-referential deformation gradient and Jacobian determinant are

$$\begin{aligned} \tilde{\boldsymbol{F}} &= \nabla_{\boldsymbol{\chi}} \tilde{\boldsymbol{\varphi}} = \frac{\partial \tilde{\boldsymbol{\varphi}}}{\partial \boldsymbol{X}} \\ &= \nabla_{\boldsymbol{\chi}} \boldsymbol{x} = \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{X}} \ , \end{aligned} \tag{65}$$

$$\tilde{J} = \det(\tilde{\boldsymbol{F}}) \ . \tag{66}$$

Using the chain rule, it easy to derive

$$\dot{\alpha}(\boldsymbol{\chi}, t) = \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{\chi}} + \tilde{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{\chi}} \alpha \ . \tag{67}$$

In perfect analogy to the Lagrangian-to-Eulerian map $\boldsymbol{\varphi}$, the reader can easily verify the following identity for the Lagrangian time derivative of $\tilde{J}$:

$$\dot{\tilde{J}} = \tilde{J}\ \nabla_{\boldsymbol{\chi}} \cdot \tilde{\boldsymbol{v}}\ . \tag{68}$$

Also in this case, a space-time Nanson's formula can be derived with arguments similar to the ones developed in Theorem 2. Derivations are left to the reader.

Homework 3 Prove (67) and (68).

2.3.1 A generalized space-time perspective on the map $\tilde{\boldsymbol{\varphi}}$

The map $\tilde{\boldsymbol{\varphi}}$, interpreted in a space-time context yields:

$$\begin{aligned} \tilde{\boldsymbol{\varphi}} : \quad \mathsf{Q}_{\boldsymbol{X}} &\rightarrow \mathsf{Q}_{\boldsymbol{\chi}} = \boldsymbol{\varphi}(\mathsf{Q}_{\boldsymbol{X}})\ , && (69)\\ (\boldsymbol{X}, t) &\mapsto (\boldsymbol{\chi}, t) = \hat{\boldsymbol{\varphi}}((\boldsymbol{X}, t)), \quad \forall (\boldsymbol{X}, t) \in \mathsf{Q}_{\boldsymbol{X}}\ . && (70) \end{aligned}$$

The *space-time deformation gradient* reads:

$$\tilde{\mathsf{F}} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \tilde{\boldsymbol{v}} & \tilde{\boldsymbol{F}} \end{bmatrix}\ , \tag{71}$$

so that

$$\begin{aligned} \tilde{\mathsf{J}} &= \det(\tilde{\mathsf{F}}) = \tilde{J}\ , && (72)\\ \tilde{\mathsf{F}}^{-1} &= \begin{bmatrix} 1 & \mathbf{0}^T \\ -\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{v}} & \tilde{\boldsymbol{F}}^{-1} \end{bmatrix}\ , && (73)\\ \nabla_{[t,\boldsymbol{X}]} &= \tilde{\mathsf{F}}^T \nabla_{[t,\boldsymbol{\chi}]}\ , && (74) \end{aligned}$$

and

Theorem 7 (Referential space-time Piola identity) The following identity holds:

$$\nabla_{[t,\boldsymbol{X}]} \cdot (\mathbf{cof}\ \tilde{\mathsf{F}}) = \mathbf{0}\ . \tag{75}$$

Proof:
Replace F, J, $\boldsymbol{F}$, J, $\boldsymbol{v}$, $\nabla_{[t,\boldsymbol{X}]}$, $\partial_t|_{\boldsymbol{X}}$, $\nabla_{\boldsymbol{X}}$ in the Lagrangian-to-Eulerian map case with $\tilde{\mathsf{F}}$, $\tilde{\mathsf{J}}$, $\tilde{\boldsymbol{F}}$, $\tilde{J}$, $\tilde{\boldsymbol{v}}$, $\nabla_{[t,\boldsymbol{X}]}$, $\partial_t|_{\boldsymbol{X}}$, $\nabla_{\boldsymbol{X}}$, respectively, and use the (spatial) Piola identity $\nabla_{\boldsymbol{X}} \cdot (\mathbf{cof}\ \tilde{\boldsymbol{F}}) = \mathbf{0}$. □

A space-time Nanson's formula can be derived with arguments similar to the ones developed in Theorem 2. Derivations are left to the reader.

Homework 4 Prove (73) and (74).

A sketch of the main results developed in this Section is presented in Figure 5.

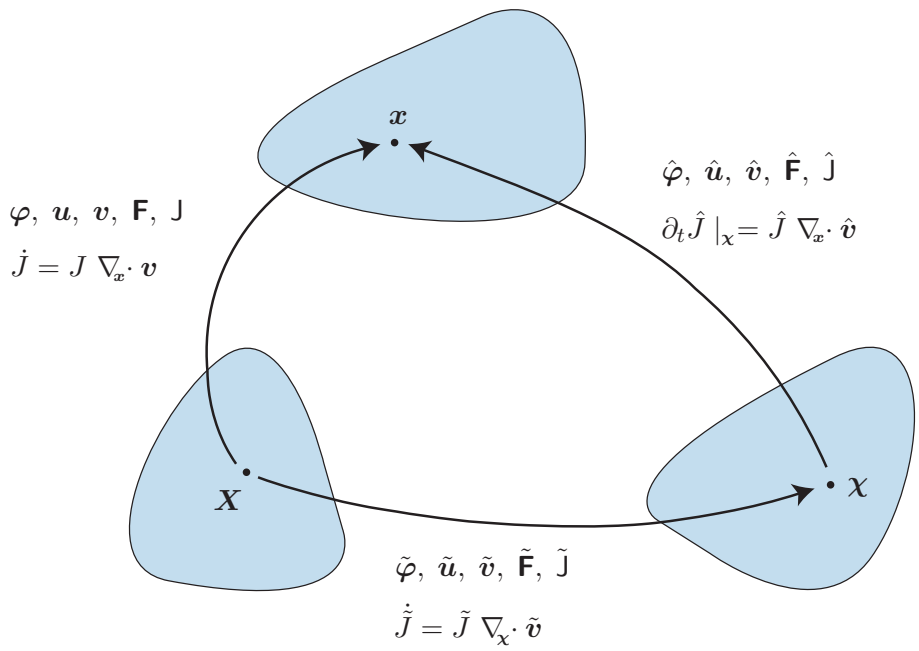

Figure 5. Sketch of the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\tilde{\boldsymbol{\varphi}}$ and their relative kinematic quantities and identities.

2.4 Two fundamental kinematic relationships

Applying (67) to the components of the position vector $\boldsymbol{x} = \hat{\boldsymbol{\varphi}}(\boldsymbol{\chi}, t)$ yields a relationship between $\boldsymbol{v}$, $\hat{\boldsymbol{v}}$, and $\tilde{\boldsymbol{v}}$:

$$\boldsymbol{v} = \hat{\boldsymbol{v}} + \hat{\boldsymbol{F}}\tilde{\boldsymbol{v}} \ , \tag{76}$$

which can be recast as

$$\boldsymbol{c} = \boldsymbol{v} - \hat{\boldsymbol{v}} = \hat{\boldsymbol{F}}\tilde{\boldsymbol{v}} \ , \tag{77}$$

where $\boldsymbol{c}$ is termed *convective velocity*, the difference between the material and mesh velocities.

Combining (30) with (50), an alternative expression for the Lagrangian time derivative can be obtained:

$$\begin{aligned} \dot{\alpha}(\boldsymbol{x}, t) &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \alpha \\ &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{\chi}} - \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}} \alpha + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \alpha \\ &= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} \alpha \end{aligned} \tag{78}$$

This expression becomes particularly useful when deriving the advective (non-conservative) version of the equations governing mechanical systems, as shown in Section 6.4.

Remark 9 Expression (78) can also be derived from (67) using (77). In fact, $\tilde{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{\chi}} \alpha = (\hat{\boldsymbol{F}}^{-1}\boldsymbol{c}) \cdot \nabla_{\boldsymbol{\chi}} \alpha = \boldsymbol{c} \cdot (\hat{\boldsymbol{F}}^{-T} \nabla_{\boldsymbol{\chi}} \alpha) = \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} \alpha$.

3 Transport theorems

To each of the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\tilde{\boldsymbol{\varphi}}$ corresponds an appropriate transport theorem, that is, an integral kinematic identity for the computation of the rate of change of integrals on lines, surfaces and volumes. Transport theorems express the time derivative of some integral quantity using the properties of maps between reference frames. In a practical sense, the aim of transport theorems is to compute the rate of change of an integral over a moving curve/surface/volume in terms of integrals over a curve/surface/volume that *coincides* with the former at a certain instant of time, without necessarily sharing the same motion. To avoid confusion in the sometimes subtle statements of the theorems, it is important to focus the attention on *what* is the motion undergone by the domain over which an integral rate of change is computed. For the sake of clarity, the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\tilde{\boldsymbol{\varphi}}$ are explicitly indicated.

3.1 Transport theorems for volume integrals

3.1.1 $\boldsymbol{\varphi}$ and the classical Reynolds transport theorem

Theorem 8 Let $\Omega_{\boldsymbol{x}}$ be a material domain which deforms according to the diffeomorphism $\boldsymbol{\varphi}$, with velocity $\boldsymbol{v} = \partial_t \boldsymbol{\varphi}|_{\boldsymbol{X}}$. Let $\partial\Omega_{\boldsymbol{x}}$ be the boundary of $\Omega_{\boldsymbol{x}}$, with outward normal $\boldsymbol{n}_{\boldsymbol{x}}$ and α a scalar field. Let $\Omega_{\boldsymbol{X}}$ be the inverse image of $\Omega_{\boldsymbol{x}}$ by means of $\boldsymbol{\varphi}$, that is, $\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \alpha \;=\; \int_{\Omega_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \int_{\partial\Omega_{\boldsymbol{x}}} \alpha\boldsymbol{v}\cdot\boldsymbol{n}_{\boldsymbol{x}} \;. \tag{79}$$

Proof:
Pulling the integrals back to the original configuration yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}\,=\,\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \alpha \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{X}}} J\alpha \;=\; \int_{\Omega_{\boldsymbol{X}}} \left.\frac{\partial(J\alpha)}{\partial t}\right|_{\boldsymbol{X}} \;=\; \int_{\Omega_X} \dot{\alpha}J + \alpha\dot{J} \;, \tag{80}$$

where the time derivative can be moved inside the integral over $\Omega_{\boldsymbol{X}}$, since $\Omega_{\boldsymbol{X}}$ is fixed in time with respect to the map $\boldsymbol{\varphi}$ (see Fig. 3). Now, by (30) and (31),

$$\begin{aligned}
\dot{\alpha}J + \alpha\dot{J} &= \dot{\alpha}J + \alpha J\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} \\
&= J\left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\alpha + \alpha\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}\right) \\
&= J\left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}}\cdot(\alpha\boldsymbol{v})\right) \;.
\end{aligned} \tag{81}$$

Pushing forward to the domain $\Omega_{\boldsymbol{x}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}} \alpha \;=\; \int_{\Omega_{\boldsymbol{x}}} \left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}}\cdot(\alpha\boldsymbol{v})\right) \;. \tag{82}$$

Applying the Gauss divergence theorem to the last integrand on the right-hand side of (82) yields (79). □

Remark 10 The point of view throughout the proof of the Reynolds transport theorem is Lagrangian, in the sense that the Lagrangian-to-Eulerian map $\boldsymbol{\varphi}$ is used, and $\Omega_{\boldsymbol{x}}$ is treated as a material domain. However, since the time derivative on the right of the equal sign of (79) is found under the integration sign, we discover that the $\Omega_{\boldsymbol{x}}$ and its boundary $\partial\Omega_{\boldsymbol{x}}$ can be interpreted as an arbitrary domain and its boundary, coinciding with $\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})$ and $\boldsymbol{\varphi}(\partial\Omega_{\boldsymbol{X}})$ at time t. In fact, the information about the motion of the the material domain $\Omega_{\boldsymbol{x}}$ is carried by the presence of the material velocity $\boldsymbol{v} = \partial_t\boldsymbol{\varphi}|_{\boldsymbol{X}}$ in the boundary integral over $\partial\Omega_{\boldsymbol{x}}$.

Remark 11 The integral over $\partial\Omega_{\boldsymbol{x}}$ can also be thought of as the flux of material across the boundary of a fixed (Eulerian) domain $\Omega_{\boldsymbol{x}}$. In this sense, the Reynolds transport theorem is the bridge between the Lagrangian and Eulerian reference frames.

3.1.2 $\hat{\boldsymbol{\varphi}}$ and the Leibnitz transport theorem

The Leibnitz transport theorem computes the rate of change of an integral quantity over a domain that undergoes an arbitrary motion, tracked by the map $\hat{\boldsymbol{\varphi}}$. In particular, if $\hat{\boldsymbol{\varphi}} \equiv \boldsymbol{\varphi}$, then the Leibnitz transport theorem collapses to the classical Reynolds transport theorem. The proof is hard to find in the available literature on ALE methods, and follows the approach presented in [30].

Theorem 9 Let $\Omega_{\boldsymbol{x}}$ be an arbitrary control volume $\Omega_{\boldsymbol{x}}$ which deforms according to the diffeomorphism $\hat{\boldsymbol{\varphi}}$, with velocity $\hat{\boldsymbol{v}} = \partial_t\hat{\boldsymbol{\varphi}}|_{\boldsymbol{\chi}}$. Let $\partial\Omega_{\boldsymbol{x}}$ be the boundary of $\Omega_{\boldsymbol{x}}$, with outward normal $\boldsymbol{n}_{\boldsymbol{x}}$, and α a scalar field. Let $\Omega_{\boldsymbol{\chi}}$ be the inverse image of $\Omega_{\boldsymbol{x}}$ by means of $\hat{\boldsymbol{\varphi}}$, that is, $\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \alpha = \int_{\Omega_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \int_{\partial\Omega_{\boldsymbol{x}}} \alpha\hat{\boldsymbol{v}}\cdot\boldsymbol{n}_{\boldsymbol{x}} \ . \tag{83}$$

Proof:
Recalling that $\hat{\boldsymbol{\varphi}}(\cdot,t) = \boldsymbol{id}$, it is clear that $\Omega_{\boldsymbol{\chi}}$ is fixed in time with respect to $\hat{\boldsymbol{\varphi}}$. Using (50) and (51),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \alpha &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{\chi}}} \alpha\hat{J} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\alpha\hat{J})}{\partial t}\right|_{\boldsymbol{\chi}} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \hat{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{x}}\alpha + \alpha\nabla_{\boldsymbol{x}}\cdot\hat{\boldsymbol{v}}\right)\hat{J} \\
&= \int_{\Omega_{\boldsymbol{x}}} \left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}}\cdot(\alpha\hat{\boldsymbol{v}})\right) \ .
\end{aligned} \tag{84}$$

The Gauss divergence theorem turns (84) into (83), which concludes the proof. □

3.1.3 $\tilde{\boldsymbol{\varphi}}$ and the generalized Reynolds transport theorem

The last transport theorem to be introduced is the one relative to the map $\tilde{\boldsymbol{\varphi}}$, sometimes called the generalized Reynolds transport theorem. This theorem is somewhat less intuitive, but it is frequently used (see, e.g., [15, 2]). The proof follows the approach presented in [29].

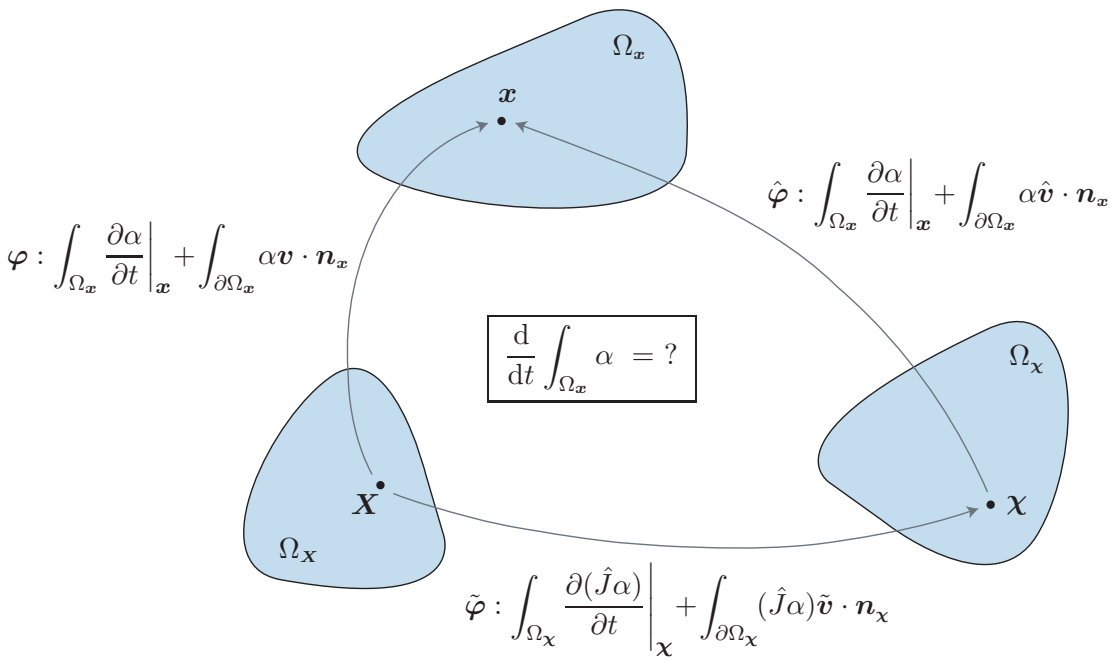

Figure 6. Sketch of the transport theorems for volume integrals, corresponding to the maps $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $\tilde{\boldsymbol{\varphi}}$. Notice that in the case of the map $\boldsymbol{\varphi}$, $\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})$ follows the deformation of the material, while in the case of the map $\hat{\boldsymbol{\varphi}}$, $\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})$ follows the deformation of the mesh. This yields the different boundary integral expressions for the corresponding transport theorems.

Theorem 10 Let $\Omega_{\boldsymbol{x}}$ be a material domain which deforms according to the diffeomorphism $\boldsymbol{\varphi}$, with velocity $\boldsymbol{v}$. Let $\Omega_{\boldsymbol{\chi}}$ be the inverse image of $\Omega_{\boldsymbol{x}}$ through $\hat{\boldsymbol{\varphi}}$, that is $\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})$. Let $\partial\Omega_{\boldsymbol{\chi}}$ be the boundary of $\Omega_{\boldsymbol{\chi}}$, with outward normal $\boldsymbol{n}_{\boldsymbol{\chi}}$. Also, let $\Omega_{\boldsymbol{X}}$ be the inverse image of $\Omega_{\boldsymbol{\chi}}$ through the diffeomorphic map $\tilde{\boldsymbol{\varphi}}$, that is, $\Omega_{\boldsymbol{\chi}} = \tilde{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{X}})$, with $\boldsymbol{\varphi} = \hat{\boldsymbol{\varphi}} \circ \tilde{\boldsymbol{\varphi}}$. Then, if α is a scalar field,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \alpha \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial (\hat{J}\alpha)}{\partial t}\right|_{\boldsymbol{\chi}} + \int_{\partial\Omega_{\boldsymbol{\chi}}} (\hat{J}\alpha)\; \tilde{\boldsymbol{v}} \cdot \boldsymbol{n}_{\boldsymbol{\chi}} \;. \tag{85}$$

Homework 5 Prove the generalized Reynolds transport theorem by pulling back to the domain $\Omega_{\boldsymbol{X}}$ in two steps, using the map $\hat{\boldsymbol{\varphi}}$ first and then the map $\tilde{\boldsymbol{\varphi}}$.

Remark 12 The generalized Reynolds transport theorem collapses to the classical Reynolds transport theorem, whenever $\hat{\boldsymbol{\varphi}} \equiv \boldsymbol{\varphi}$, so that $\hat{J} = J$. If this is the case, $\tilde{\boldsymbol{\varphi}} \equiv \boldsymbol{id}$, $\tilde{\boldsymbol{v}} = \boldsymbol{0}$, and it is easy to see that (85) reduces to the second integral expression in (80).

Remark 13 Recalling that $\tilde{\boldsymbol{v}} = \hat{\boldsymbol{F}}^{-1}\boldsymbol{c}$, it is easy to derive $(\hat{J}\alpha)\; \tilde{\boldsymbol{v}} \cdot \boldsymbol{n}_{\boldsymbol{\chi}} = \alpha \boldsymbol{c} \cdot (\hat{J}\hat{\boldsymbol{F}}^{-T}\boldsymbol{n}_{\boldsymbol{\chi}}) = \alpha \boldsymbol{c} \cdot (\mathbf{cof}\,\hat{\boldsymbol{F}}\boldsymbol{n}_{\boldsymbol{\chi}}) = \boldsymbol{c} \cdot \boldsymbol{n}_{\boldsymbol{x}}$. Then (85) takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \alpha \;=\; \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \alpha \;+\; \int_{\partial\Omega_{\boldsymbol{x}}} \alpha\; \boldsymbol{c} \cdot \boldsymbol{n}_{\boldsymbol{x}} \;, \tag{86}$$

which is often used in finite-volume computations [11]. In this context, the volume integral on the left hand side of (86) is termed the control-mass integral, while the volume integral on the right hand side is termed the control-volume integral.

3.2 Transport theorems for surface integrals

Especially in the context of electromagnetics, the rate of change of integrals over surfaces are frequently used in developing integral and variational formulations. These results are presented next.

3.2.1 Time rate of integrals over a surface with motion $\boldsymbol{\varphi}$

Theorem 11 Let $\mathcal{S}_{\boldsymbol{x}}$ be a material surface deforming with velocity $\boldsymbol{v}$, through the invertible map $\boldsymbol{\varphi}$. Let $\boldsymbol{n}_{\boldsymbol{x}}$ be the normal to the surface $\mathcal{S}_{\boldsymbol{x}}$ according to the right-hand convention (the outward normal if the surface is closed). Let $\mathcal{S}_{\boldsymbol{X}}$ be the inverse image of $\mathcal{S}_{\boldsymbol{x}}$ through $\boldsymbol{\varphi}$, that is, $\mathcal{S}_{\boldsymbol{x}} = \boldsymbol{\varphi}(\mathcal{S}_{\boldsymbol{X}})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathcal{S}_{\boldsymbol{X}})} \alpha \boldsymbol{n}_{\boldsymbol{x}} = \int_{\mathcal{S}_{\boldsymbol{x}}} \left(\left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \cdot (\alpha \boldsymbol{v}) \right) \boldsymbol{n}_{\boldsymbol{x}} - \int_{\mathcal{S}_{\boldsymbol{x}}} \alpha (\nabla_{\boldsymbol{x}} \boldsymbol{v})^T \boldsymbol{n}_{\boldsymbol{x}} \ . \tag{87}$$

Proof:
Before proceeding with the details of the proof, let us recall two important results. First,

$$\begin{aligned} \dot{\boldsymbol{F}} &= \left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} \left(\frac{\partial \varphi_i}{\partial X_j} \right) \\ &= \frac{\partial v_i}{\partial X_j} = \nabla_{\boldsymbol{X}} \boldsymbol{v} \\ &= \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial X_j} = (\nabla_{\boldsymbol{x}} \boldsymbol{v}) \boldsymbol{F} \ . \end{aligned} \tag{88}$$

In addition, by differentiating the tensor $\boldsymbol{I} = \boldsymbol{F}\boldsymbol{F}^{-1}$, it is easy to verify

$$\partial_t|_{\boldsymbol{X}} \left(\boldsymbol{F}^{-1} \right) = -\boldsymbol{F}^{-1} \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} \tag{89}$$

Applying Nanson's formula (9) to the map $\boldsymbol{\varphi}$, recalling (31), and using (88)–(89),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathcal{S}_{\boldsymbol{X}})} \alpha \boldsymbol{n}_{\boldsymbol{x}} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}_{\boldsymbol{X}}} \alpha J \boldsymbol{F}^{-T} \boldsymbol{n}_{\boldsymbol{X}} \\ &= \int_{\mathcal{S}_{\boldsymbol{X}}} \left(\dot{\alpha} J \boldsymbol{F}^{-T} + \alpha \dot{J} \boldsymbol{F}^{-T} + \alpha J \ \partial_t|_{\boldsymbol{X}} \left(\boldsymbol{F}^{-T} \right) \right) \boldsymbol{n}_{\boldsymbol{X}} \\ &= \int_{\mathcal{S}_{\boldsymbol{X}}} \left(\dot{\alpha} J \boldsymbol{F}^{-T} + \alpha J (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}) \boldsymbol{F}^{-T} - \alpha J \left(\boldsymbol{F}^{-1} \dot{\boldsymbol{F}} \boldsymbol{F}^{-1} \right)^T \right) \boldsymbol{n}_{\boldsymbol{X}} \\ &= \int_{\mathcal{S}_{\boldsymbol{X}}} \left(\dot{\alpha} J \boldsymbol{F}^{-T} + \alpha J (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}) \boldsymbol{F}^{-T} - \alpha J \left(\boldsymbol{F}^{-1} \nabla_{\boldsymbol{x}} \boldsymbol{v} \right)^T \right) \boldsymbol{n}_{\boldsymbol{X}} \\ &= \int_{\mathcal{S}_{\boldsymbol{X}}} \left(\dot{\alpha} + \alpha \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} - \alpha (\nabla_{\boldsymbol{x}} \boldsymbol{v})^T \right) J \boldsymbol{F}^{-T} \boldsymbol{n}_{\boldsymbol{X}} \\ &= \int_{\mathcal{S}_{\boldsymbol{x}}} \left(\dot{\alpha} + \alpha \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} \right) \boldsymbol{n}_{\boldsymbol{x}} - \int_{\mathcal{S}_{\boldsymbol{x}}} \alpha (\nabla_{\boldsymbol{x}} \boldsymbol{v})^T \boldsymbol{n}_{\boldsymbol{x}} \\ &= \int_{\mathcal{S}_{\boldsymbol{x}}} \left(\left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \alpha + \alpha \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} \right) \boldsymbol{n}_{\boldsymbol{x}} - \int_{\mathcal{S}_{\boldsymbol{x}}} \alpha (\nabla_{\boldsymbol{x}} \boldsymbol{v})^T \boldsymbol{n}_{\boldsymbol{x}} \\ &= \int_{\mathcal{S}_{\boldsymbol{x}}} \left(\left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \cdot (\alpha \boldsymbol{v}) \right) \boldsymbol{n}_{\boldsymbol{x}} - \int_{\mathcal{S}_{\boldsymbol{x}}} \alpha (\nabla_{\boldsymbol{x}} \boldsymbol{v})^T \boldsymbol{n}_{\boldsymbol{x}} \ , \end{aligned} \tag{90}$$

which concludes the proof. □

3.2.2 Time rate of integrals over a surface with motion $\hat{\boldsymbol{\varphi}}$

Theorem 12 Let $\mathcal{S}_{\boldsymbol{x}}$ be a material surface deforming with velocity $\hat{\boldsymbol{v}}$, through the invertible map $\hat{\boldsymbol{\varphi}}$. Let $\boldsymbol{n}_{\boldsymbol{x}}$ be the normal to the surface $\mathcal{S}_{\boldsymbol{x}}$ according to the right-hand convention (the outward normal if the surface is closed). Let $\mathcal{S}_{\boldsymbol{\chi}}$ be the inverse image of $\mathcal{S}_{\boldsymbol{x}}$ through $\hat{\boldsymbol{\varphi}}$, that is, $\mathcal{S}_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathcal{S}_{\boldsymbol{\chi}})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\mathcal{S}_{\boldsymbol{\chi}})} \alpha \boldsymbol{n}_{\boldsymbol{x}} = \int_{\mathcal{S}_{\boldsymbol{x}}} \left(\left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \cdot (\alpha \hat{\boldsymbol{v}}) \right) \boldsymbol{n}_{\boldsymbol{x}} - \int_{\mathcal{S}_{\boldsymbol{x}}} \alpha (\nabla_{\boldsymbol{x}} \hat{\boldsymbol{v}})^T \boldsymbol{n}_{\boldsymbol{x}} \,. \tag{91}$$

Proof:
The proof is virtually identical to the proof of the Theorem 11, once $\mathcal{S}_{\boldsymbol{\chi}}$, $\hat{\boldsymbol{\varphi}}$, $\hat{\boldsymbol{F}}$, $\hat{J}$, $\hat{\boldsymbol{v}}$, $\boldsymbol{n}_{\boldsymbol{\chi}}$, and $\partial_t|_{\boldsymbol{\chi}}$ are substituted in place of $\mathcal{S}_{\boldsymbol{X}}$, $\boldsymbol{\varphi}$, $\boldsymbol{F}$, J, $\boldsymbol{v}$, $\boldsymbol{n}_{\boldsymbol{X}}$, and $\partial_t|_{\boldsymbol{X}}$, respectively. The proof requires the use of the identities $\partial_t \hat{\boldsymbol{F}}|_{\boldsymbol{\chi}} = (\nabla_{\boldsymbol{x}} \hat{\boldsymbol{v}})\hat{\boldsymbol{F}}$ and $\partial_t \hat{\boldsymbol{F}}^{-1}|_{\boldsymbol{\chi}} = -\hat{\boldsymbol{F}}^{-1} \partial_t \hat{\boldsymbol{F}}|_{\boldsymbol{\chi}} \hat{\boldsymbol{F}}^{-1}$, analogous to (88) and (89), respectively. □

3.3 Transport theorems for line integrals

Again in the context of electromagnetics, the rate of change of integrals over closed curves are of great importance. Let $\mathcal{C}_{\boldsymbol{x}}$ be a closed curve in $\mathbb{R}^{n_d}$, that is, an almost everywhere smooth map (hence, almost everywhere continuous), such that

$$\mathcal{C}_{\boldsymbol{x}} : [0,1] \rightarrow \mathbb{R}^{n_d} \,, \tag{92}$$
$$s \mapsto \mathcal{C}_{\boldsymbol{x}}(s) = \boldsymbol{x}(s) \in \mathbb{R}^{n_d} \,, \tag{93}$$

and $\boldsymbol{x}(0) = \boldsymbol{x}(1)$. $\mathcal{C}_{\boldsymbol{x}}$ is defined with respect the the Eulerian reference frame. Analogously, closed curves can be defined in the Lagrangian and referential frames as follows:

$$\mathcal{C}_{\boldsymbol{X}} : [0,1] \rightarrow \mathbb{R}^{n_d} \,, \tag{94}$$
$$s \mapsto \mathcal{C}_{\boldsymbol{X}}(s) = \boldsymbol{X}(s) \in \mathbb{R}^{n_d} \,, \tag{95}$$

such that $\boldsymbol{X}(0) = \boldsymbol{X}(1)$, and

$$\mathcal{C}_{\boldsymbol{\chi}} : [0,1] \rightarrow \mathbb{R}^{n_d} \,, \tag{96}$$
$$s \mapsto \mathcal{C}_{\boldsymbol{\chi}}(s) = \boldsymbol{\chi}(s) \in \mathbb{R}^{n_d} \,, \tag{97}$$

such that $\boldsymbol{\chi}(0) = \boldsymbol{\chi}(1)$. $\mathcal{C}_{\boldsymbol{x}}([0,1])$, $\mathcal{C}_{\boldsymbol{X}}([0,1])$, and $\mathcal{C}_{\boldsymbol{\chi}}([0,1])$, are usually named the *traces* of the curve maps $\mathcal{C}_{\boldsymbol{x}}$, $\mathcal{C}_{\boldsymbol{X}}$, and $\mathcal{C}_{\boldsymbol{\chi}}$, respectively. Tangent vectors to $\mathcal{C}_{\boldsymbol{x}}$, $\mathcal{C}_{\boldsymbol{X}}$, and $\mathcal{C}_{\boldsymbol{\chi}}$ are obtained differentiating with respect to the parameter s:

$$\boldsymbol{t}_{\boldsymbol{x}} = \frac{\partial \mathcal{C}_{\boldsymbol{x}}(s)}{\partial s} = \frac{\partial \boldsymbol{x}(s)}{\partial s} \,, \tag{98}$$
$$\boldsymbol{t}_{\boldsymbol{X}} = \frac{\partial \mathcal{C}_{\boldsymbol{X}}(s)}{\partial s} = \frac{\partial \boldsymbol{X}(s)}{\partial s} \,, \tag{99}$$
$$\boldsymbol{t}_{\boldsymbol{\chi}} = \frac{\partial \mathcal{C}_{\boldsymbol{\chi}}(s)}{\partial s} = \frac{\partial \boldsymbol{\chi}(s)}{\partial s} \,. \tag{100}$$

In what follows, the notation $\mathcal{C}_{\boldsymbol{x}}$, $\mathcal{C}_{\boldsymbol{X}}$, $\mathcal{C}_{\boldsymbol{\chi}}$, may be used, loosely, for $\mathcal{C}_{\boldsymbol{x}}([0,1])$, $\mathcal{C}_{\boldsymbol{X}}([0,1])$, $\mathcal{C}_{\boldsymbol{\chi}}([0,1])$. Similarly, we indicate

$$\oint_{\mathcal{C}_{\boldsymbol{x}}} (\cdot)\ \mathrm{d}\boldsymbol{x} = \int_0^1 (\cdot)\frac{\partial \boldsymbol{x}(s)}{\partial s}\ \mathrm{d}s = \int_0^1 (\cdot)\frac{\partial \mathcal{C}_{\boldsymbol{x}}(s)}{\partial s}\ \mathrm{d}s = \int_0^1 (\cdot)\boldsymbol{t}_{\boldsymbol{x}}\ \mathrm{d}s\ . \tag{101}$$

Analogous expressions hold for the integrals on $\mathcal{C}_{\boldsymbol{X}}$ and $\mathcal{C}_{\boldsymbol{\chi}}$.

3.3.1 Time rate of integrals over a closed curve with motion $\boldsymbol{\varphi}$

Theorem 13 Let $\mathcal{C}_{\boldsymbol{x}}$ be a material closed curve deforming with velocity $\boldsymbol{v}$, through the invertible map $\boldsymbol{\varphi}$. Let $\mathcal{C}_{\boldsymbol{X}}$ be the inverse image of $\mathcal{C}_{\boldsymbol{x}}$ through $\boldsymbol{\varphi}$, that is, $\mathcal{C}_{\boldsymbol{x}} = \boldsymbol{\varphi}(\mathcal{C}_{\boldsymbol{X}})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathcal{C}_{\boldsymbol{X}})} \alpha\ \mathrm{d}\boldsymbol{x} = \oint_{\mathcal{C}_{\boldsymbol{x}}} \dot{\alpha}\ \mathrm{d}\boldsymbol{x} + \oint_{\mathcal{C}_{\boldsymbol{x}}} \alpha(\nabla_{\boldsymbol{x}}\boldsymbol{v})\mathrm{d}\boldsymbol{x}\ , \tag{102}$$

where $(\nabla_{\boldsymbol{x}}\boldsymbol{v})\mathrm{d}\boldsymbol{x} = (\nabla_{\boldsymbol{x}}\boldsymbol{v})\boldsymbol{t}_{\boldsymbol{x}}\mathrm{d}s$.

Proof:
The theorem can be easily proven by pulling back $\mathcal{C}_{\boldsymbol{x}}$ to its original configuration $\mathcal{C}_{\boldsymbol{X}}$ through the map $\boldsymbol{\varphi}$, recalling the chain rule $\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\mathrm{d}\boldsymbol{X}$, and applying (88):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathcal{C}_{\boldsymbol{X}})} \alpha\ \mathrm{d}\boldsymbol{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \alpha\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}\frac{\partial \boldsymbol{X}}{\partial s}\ \mathrm{d}s \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \alpha\boldsymbol{F}\boldsymbol{t}_{\boldsymbol{X}}\ \mathrm{d}s \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_{\boldsymbol{X}}} \alpha\boldsymbol{F}\ \mathrm{d}\boldsymbol{X} \\
&= \oint_{\mathcal{C}_{\boldsymbol{X}}} \left(\dot{\alpha}\boldsymbol{F} + \alpha\dot{\boldsymbol{F}}\right)\mathrm{d}\boldsymbol{X} \\
&= \oint_{\mathcal{C}_{\boldsymbol{X}}} \left(\dot{\alpha}\boldsymbol{F} + \alpha(\nabla_{\boldsymbol{x}}\boldsymbol{v})\boldsymbol{F}\right)\mathrm{d}\boldsymbol{X} \\
&= \oint_{\mathcal{C}_{\boldsymbol{x}}} \dot{\alpha}\ \mathrm{d}\boldsymbol{x} + \oint_{\mathcal{C}_{\boldsymbol{x}}} \alpha(\nabla_{\boldsymbol{x}}\boldsymbol{v})\ \mathrm{d}\boldsymbol{x}\ .
\end{aligned} \tag{103}$$

□

3.3.2 Time rate of integrals over a closed curve with motion $\hat{\boldsymbol{\varphi}}$

Theorem 14 Let $\mathcal{C}_{\boldsymbol{x}}$ be a closed curve deforming with velocity $\hat{\boldsymbol{v}}$, through the invertible map $\hat{\boldsymbol{\varphi}}$. Let $\mathcal{C}_{\boldsymbol{\chi}}$ be the inverse image of $\mathcal{C}_{\boldsymbol{x}}$ through $\hat{\boldsymbol{\varphi}}$, that is, $\mathcal{C}_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathcal{C}_{\boldsymbol{\chi}})$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathcal{C}_{\boldsymbol{X}})} \alpha\ \mathrm{d}\boldsymbol{x} = \oint_{\mathcal{C}_{\boldsymbol{x}}} \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{\chi}}\ \mathrm{d}\boldsymbol{x} + \oint_{\mathcal{C}_{\boldsymbol{x}}} \alpha(\nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}})\mathrm{d}\boldsymbol{x}\ . \tag{104}$$

Homework 6 Prove the previous theorem by mimicking the proof of Theorem 13, and replacing the map $\boldsymbol{\varphi}$ by $\hat{\boldsymbol{\varphi}}$.

4 Master balance equations

A master balance equation abstractly expresses the budget of fluxes and source terms underlying a conservation law. In this sense, the master balance equation is the fundamental template for conservation laws. If α indicates a conserved scalar variable, the master balance law for a *material* domain takes the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})}\alpha=\int_{\partial\Omega_{\boldsymbol{x}}}\boldsymbol{\gamma}\cdot\boldsymbol{n}_{\boldsymbol{x}}+\int_{\Omega_{\boldsymbol{x}}}\beta\,, \tag{105}$$

where the vector $\boldsymbol{\gamma}$ is the flux of the conserved variable α, and β represents a source/sink term.

Remark 14 The master balance law (105), as a template for a physics principle, holds even if $\boldsymbol{\varphi}$ is not a smooth map (e.g., a piecewise smooth map). In this case, $\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})$ simply indicates that $\Omega_{\boldsymbol{x}}$ is a control mass and that its boundary $\partial\Omega_{\boldsymbol{x}}$ moves with the material.

The classical and generalized transport theorems developed in Section 3 provide different ways of expressing the left-hand side of (105), and, consequently, different ways of formulating the master balance law. Because of the use of $\boldsymbol{\varphi}$, (105) is presented from a Lagrangian point of view. As we will see momentarily, the Leibnitz transport theorem cannot be directly applied, but has to be leveraged in a very specific way. Before we proceed, let us extend the master balance law to the case of a vector conserved quantity $\boldsymbol{\alpha}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})}\boldsymbol{\alpha}=\int_{\partial\Omega_{\boldsymbol{x}}}\boldsymbol{\Gamma}\boldsymbol{n}_{\boldsymbol{x}}+\int_{\Omega_{\boldsymbol{x}}}\boldsymbol{\beta}\,, \tag{106}$$

where $\boldsymbol{\Gamma}$ is a second-order tensor, representing the flux of $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ is the source/sink vector.

4.1 Master balance law using the classical Reynolds transport theorem

Assuming smoothness of $\boldsymbol{\varphi}$ and applying (79), the master balance laws (105)–(106) become

$$\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})}\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}}=\int_{\partial\Omega_{\boldsymbol{x}}}(\boldsymbol{\gamma}-\alpha\boldsymbol{v})\cdot\boldsymbol{n}_{\boldsymbol{x}}+\int_{\Omega_{\boldsymbol{x}}}\beta\,, \tag{107}$$

$$\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})}\left.\frac{\partial\boldsymbol{\alpha}}{\partial t}\right|_{\boldsymbol{x}}=\int_{\partial\Omega_{\boldsymbol{x}}}(\boldsymbol{\Gamma}-\boldsymbol{\alpha}\otimes\boldsymbol{v})\,\boldsymbol{n}_{\boldsymbol{x}}+\int_{\Omega_{\boldsymbol{x}}}\boldsymbol{\beta}\,, \tag{108}$$

where $\otimes$ represents the tensor dyadic product, (i.e., $\boldsymbol{\alpha}\otimes\boldsymbol{v}=\alpha_i v_j$). Using the Gauss divergence theorem on the boundary integrals,

$$0=\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})}\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}}+\nabla_{\boldsymbol{x}}\cdot(\alpha\boldsymbol{v}-\boldsymbol{\gamma})-\beta\,, \tag{109}$$

$$\mathbf{0}=\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})}\left.\frac{\partial\boldsymbol{\alpha}}{\partial t}\right|_{\boldsymbol{x}}+\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{\alpha}\otimes\boldsymbol{v}-\boldsymbol{\Gamma})-\boldsymbol{\beta}\,. \tag{110}$$

4.2 Rankine-Hugoniot jump conditions

The derivation of the Rankine-Hugoniot jump conditions is a very important and instructive application of the Leibnitz transport theorem, and the master balance laws (105) and (107) (resp., (106) and (108)). Let

us assume that a discontinuity of α, $\boldsymbol{\gamma}$, and, possibly, $\boldsymbol{v}$ is represented by a surface $\Gamma_{\boldsymbol{s}}$ moving in space with velocity $\boldsymbol{s}(\boldsymbol{x},t)$. As illustrated in Figure 7, let us construct an open domain $\Omega_{\boldsymbol{x}}$ divided by $\Gamma_{\boldsymbol{s}}$ into two open sub-domains $\Omega_{\boldsymbol{x}}^{(1)}$ and $\Omega_{\boldsymbol{x}}^{(2)}$, so that $\Omega_{\boldsymbol{x}} = \Omega_{\boldsymbol{x}}^{(1)} \bigcup \Omega_{\boldsymbol{x}}^{(2)} \bigcup \Gamma_{\boldsymbol{s}}$. Let us think for a moment about $\Omega_{\boldsymbol{x}}^{(1)}$ (resp., $\Omega_{\boldsymbol{x}}^{(2)}$) as a domain with boundary $\partial\Omega^{(1)} \setminus \Gamma_{\boldsymbol{s}}$ (resp. $\partial\Omega^{(2)} \setminus \Gamma_{\boldsymbol{s}}$) deforming with the material, and boundary $\partial\Omega^{(1)} \bigcap \Gamma_{\boldsymbol{s}} = \Gamma_{\boldsymbol{s}}$ (resp. $\partial\Omega^{(2)} \bigcap \Gamma_{\boldsymbol{s}}$) deforming with velocity $\boldsymbol{s}$ ($\boldsymbol{s} \neq \boldsymbol{v}$ in general). By construction,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \alpha \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}^{(1)}=\hat{\boldsymbol{\varphi}}\left(\Omega_{\boldsymbol{X}}^{(1)}\right)} \alpha + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}^{(2)}=\hat{\boldsymbol{\varphi}}\left(\Omega_{\boldsymbol{X}}^{(2)}\right)} \alpha \,. \tag{111}$$

Remark 15 It is the motion of the boundary $\partial\Omega_{\boldsymbol{x}}$ which defines the time derivative on the left hand side of (111), and not the motion of the internal interface $\Gamma_{\boldsymbol{s}}$.

Then, applying the Leibnitz transport theorem over $\Omega_{\boldsymbol{x}}^{(1)}$ and $\Omega_{\boldsymbol{x}}^{(2)}$, where deformation mappings are smooth, yields:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \alpha \;&=\; \int_{\Omega_{\boldsymbol{x}}^{(1)}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \int_{\partial\Omega_{\boldsymbol{x}}^{(1)}\setminus\Gamma_{\boldsymbol{s}}} \alpha\boldsymbol{v}\cdot\boldsymbol{n}_{\boldsymbol{x}} + \int_{\partial\Omega^{(1)}\bigcap\Gamma_{\boldsymbol{s}}} \alpha\boldsymbol{s}\cdot\boldsymbol{n}_{\boldsymbol{x}} \\
&\quad + \int_{\Omega_{\boldsymbol{x}}^{(2)}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \int_{\partial\Omega_{\boldsymbol{x}}^{(2)}\setminus\Gamma_{\boldsymbol{s}}} \alpha\boldsymbol{v}\cdot\boldsymbol{n}_{\boldsymbol{x}} + \int_{\partial\Omega^{(1)}\bigcap\Gamma_{\boldsymbol{s}}} \alpha\boldsymbol{s}\cdot\boldsymbol{n}_{\boldsymbol{x}} \\
&=\; \int_{\Omega_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \int_{\partial\Omega_{\boldsymbol{x}}} \alpha\boldsymbol{v}\cdot\boldsymbol{n}_{\boldsymbol{x}} + \int_{\Gamma_{\boldsymbol{s}}} [\![\alpha]\!]\cdot\boldsymbol{s} \,,
\end{aligned} \tag{112}$$

with $[\![\alpha]\!] = \alpha^{(1)}\boldsymbol{n}_{\boldsymbol{x}}^{(1)} + \alpha^{(2)}\boldsymbol{n}_{\boldsymbol{x}}^{(2)}$, and $\boldsymbol{n}_{\boldsymbol{x}}^{(1)} = -\boldsymbol{n}_{\boldsymbol{x}}^{(2)}$. Equation (112) represents the generalized form of the Reynolds transport theorem for discontinuous fields across $\Gamma_{\boldsymbol{s}}$, and collapses to (79) if $[\![\alpha]\!] = \boldsymbol{0}$.

Remark 16 The general form of the Leibnitz transport theorem for discontinuous fields is obtained replacing $\boldsymbol{v}$ with $\hat{\boldsymbol{v}}$ in the second integral on the right hand side of (112).

Let us now consider the sum of (105) written over $\Omega_{\boldsymbol{x}}^{(1)}$ and $\Omega_{\boldsymbol{x}}^{(2)}$ (thought of as Lagrangian domains), and recall the fact that (107) is equivalent to (105) for smooth deformation maps:

$$\begin{aligned}
0 \;&=\; \int_{\Omega_{\boldsymbol{x}}^{(1)}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} - \int_{\partial\Omega_{\boldsymbol{x}}^{(1)}} (\boldsymbol{\gamma}-\alpha\boldsymbol{v})\cdot\boldsymbol{n}_{\boldsymbol{x}} - \int_{\Omega_{\boldsymbol{x}}^{(1)}} \beta \\
&\quad + \int_{\Omega_{\boldsymbol{x}}^{(2)}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} - \int_{\partial\Omega_{\boldsymbol{x}}^{(2)}} (\boldsymbol{\gamma}-\alpha\boldsymbol{v})\cdot\boldsymbol{n}_{\boldsymbol{x}} - \int_{\Omega_{\boldsymbol{x}}^{(2)}} \beta \\
&=\; \int_{\Omega_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} - \int_{\partial\Omega_{\boldsymbol{x}}} (\boldsymbol{\gamma}-\alpha\boldsymbol{v})\cdot\boldsymbol{n}_{\boldsymbol{x}} - \int_{\Gamma_{\boldsymbol{s}}} [\![\boldsymbol{\gamma}-\alpha\boldsymbol{v}]\!] - \int_{\Omega_{\boldsymbol{x}}} \beta \,,
\end{aligned} \tag{113}$$

where $[\![\boldsymbol{\gamma}-\alpha\boldsymbol{v}]\!] = (\boldsymbol{\gamma}-\alpha\boldsymbol{v})^{(1)}\cdot\boldsymbol{n}_{\boldsymbol{x}}^{(1)} + (\boldsymbol{\gamma}-\alpha\boldsymbol{v})^{(2)}\cdot\boldsymbol{n}_{\boldsymbol{x}}^{(2)}$, and β is assumed bounded over $\Omega_{\boldsymbol{x}}$ (in particular, β is *not* a Dirac distribution). Subtracting (113) from (107) written over $\Omega_{\boldsymbol{x}}$, and using (112) yields

$$\begin{aligned}
0 \;&=\; \int_{\Omega_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} - \int_{\partial\Omega_{\boldsymbol{x}}} (\boldsymbol{\gamma}-\alpha\boldsymbol{v})\cdot\boldsymbol{n}_{\boldsymbol{x}} - \int_{\Omega_{\boldsymbol{x}}} \beta + \int_{\Gamma_{\boldsymbol{s}}} [\![\alpha]\!]\cdot\boldsymbol{s} \\
&\quad - \int_{\Omega_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{x}} + \int_{\partial\Omega_{\boldsymbol{x}}} (\boldsymbol{\gamma}-\alpha\boldsymbol{v})\cdot\boldsymbol{n}_{\boldsymbol{x}} + \int_{\Gamma_{\boldsymbol{s}}} [\![\boldsymbol{\gamma}-\alpha\boldsymbol{v}]\!] + \int_{\Omega_{\boldsymbol{x}}} \beta \\
&=\; \int_{\Gamma_{\boldsymbol{s}}} [\![\alpha]\!]\cdot\boldsymbol{s} + \int_{\Gamma_{\boldsymbol{s}}} [\![\boldsymbol{\gamma}-\alpha\boldsymbol{v}]\!] \,,
\end{aligned} \tag{114}$$

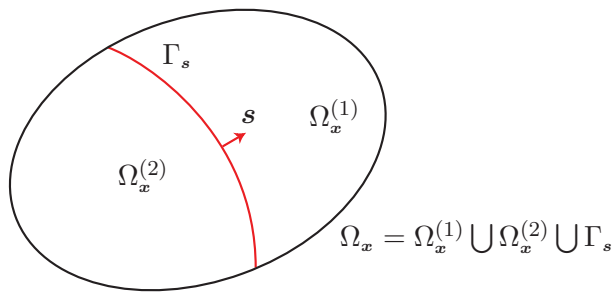

Figure 7. Sketch of the domain $\Omega_{\boldsymbol{x}} = \Omega_{\boldsymbol{x}}^{(1)} \bigcup \Omega_{\boldsymbol{x}}^{(2)} \bigcup \Gamma_{\boldsymbol{s}}$, with a discontinuity along the surface $\Gamma_{\boldsymbol{s}}$, moving with velocity $\boldsymbol{s}$.

which must hold for any material domain $\Omega_{\boldsymbol{x}}$ intersecting $\Gamma_{\boldsymbol{s}}$. Assuming $\boldsymbol{n}_{\boldsymbol{x}}^{(1)} = -\boldsymbol{n}_{\boldsymbol{x}}^{(2)}$ is a continuous function over $\Gamma_{\boldsymbol{s}}$, and $\boldsymbol{\gamma}$, α, and $\boldsymbol{v}$ are smooth over the interiors of $\Omega_{\boldsymbol{x}}^{(1)}$ and $\Omega_{\boldsymbol{x}}^{(2)}$, an application of the localization theorem over $\Gamma_{\boldsymbol{s}}$ yields the scalar Rankine-Hugoniot jump conditions:

$$[\![\alpha]\!] \cdot \boldsymbol{s} = [\![\alpha \boldsymbol{v} - \boldsymbol{\gamma}]\!] \,. \tag{115}$$

It is not difficult to show, using (106), that the vector form of the Rankine-Hugoniot jump conditions is given by

$$[\![\boldsymbol{\alpha}]\!] \boldsymbol{s} = [\![\boldsymbol{\alpha} \otimes \boldsymbol{v} - \boldsymbol{\Gamma}]\!] \,, \tag{116}$$

with $[\![\boldsymbol{\Gamma}]\!] = \boldsymbol{\Gamma}^{(1)} \boldsymbol{n}_{\boldsymbol{x}}^{(1)} + \boldsymbol{\Gamma}^{(2)} \boldsymbol{n}_{\boldsymbol{x}}^{(2)}$, and analogously for $[\![\boldsymbol{\alpha} \otimes \boldsymbol{v}]\!]$. Given specific conservations laws, it is possible to use the Rankine-Hugoniot conditions to compute the normal speed $\boldsymbol{s} \cdot \boldsymbol{n}_{\boldsymbol{x}}^{(2)}$ of the discontinuity in terms of the states before and past it.

Remark 17 The derivations of the Rankine-Hugoniot jump conditions have been carried out in the most general case, without assuming a steady or zero discontinuity velocity. Notice also that the jump conditions apply to quantities normal to the discontinuity surface.

4.3 Master balance law using the Leibnitz transport theorem

It is possible to recast the master balance law (107) (or, resp., (108)) applying the Leibnitz transport theorem. This approach is very common in computational mechanics applications. Let us start from (107), and replace the integral of the Eulerian time derivative using (83), that is,

$$\int_{\Omega_{\boldsymbol{x}}} \left. \frac{\partial \alpha}{\partial t} \right|_{\boldsymbol{x}} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \alpha - \int_{\partial \Omega_{\boldsymbol{x}}} \alpha \hat{\boldsymbol{v}} \cdot \boldsymbol{n}_{\boldsymbol{x}} = \int_{\Omega_{\boldsymbol{x}}} \hat{J}^{-1} \left. \frac{\partial (\hat{J} \alpha)}{\partial t} \right|_{\boldsymbol{\chi}} - \int_{\partial \Omega_{\boldsymbol{x}}} \alpha \hat{\boldsymbol{v}} \cdot \boldsymbol{n}_{\boldsymbol{x}} \,. \tag{117}$$

Hence, recalling that $\boldsymbol{c} = \boldsymbol{v} - \hat{\boldsymbol{v}}$, (107) and (108) yield

$$\int_{\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left. \frac{\partial (\hat{J} \alpha)}{\partial t} \right|_{\boldsymbol{\chi}} = \int_{\partial \Omega_{\boldsymbol{x}}} (\boldsymbol{\gamma} - \alpha \boldsymbol{c}) \cdot \boldsymbol{n}_{\boldsymbol{x}} + \int_{\Omega_{\boldsymbol{x}}} \beta \,, \tag{118}$$

$$\int_{\Omega_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left. \frac{\partial (\hat{J} \boldsymbol{\alpha})}{\partial t} \right|_{\boldsymbol{\chi}} = \int_{\partial \Omega_{\boldsymbol{x}}} (\boldsymbol{\Gamma} - \boldsymbol{\alpha} \otimes \boldsymbol{c}) \, \boldsymbol{n}_{\boldsymbol{x}} + \int_{\Omega_{\boldsymbol{x}}} \boldsymbol{\beta} \,, \tag{119}$$

or, using the Gauss divergence theorem,

$$0 = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left.\frac{\partial(\hat{J}\alpha)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{x}} \cdot (\alpha \boldsymbol{c} - \boldsymbol{\gamma}) - \beta \,, \tag{120}$$

$$\mathbf{0} = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left.\frac{\partial(\hat{J}\boldsymbol{\alpha})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{\alpha} \otimes \boldsymbol{c} - \boldsymbol{\Gamma}) - \boldsymbol{\beta} \,. \tag{121}$$

This approach can be considered *hybrid*, in the sense that all the terms are evaluated in the current configuration, but the time-derivative term still contains the Jacobian $\hat{J}$. In spite of this fact, (120)–(121) are preferred in many large-scale computational methods [18], since specific discretizations in time allow to simplify $\hat{J}$ from the discrete equations.

4.4 Master balance law using the generalized Reynolds transport theorem

Applying (9) to the map $\hat{\boldsymbol{\varphi}}$, yields

$$\boldsymbol{n}_{\boldsymbol{x}} \mathrm{d}(\partial\Omega_{\boldsymbol{x}}) = \hat{J}\hat{\boldsymbol{F}}^{-T} \boldsymbol{n}_{\boldsymbol{\chi}} \mathrm{d}(\partial\Omega_{\boldsymbol{\chi}}) \,. \tag{122}$$

Using the previous identity, applying the transport theorem (85), and recalling from (77) that $\tilde{\boldsymbol{v}} = \hat{\boldsymbol{F}}^{-1}\boldsymbol{c}$, the scalar and vector master balance laws read

$$\int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\alpha)}{\partial t}\right|_{\boldsymbol{\chi}} = \int_{\partial\Omega_{\boldsymbol{\chi}}} \left(\hat{J}\hat{\boldsymbol{F}}^{-1}(\boldsymbol{\gamma} - \alpha\boldsymbol{c})\right) \cdot \boldsymbol{n}_{\boldsymbol{\chi}} + \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\beta \,, \tag{123}$$

$$\int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\boldsymbol{\alpha})}{\partial t}\right|_{\boldsymbol{\chi}} = \int_{\partial\Omega_{\boldsymbol{\chi}}} (\boldsymbol{\Gamma} - \boldsymbol{\alpha} \otimes \boldsymbol{c})\hat{J}\hat{\boldsymbol{F}}^{-T} \boldsymbol{n}_{\boldsymbol{\chi}} + \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\boldsymbol{\beta} \,, \tag{124}$$

and, by the Gauss divergence theorem,

$$0 = \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\alpha)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J}\hat{\boldsymbol{F}}^{-1}(\alpha\boldsymbol{c} - \boldsymbol{\gamma})\right) - \hat{J}\beta \,, \tag{125}$$

$$\mathbf{0} = \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\boldsymbol{\alpha})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left((\boldsymbol{\alpha} \otimes \boldsymbol{c} - \boldsymbol{\Gamma})\hat{J}\hat{\boldsymbol{F}}^{-T}\right) - \hat{J}\boldsymbol{\beta} \,. \tag{126}$$

Note also that (125)–(126) are obtained using $\hat{\boldsymbol{\varphi}}$ to pull back (120)–(121) to the domain $\Omega_{\boldsymbol{\chi}}$.

5 A generalized space-time approach to the master balance laws

A more general and compact presentation of the results outlined in the previous section can be developed by means of a space-time approach. This methodology has also the advantage of highlighting key symmetries in the transformations.

5.1 Space-time master balance law using the $\boldsymbol{\varphi}$ map

Integrating over the time interval $[0, t]$, it is possible to recast (109) in space-time divergence form:

$$0 = \int_0^t \int_{\Omega_{\boldsymbol{x}}=\boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})} \nabla_{[t,\boldsymbol{x}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{x}]} - \beta \,, \tag{127}$$

where the space-time flux

$$\boldsymbol{\gamma}_{[t,\boldsymbol{x}]} = \begin{Bmatrix} \alpha \\ \alpha \boldsymbol{v} - \boldsymbol{\gamma} \end{Bmatrix} \tag{128}$$

has been introduced. Equation (127) can be also presented for a general space-time domain $\mathsf{Q}_{\boldsymbol{x}}$ as

$$0 = \int_{\mathsf{Q}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathsf{Q}_{\boldsymbol{X}})} \nabla_{[t,\boldsymbol{x}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{x}]} - \beta \,. \tag{129}$$

Using (37), equation (129) can be pushed forward to the original configuration of the body:

$$\begin{aligned} 0 &= \int_{\mathsf{Q}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathsf{Q}_{\boldsymbol{X}})} \nabla_{[t,\boldsymbol{x}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{x}]} - \beta \\ &= \int_{\mathsf{Q}_{\boldsymbol{X}}} \left(\mathsf{J}\mathbf{F}^{-T} \nabla_{[t,\boldsymbol{X}]} \right) \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{x}]} - \mathsf{J}\beta \\ &= \int_{\mathsf{Q}_{\boldsymbol{X}}} \nabla_{[t,\boldsymbol{X}]} \cdot \left(\mathsf{J}\mathbf{F}^{-1} \boldsymbol{\gamma}_{[t,\boldsymbol{x}]} \right) - \mathsf{J}\beta \\ &= \int_{\mathsf{Q}_{\boldsymbol{X}}} \nabla_{[t,\boldsymbol{X}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{X}]} - \mathsf{J}\beta \,, \end{aligned} \tag{130}$$

where

$$\begin{aligned} \boldsymbol{\gamma}_{[t,\boldsymbol{X}]} &= \mathsf{J}\mathbf{F}^{-1} \boldsymbol{\gamma}_{[t,\boldsymbol{x}]} \\ &= J \begin{bmatrix} 1 & \mathbf{0}^T \\ -\boldsymbol{F}^{-1}\boldsymbol{v} & \boldsymbol{F}^{-1} \end{bmatrix} \begin{Bmatrix} \alpha \\ \alpha \boldsymbol{v} - \boldsymbol{\gamma} \end{Bmatrix} \\ &= \begin{Bmatrix} J\alpha \\ -J\boldsymbol{F}^{-1}\boldsymbol{\gamma} \end{Bmatrix} \,. \end{aligned} \tag{131}$$

Notice that the second row of the vector $\boldsymbol{\gamma}_{[t,\boldsymbol{X}]}$, is consistent with the definition of the *spatial* Piola flux vector. To derive (130), we have used the Piola identity (40). In fact, for any $(n_d + 1)$-dimensional vector

$\mathbf{w}$,

$$
\begin{aligned}
\left(\mathsf{J}\mathbf{F}^{-T}\nabla_{[t,\boldsymbol{X}]}\right)\cdot\mathbf{w} &= \mathsf{J}\mathsf{F}^{-T}_{ij}\ \partial_j \mathsf{w}_i \\
&= \mathsf{J}\mathsf{F}^{-T}_{ij}\ \partial_j \mathsf{w}_i + \underbrace{\mathsf{w}_i\ \partial_j(\mathsf{J}\mathsf{F}^{-T}_{ij})}_{=0 \text{ by } (40)} \\
&= \partial_j(\mathsf{J}\mathsf{F}^{-T}_{ij}\mathsf{w}_i) \\
&= \nabla_{[t,\boldsymbol{X}]}\cdot(\mathsf{J}\mathbf{F}^{-1}\mathbf{w})\ . \qquad (132)
\end{aligned}
$$

5.2 Space-time master balance law using the $\hat{\boldsymbol{\varphi}}$ map

Also equation (125) can be cast in space-time divergence form:

$$
0 = \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \nabla_{[t,\boldsymbol{\chi}]}\cdot\boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]} - \hat{\mathsf{J}}\beta\ , \qquad (133)
$$

where

$$
\boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]} = \left\{\begin{array}{c} \hat{J}\alpha \\ \hat{J}\hat{\boldsymbol{F}}^{-1}(\alpha\boldsymbol{c}-\boldsymbol{\gamma}) \end{array}\right\}\ . \qquad (134)
$$

Notice the very important identity

$$
\boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]} = \hat{\mathsf{J}}\hat{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{x}]}\ , \qquad (135)
$$

which is very instructive to derive:

$$
\begin{aligned}
\hat{\mathsf{J}}\hat{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{x}]} &= \hat{J}\begin{bmatrix} 1 & \mathbf{0}^T \\ -\hat{\boldsymbol{F}}^{-1}\hat{\boldsymbol{v}} & \hat{\boldsymbol{F}}^{-1} \end{bmatrix}\left\{\begin{array}{c} \alpha \\ \alpha\boldsymbol{v}-\boldsymbol{\gamma} \end{array}\right\} \\
&= \left\{\begin{array}{c} \hat{J}\alpha \\ \hat{J}\hat{\boldsymbol{F}}^{-1}(\alpha(\boldsymbol{v}-\hat{\boldsymbol{v}})-\boldsymbol{\gamma}) \end{array}\right\} \\
&= \left\{\begin{array}{c} \hat{J}\alpha \\ \hat{J}\hat{\boldsymbol{F}}^{-1}(\alpha\boldsymbol{c}-\boldsymbol{\gamma}) \end{array}\right\} \\
&= \boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]}\ . \qquad (136)
\end{aligned}
$$

Thus, it is possible to transform (133) into (127). In fact,

$$
\begin{aligned}
0 &= \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \nabla_{[t,\boldsymbol{\chi}]}\cdot\boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]} - \hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \nabla_{[t,\boldsymbol{\chi}]}\cdot(\hat{\mathsf{J}}\hat{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{x}]}) - \hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \left(\hat{\mathsf{J}}\hat{\mathbf{F}}^{-T}\nabla_{[t,\boldsymbol{\chi}]}\right)\cdot\boldsymbol{\gamma}_{[t,\boldsymbol{x}]} - \hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\mathsf{Q}_{\boldsymbol{\chi}})} \nabla_{[t,\boldsymbol{x}]}\cdot\boldsymbol{\gamma}_{[t,\boldsymbol{x}]} - \beta\ , \qquad (137)
\end{aligned}
$$

where a result for $\hat{\boldsymbol{\varphi}}$ analogous to (132) has been used in the algebraic manipulations.

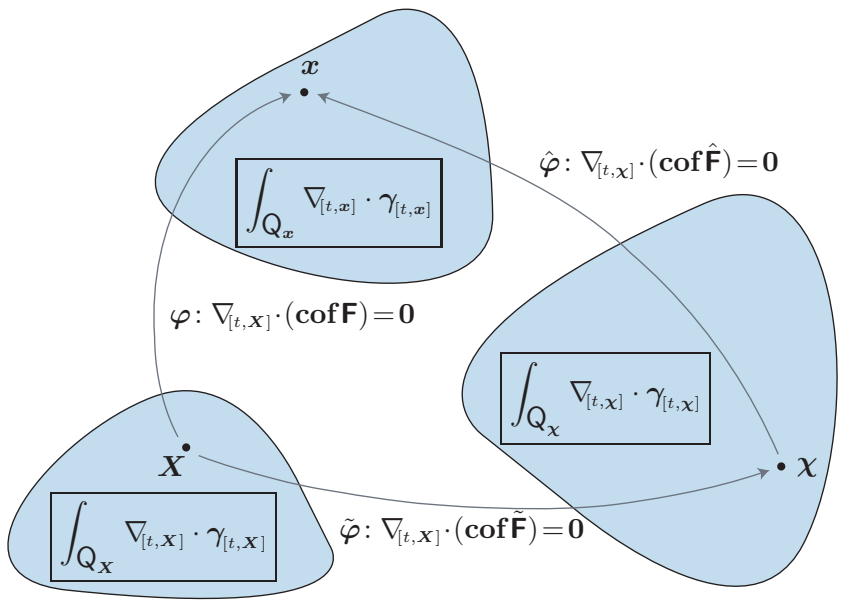

Figure 8. Sketch of the space-time divergence integral associated with each map and corresponding transformation rules.

5.3 Space-time master balance law using the $\tilde{\boldsymbol{\varphi}}$ map

Homework 7 Starting from the space-time balance law (133), and using either the space-time Piola identity (75), or the composition rule $\boldsymbol{\varphi} = \hat{\boldsymbol{\varphi}} \circ \tilde{\boldsymbol{\varphi}}$, prove that

$$\begin{aligned} 0 &= \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \nabla_{[t,\boldsymbol{\chi}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]} - \hat{\mathsf{J}}\beta \\ &= \int_{\mathsf{Q}_{\boldsymbol{X}}} \nabla_{[t,\boldsymbol{X}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{X}]} - \mathsf{J}\beta \,. \end{aligned} \tag{138}$$

The main results of the space-time appraoch are summarized in the diagram of Figure 8.

Conservation equation	α / $\boldsymbol{\alpha}$	β / $\boldsymbol{\beta}$	γ / $\boldsymbol{\Sigma}$
Mass	ρ	0	$\mathbf{0}$
Momentum	$\rho\boldsymbol{v}$	$\rho\boldsymbol{b}$	$\boldsymbol{\sigma}$
Angular momentum	$\boldsymbol{x}\times\rho\boldsymbol{v}$	$\boldsymbol{x}\times\rho\boldsymbol{b}$	$\boldsymbol{x}\times\boldsymbol{\sigma}=\varepsilon_{ikl}x_k\sigma_{lj}$
Total energy	$\rho e+\rho\boldsymbol{v}\cdot\boldsymbol{v}/2$	$\rho r+\rho\boldsymbol{v}\cdot\boldsymbol{b}$	$\boldsymbol{\sigma}^T\boldsymbol{v}-\boldsymbol{q}$
Entropy	$\rho\eta$	$\rho r/\theta$	$-\boldsymbol{q}/\theta$

Table 1. Conservation equations and corresponding substitutions in the scalar or vector master balance law.

6 Conservation laws

Using the templates of the scalar and vector master balance laws, the conservation principles for mass, momentum, angular momentum, and total energy and the entropy inequality are presented next. In addition, also a space-time divergence form of the equations is presented. Since the equations for arbitrary moving domains are pursued, the Lagrangian and Eulerian forms of the equations are omitted. These forms can be obtained by setting $\hat{\boldsymbol{v}}=\boldsymbol{v}$ and $\hat{\boldsymbol{v}}=\mathbf{0}$ in the ALE equations, respectively.

6.1 Equation in conservation form

Each of the equations in conservation form is developed from the scalar or vector master balance laws, by substituting the appropriate flux and source terms, as shown in Table 1. Local, differential statements can be derived from the integral forms of the equations, once the localization theorem for smooth integrands is applied.

6.1.1 Conservation of mass

Using the substitutions from the second line of Table 1 for equation (125),

$$0 \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot(\hat{J}\hat{\boldsymbol{F}}^{-1}\rho\boldsymbol{c})\,. \tag{139}$$

Using the definition of $\tilde{\boldsymbol{v}}$, it is possible to rewrite (139) as

$$0 \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot(\hat{J}\rho\tilde{\boldsymbol{v}})\,. \tag{140}$$

Alternatively, using the Piola identity (13) applied to the map $\hat{\boldsymbol{\varphi}}$, it is possible to push (139) forward to the current configuration, obtaining

$$0 \;=\; \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1}\left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{c})\,. \tag{141}$$

6.1.2 Conservation of momentum

Using the substitutions from the third line of Table 1 for equation (126),

$$\mathbf{0} \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left((\rho\boldsymbol{v}\otimes\boldsymbol{c} - \boldsymbol{\sigma})\hat{J}\hat{\boldsymbol{F}}^{-T} \right) + \hat{J}\rho\boldsymbol{b}\,, \tag{142}$$

where $\boldsymbol{b}$ is the body force per unit mass, and $\boldsymbol{\sigma}$ is the Cauchy stress tensor (depending on the specific constitutive law of the material). Defining by $\hat{\boldsymbol{P}} = \hat{J}\boldsymbol{\sigma}\hat{\boldsymbol{F}}^{-T}$ the *first Piola-Kirchhoff stress* tensor with respect to the map $\hat{\boldsymbol{\varphi}}$, and using the definition of $\tilde{\boldsymbol{v}}$, (142) yields

$$\mathbf{0} \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot ((\hat{J}\rho\boldsymbol{v})\otimes\tilde{\boldsymbol{v}} - \hat{\boldsymbol{P}}) + \hat{J}\rho\boldsymbol{b}\,. \tag{143}$$

Pushing forward to the current configuration,

$$\mathbf{0} \;=\; \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left.\frac{\partial(\hat{J}\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{x}} \cdot (\rho\boldsymbol{v}\otimes\boldsymbol{c} - \boldsymbol{\sigma}) + \rho\boldsymbol{b}\,. \tag{144}$$

6.1.3 Conservation of angular momentum

Using the substitutions from the fourth line of Table 1 for equation (126),

$$\mathbf{0} \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\boldsymbol{x}\times\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left((\boldsymbol{x}\times\rho\boldsymbol{v})\otimes\boldsymbol{c} - \boldsymbol{x}\times\boldsymbol{\sigma})\hat{J}\hat{\boldsymbol{F}}^{-T} \right) + \hat{J}\boldsymbol{x}\times\rho\boldsymbol{b}\,. \tag{145}$$

Homework 8 Show that conservation of angular momentum (145) is equivalent to the tensor $\boldsymbol{\sigma}$ being symmetric. Hint: Use index notation, and collect the cross product of the position times the momentum equation. Use then the definition of $\hat{F}_{ij}$ and the fact that $\boldsymbol{v}\times\boldsymbol{v} = \mathbf{0}$, $\forall\boldsymbol{v}$.

6.1.4 Conservation of total energy

Using the substitutions of the fifth line of Table 1 for equation (125),

$$0 \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J}\hat{\boldsymbol{F}}^{-1}(\rho E\boldsymbol{c} - \boldsymbol{\sigma}^T\boldsymbol{v} + \boldsymbol{q}) \right) + \hat{J}\rho(\boldsymbol{b}\cdot\boldsymbol{v} + r)\,, \tag{146}$$

where $E = (e + \boldsymbol{v}\cdot\boldsymbol{v}/2)$ is the total energy per unit mass, e is the internal per unit mass, $\boldsymbol{q}$ is the heat flux, and r is the energy source/sink per unit mass. The specific expressions for $\boldsymbol{q}$ and r depend on the

material constitutive law. If a *Piola-Kirchhoff heat flux* $\hat{\boldsymbol{Q}} = \hat{J}\boldsymbol{F}^{-1}\boldsymbol{q}$ with respect to the map $\hat{\boldsymbol{\varphi}}$ is defined, (146) reduces to

$$0 = \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left((\hat{J}\rho E)\tilde{\boldsymbol{v}} - \hat{\boldsymbol{P}}^T\boldsymbol{v} + \hat{\boldsymbol{Q}}\right) + \hat{J}\rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \,. \tag{147}$$

Pushing forward to the current configuration,

$$0 = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{x}} \cdot \left(\rho E\boldsymbol{c} - \boldsymbol{\sigma}^T\boldsymbol{v} + \boldsymbol{q}\right) + \rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \,. \tag{148}$$

6.1.5 Entropy balance

Using the substitutions of the sixth line of Table 1 for equation (125),

$$\begin{aligned} 0 &\leq \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\eta)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J}\hat{\boldsymbol{F}}^{-1}\left(\rho\eta\boldsymbol{c} + \frac{\boldsymbol{q}}{\theta}\right)\right) + \hat{J}\rho\frac{r}{\theta} \\ &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\eta)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left(\left(\hat{J}\rho\eta\right)\tilde{\boldsymbol{v}} + \frac{\hat{\boldsymbol{Q}}}{\theta}\right) + \hat{J}\rho\frac{r}{\theta} \,, \end{aligned} \tag{149}$$

where η is the entropy, and θ is the thermodynamic temperature. Pushing forward to the current configuration,

$$0 \leq \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1} \left.\frac{\partial(\hat{J}\rho\eta)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{x}} \cdot \left(\rho\eta\boldsymbol{c} + \frac{\boldsymbol{q}}{\theta}\right) + \rho\frac{r}{\theta} \,. \tag{150}$$

Remark 18 The entropy condition is of fundamental importance when seeking solutions of the conservation principles which are consistent with the second law of thermodynamics. A typical example is the case of shock waves, for which at least two solutions satisfy the mass, momentum, angular momentum and total energy equations, but only one of them satisfies the entropy inequality.

6.2 Equations in conservative vector form

The mass, momentum, and total energy equations can be compactly expressed by means of a *conservative vector form*. For this purpose, let us define:

$$\mathsf{U} = \begin{Bmatrix} \rho \\ \rho\boldsymbol{v} \\ \rho E \end{Bmatrix}, \qquad \mathsf{G} = \mathsf{U} \otimes \boldsymbol{c} - \begin{bmatrix} \boldsymbol{0}^T \\ \boldsymbol{\sigma} \\ \boldsymbol{v}^T\boldsymbol{\sigma} - \boldsymbol{q}^T \end{bmatrix}, \qquad \mathsf{Z} = \begin{Bmatrix} \boldsymbol{0}^T \\ \rho\boldsymbol{b} \\ \rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \end{Bmatrix} \,, \tag{151}$$

where the notation $\mathsf{U} \otimes \boldsymbol{c} = \mathsf{U}\boldsymbol{c}^T = \mathsf{U}_i c_j$ for the dyadic (tensor) product has been used. Then (139), (142), and (146) can be written as

$$\boldsymbol{0} = \int_{\Omega_{\boldsymbol{\chi}}} \partial_t|_{\chi}(\hat{J}\mathsf{U}) + \nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J}\mathsf{G}\hat{\boldsymbol{F}}^{-T}\right) + \hat{J}\mathsf{Z} \,, \tag{152}$$

where the divergence operator $\nabla_{\boldsymbol{\chi}} \cdot$ is understood to apply to the second index of the $((n_d+2)\times n_d)$-matrix $\hat{J}\mathbf{G}\hat{\boldsymbol{F}}^{-T}$, that is,

$$\nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J}\mathbf{G}\hat{\boldsymbol{F}}^{-T}\right) = \partial_{\chi_j}\left(\hat{J}\mathsf{G}_{ik}\hat{F}^{-T}_{kj}\right) \ . \tag{153}$$

Alternatively, using the Piola identity in space for the map $\hat{\boldsymbol{\varphi}}$, (152) can be arranged as

$$\mathbf{0} = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \hat{J}^{-1}\partial_t\,|_{\boldsymbol{\chi}}(\hat{J}\mathbf{U}) + \nabla_{\boldsymbol{x}} \cdot \mathbf{G} + \mathbf{Z} \ . \tag{154}$$

6.3 Space-time conservative vector form

Equation (152) can also be presented in space-time form,

$$\mathbf{0} = \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \nabla_{[t,\boldsymbol{\chi}]} \cdot \mathbf{G}_{[t,\boldsymbol{\chi}]} + \hat{\mathsf{J}}\mathbf{Z} \ , \tag{155}$$

where, in perfect analogy with the definition of the flux $\boldsymbol{\gamma}_{[t,\boldsymbol{\chi}]}$ for the space-time balance law,

$$\mathbf{G}_{[t,\boldsymbol{\chi}]} \ = \ \left[\hat{J}\mathbf{U} \,\middle|\, \hat{J}\mathbf{G}\hat{\boldsymbol{F}}^{-T}\right] \ . \tag{156}$$

In particular,

$$\mathbf{G}_{[t,\boldsymbol{\chi}]} \ = \ \hat{\mathsf{J}}\mathbf{G}_{[t,\boldsymbol{x}]}\hat{\mathbf{F}}^{-T} \ , \tag{157}$$

with

$$\mathbf{G}_{[t,\boldsymbol{x}]} \ = \ \left[\mathbf{U} \,\middle|\, \mathbf{U}\boldsymbol{v}^T - \begin{bmatrix} \mathbf{0}^T \\ \boldsymbol{\sigma} \\ \boldsymbol{v}^T\boldsymbol{\sigma} - \boldsymbol{q} \end{bmatrix}\right] \ . \tag{158}$$

Homework 9 Derive (157).

Using (157), equation (155) can be recast in the current configuration as

$$\mathbf{0} = \int_{\mathsf{Q}_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\mathsf{Q}_{\boldsymbol{\chi}})} \nabla_{[t,\boldsymbol{x}]} \cdot \mathbf{G}_{[t,\boldsymbol{x}]} + \mathbf{Z} \ . \tag{159}$$

6.4 Equations in advective form

In many practical instances, the so-called *advective* form of the balance equations is used. One of the advantages of the advective form is the ability to satisfy exactly a Geometric Conservation Law, to be discussed in Section 8.

6.4.1 Advective form of the mass conservation equation

The advective form of the mass conservation equation can be obtained with a number of manipulations involving the Piola identity relative to the map $\hat{\boldsymbol{\varphi}}$, and equation (51), expressing the referential time derivative of the Jacobian determinant $\hat{J}$. Starting from the mass conservation equation (139),

$$
\begin{aligned}
0 &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot \left(\hat{J}\hat{\boldsymbol{F}}^{-1}\rho\boldsymbol{c}\right) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \rho\left.\frac{\partial\hat{J}}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_i}\left(\hat{J}\hat{F}^{-1}_{ij}\rho c_j\right) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \rho\left.\frac{\partial\hat{J}}{\partial t}\right|_{\boldsymbol{\chi}} + \rho c_j \underbrace{\partial_{\chi_i}\left(\hat{J}\hat{F}^{-T}_{ji}\right)}_{=0 \text{ by } (13)} + c_j\hat{J}\hat{F}^{-1}_{ij}\partial_{\chi_i}\rho - \hat{J}\rho\hat{F}^{-1}_{ij}\partial_{\chi_i}\hat{v}_j + \hat{J}\rho\hat{F}^{-1}_{ij}\partial_{\chi_i}v_j \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}(\hat{\boldsymbol{F}}^{-1}\boldsymbol{c})\cdot\nabla_{\boldsymbol{\chi}}\rho + \hat{J}\rho\hat{\boldsymbol{F}}^{-T} : \nabla_{\boldsymbol{\chi}}\boldsymbol{v} + \underbrace{\rho\left.\frac{\partial\hat{J}}{\partial t}\right|_{\boldsymbol{\chi}} - \hat{J}\rho\hat{\boldsymbol{F}}^{-T} : \nabla_{\boldsymbol{\chi}}\hat{\boldsymbol{v}}}_{=0 \text{ by } (51)} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\tilde{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{\chi}}\rho + \hat{J}\rho\hat{\boldsymbol{F}}^{-T} : \nabla_{\boldsymbol{\chi}}\boldsymbol{v}\ ,
\end{aligned}
\tag{160}
$$

where we have used (77), to substitute $\hat{\boldsymbol{F}}\boldsymbol{c} = \tilde{\boldsymbol{v}}$. Pushing (160) forward to the current configuration yields the more intuitive form

$$
0 = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c}\cdot\nabla_{\boldsymbol{x}}\rho + \rho\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}\ . \tag{161}
$$

It is worth noticing in (161) the use of (78) for the Lagrangian time derivative of ρ, namely:

$$
\dot{\rho} = \left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c}\cdot\nabla_{\boldsymbol{x}}\rho\ . \tag{162}
$$

6.4.2 Advective form of the momentum equation

The advective form of the momentum equation can be derived by noticing that (142), namely,

$$
\begin{aligned}
\mathbf{0} &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot\left((\rho\boldsymbol{v}\otimes\boldsymbol{c} - \boldsymbol{\sigma})\hat{J}\hat{\boldsymbol{F}}^{-T}\right) + \hat{J}\rho\boldsymbol{b} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho\left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + v_i\left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_k}\left(\hat{J}\rho v_i c_j\hat{F}^{-T}_{jk}\right) - \partial_{\chi_k}\left(\hat{J}\sigma_{ij}\hat{F}^{-T}_{jk}\right) + \hat{J}\rho b_i\ ,
\end{aligned}
\tag{163}
$$

contains the product of the velocity $\boldsymbol{v}$ times the mass conservation equation (139). Thus, simplifying this term,

$$
\begin{aligned}
\mathbf{0} &= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + v_i \underbrace{\left(\left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_k}\left(\hat{J}\rho c_j \hat{F}^{-T}_{jk}\right)\right)}_{=0 \text{ by } (139)} \\
&\quad + \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho c_j \hat{F}^{-T}_{jk}\partial_{\chi_k} v_i - \partial_{\chi_k}\left(\hat{J}\sigma_{ij}\hat{F}^{-T}_{jk}\right) + \hat{J}\rho b_i \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho c_j \hat{F}^{-T}_{jk}\partial_{\chi_k} v_i - \partial_{\chi_k}\left(\hat{J}\sigma_{ij}\hat{F}^{-T}_{jk}\right) + \hat{J}\rho b_i \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho\nabla_{\boldsymbol{\chi}}\boldsymbol{v}(\hat{\boldsymbol{F}}^{-1}\boldsymbol{c}) - \nabla_{\boldsymbol{\chi}}\cdot\left(\hat{J}\boldsymbol{\sigma}\hat{\boldsymbol{F}}^{-T}\right) + \hat{J}\rho\boldsymbol{b}\,. \qquad (164)
\end{aligned}
$$

Using the definition of the first Piola-Kirchhoff stress tensor $\hat{\boldsymbol{P}}$ with respect to the map $\hat{\boldsymbol{\varphi}}$, it possible to recast (164) as

$$
\mathbf{0} = \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho(\nabla_{\boldsymbol{\chi}}\boldsymbol{v})\tilde{\boldsymbol{v}} - \nabla_{\boldsymbol{\chi}}\cdot\hat{\boldsymbol{P}} + \hat{J}\rho\boldsymbol{b}\,, \qquad (165)
$$

Pushing (164) forward to the current configuration, we obtain:

$$
\mathbf{0} = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \rho \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{\chi}} + \rho(\nabla_{\boldsymbol{x}}\boldsymbol{v})\boldsymbol{c} - \nabla_{\boldsymbol{x}}\cdot\boldsymbol{\sigma} + \rho\boldsymbol{b}\,. \qquad (166)
$$

6.4.3 Advective form of the total energy equation

The advective form of the total energy equation can be derived by noticing that (146) contains the product of the velocity $\boldsymbol{v}$ times the momentum equation (142) (also called the *kinetic energy* equation), and the product of the internal energy e times the mass conservation equation.

Homework 10 Show that

$$
\begin{aligned}
0 &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot(\hat{J}\hat{\boldsymbol{F}}^{-1}(\rho E\boldsymbol{c} - \boldsymbol{\sigma}^T\boldsymbol{v} + \boldsymbol{q})) + \hat{J}\rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho\tilde{\boldsymbol{v}}\nabla_{\boldsymbol{\chi}} e - \left(\hat{J}\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right) : \nabla_{\boldsymbol{\chi}}\boldsymbol{v} + \nabla_{\boldsymbol{\chi}}\cdot(\hat{J}\boldsymbol{F}^{-1}\boldsymbol{q}) + \hat{J}\rho r\,. \qquad (167)
\end{aligned}
$$

Using the definition of the Piola-Kirchhoff heat flux $\hat{\boldsymbol{Q}}$ and the first Piola-Kirchhoff stress tensor $\hat{\boldsymbol{P}}$, yields

$$
0 = \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho\tilde{\boldsymbol{v}}\nabla_{\boldsymbol{\chi}} e - \hat{\boldsymbol{P}} : \nabla_{\boldsymbol{\chi}}\boldsymbol{v} + \nabla_{\boldsymbol{\chi}}\cdot\hat{\boldsymbol{Q}} + \hat{J}\rho r\,. \qquad (168)
$$

Pushing (167) forward to the current configuration, and applying the Piola identity (13) for the map $\hat{\boldsymbol{\varphi}}$,

$$0 = \int_{\Omega_{\boldsymbol{x}}=\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})} \rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c}\cdot\nabla_{\boldsymbol{x}} e - \boldsymbol{\sigma} : \nabla_{\boldsymbol{x}}\boldsymbol{v} + \nabla_{\boldsymbol{x}}\cdot\boldsymbol{q} + \rho r \,. \tag{169}$$

6.4.4 Advective form of the entropy equation

Equation (149) contains the product of the entropy η times the mass conservation equation (139). Hence,

$$\begin{aligned}
0 &\leq \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\eta)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot\left(\hat{J}\hat{\boldsymbol{F}}^{-1}\left(\rho\eta\boldsymbol{c}+\frac{\boldsymbol{q}}{\theta}\right)\right) + \hat{J}\rho\frac{r}{\theta} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial\eta}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho c_j \hat{F}^{-1}_{ij}\partial_{\chi_i}\eta + \partial_{\chi_i}\left(\hat{J}\hat{F}^{-1}_{ij}\frac{q_j}{\theta}\right) + \hat{J}\rho\frac{r}{\theta} \\
&\quad + \eta \underbrace{\left(\left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_i}(\hat{J}\hat{F}^{-1}_{ij}\rho c_j)\right)}_{=0 \text{ by } (139)} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial\eta}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho(\hat{\boldsymbol{F}}^{-1}\boldsymbol{c})\cdot\nabla_{\boldsymbol{\chi}}\eta + \nabla_{\boldsymbol{\chi}}\cdot\left(\hat{J}\hat{\boldsymbol{F}}^{-1}\frac{\boldsymbol{q}}{\theta}\right) + \hat{J}\rho\frac{r}{\theta} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial\eta}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho\tilde{\boldsymbol{v}}\nabla_{\boldsymbol{\chi}}\eta + \nabla_{\boldsymbol{\chi}}\cdot\left(\frac{\hat{\boldsymbol{Q}}}{\theta}\right) + \hat{J}\rho\frac{r}{\theta} \,.
\end{aligned} \tag{170}$$

Pushing forward to the current configuration,

$$0 \leq \int_{\Omega_{\boldsymbol{\chi}}} \rho \left.\frac{\partial\eta}{\partial t}\right|_{\boldsymbol{\chi}} + \rho\boldsymbol{c}\cdot\nabla_{\boldsymbol{x}}\eta + \nabla_{\boldsymbol{x}}\cdot\left(\frac{\boldsymbol{q}}{\theta}\right) + \rho\frac{r}{\theta} \,. \tag{171}$$

7 Variational equations and constitutive laws

This section is devoted to the development of variational forms of use in computational methods (particularly, the finite element method). Three examples will be considered: The compressible Navier-Stokes equations, the incompressible Navier-Stokes equations and the equations of compressible, isothermal, nonlinear elasticity. Only space-time variational formulations are considered, with the exception of the incompressible Navier-Stokes equations, for which a semi-discrete advective weak form is also presented. The space-time approach has the advantage of a broader generality, with respect to semi-discrete implementations, since, in many cases of practical interest, semi-discrete variational formulations can be interpreted as space-time formulations with approximate integration quadratures in time.

7.1 Compressible Navier-Stokes equations

7.1.1 Constitutive laws and initial/boundary value problem

The motion of a viscous, conductive, compressible fluid is described by the compressible Navier-Stokes equations, for which the following consitutive relationships hold:

$$\boldsymbol{\sigma} = 2\rho\nu\nabla_{\boldsymbol{x}}^{s}\boldsymbol{v} - p\boldsymbol{I} \ , \tag{172}$$

$$\boldsymbol{q} = -\rho\kappa c_p\nabla_{\boldsymbol{x}}\theta \ , \tag{173}$$

$$p = \mathsf{p}(\rho, e) \ , \tag{174}$$

where ν is the *kinematic viscosity*, $\nabla_{\boldsymbol{x}}^{s} = 1/2(\nabla_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}}{}^{T})$ is the symmetric part of the gradient, p is the pressure of the fluid, $\boldsymbol{I} = \delta_{ij}$ is the identity (or Kronecker) tensor, c_p is the specific heat at constant pressure, and κ is the thermal diffusivity coefficient (in a fluid, thermal diffusion is typically assumed to be an isotropic process). Equation (174) is the equation of state for the fluid, and expresses the fact that the thermodynamic pressure, the internal energy and the density are not independent of one another. For most fluids, it is possible to express the internal energy e in terms of the temperature θ as follows

$$e = c_v(\theta)\theta \ , \tag{175}$$

with c_v the specific heat at constant volume. Typically, c_v and c_p are functions of θ. Let

$$\mathsf{U} = \left\{ \begin{array}{c} \rho \\ \rho\boldsymbol{v} \\ \rho(e + \boldsymbol{v}\cdot\boldsymbol{v}/2) \end{array} \right\} \ , \tag{176}$$

$$\mathsf{G} = \mathsf{U} \otimes \boldsymbol{c} + \mathsf{G}_{\mathsf{L}} \ , \tag{177}$$

$$\mathsf{G}_{\mathsf{L}} = -\left[\begin{array}{c} \boldsymbol{0}^T \\ 2\rho\nu\nabla_{\boldsymbol{x}}^{s}\boldsymbol{v} - p\boldsymbol{I} \\ 2\rho\nu\boldsymbol{v}^T\nabla_{\boldsymbol{x}}^{s}\boldsymbol{v} - \boldsymbol{v}^T p + \rho c_p\kappa(\nabla_{\boldsymbol{x}}\theta)^T \end{array} \right] \ , \tag{178}$$

$$\mathsf{Z} = \left\{ \begin{array}{c} 0 \\ \rho\boldsymbol{b} \\ \rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \end{array} \right\} \ , \tag{179}$$

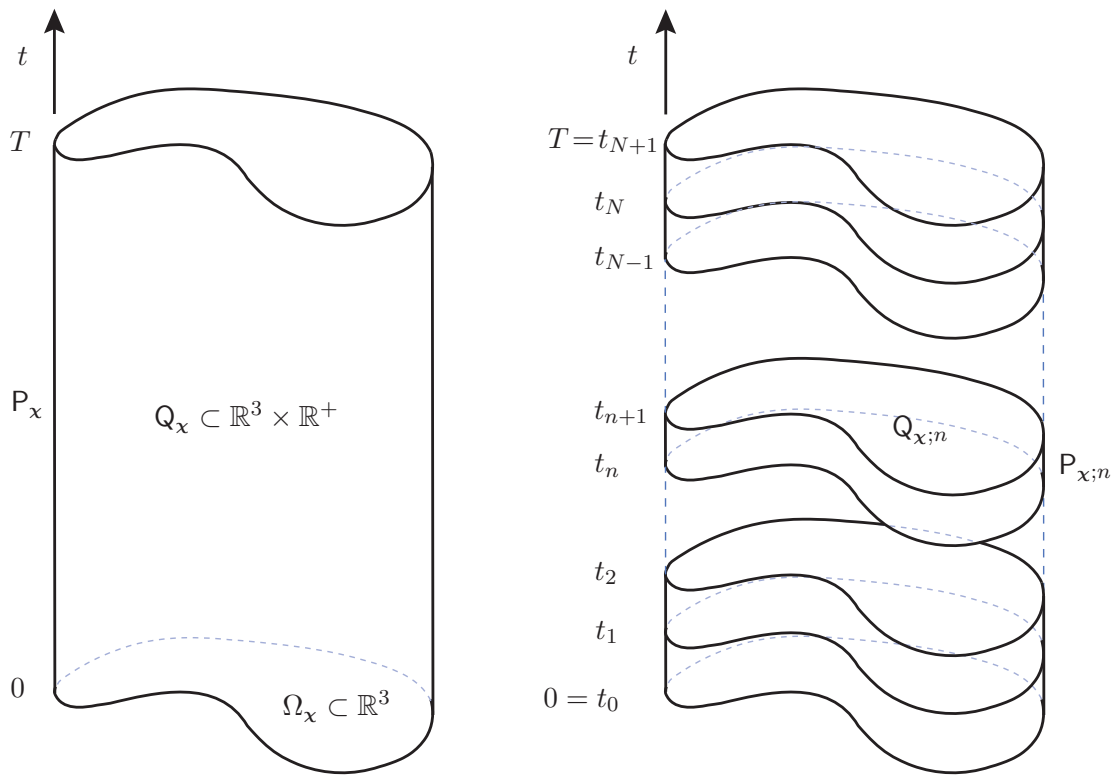

Figure 9. Space-time domain (left) and slicing into space-time slabs (right).

where the matrix $\mathbf{G}_{\mathsf{L}}$ represents the Lagrangian part of the spatial fluxes ($\mathbf{G} = \mathbf{G}_{\mathsf{L}}$ when $\boldsymbol{c} = \mathbf{0}$). Hence, the following initial/boundary value problem can be posed:

$$\partial_t|_{\boldsymbol{\chi}}(\hat{J}\mathbf{U}) + \nabla_{\boldsymbol{\chi}} \cdot \left(\mathbf{G}\ \mathbf{cof}\,\hat{\boldsymbol{F}}\right) + \hat{J}\mathbf{Z} = \mathbf{0}, \quad \text{in } \Omega_{\boldsymbol{\chi}}\ , \tag{180}$$

$$\mathcal{U}(\mathbf{U}) = \mathbf{U}_g\ , \quad \text{on } \partial\Omega^g_{\boldsymbol{\chi}}\ , \tag{181}$$

$$\mathbf{G}\ \mathbf{cof}\,\hat{\boldsymbol{F}}\ \boldsymbol{n}_{\boldsymbol{\chi}} = \mathbf{h}\ , \quad \text{on } \partial\Omega^h_{\boldsymbol{\chi}}\ , \tag{182}$$

$$\mathbf{U}(\boldsymbol{\chi}, 0) = \mathbf{U}_0(\boldsymbol{\chi})\ , \tag{183}$$

where $\partial\Omega^g_{\boldsymbol{\chi}}$ is the *Dirichlet* boundary, $\mathcal{U}(\cdot)$ is a boundary operator that may mask some of the entries of $\mathbf{U}$, and $\mathbf{U}_g$ is the vector of Dirichlet boundary conditions, which, in the most general case, may be a function of the solution itself. Analogously, $\partial\Omega^h_{\boldsymbol{\chi}}$ is the *Neumann* boundary, and $\mathbf{h}$ is the (n_d+2)-dimensional vector of Neumann conditions. Typically, $\boldsymbol{b}$, the body force per unit mass, is due to a gravitational field. Using the intuitive notation outlined in (154), the initial/boundary value problem (180)–(183) can be restated as

$$\hat{J}^{-1}\partial_t|_{\boldsymbol{\chi}}(\hat{J}\mathbf{U}) + \nabla_{\boldsymbol{x}} \cdot \mathbf{G} + \mathbf{Z} = \mathbf{0}\ , \quad \text{in } \Omega_{\boldsymbol{x}}\ , \tag{184}$$

$$\mathcal{U}(\mathbf{U}) = \mathbf{U}_g\ , \quad \text{on } \partial\Omega^g_{\boldsymbol{x}}\ , \tag{185}$$

$$\mathbf{G}\boldsymbol{n}_{\boldsymbol{x}} = \mathbf{h}\ , \quad \text{on } \partial\Omega^h_{\boldsymbol{x}}\ , \tag{186}$$

$$\mathbf{U}(\boldsymbol{x}, 0) = \mathbf{U}_0(\boldsymbol{x})\ , \tag{187}$$

7.1.2 Space-time variational formulation

We consider a space-time domain $\mathsf{Q}_{\boldsymbol{\chi}} = \Omega_{\boldsymbol{\chi}} \times]0,T[\subset \mathbb{R}^{n_d} \times \mathbb{R}^+$ with lateral boundary $\mathsf{P}_{\boldsymbol{\chi}} = \partial\Omega_{\boldsymbol{\chi}} \times]0,T[$, as illustrated in the left-hand side of Figure 9. $\mathsf{P}_{\boldsymbol{\chi}}$ is further divided into the Dirichlet portion of the lateral boundary $\mathsf{P}^g_{\boldsymbol{\chi}}$, and the Neumann portion of the lateral boundary $\mathsf{P}^h_{\boldsymbol{\chi}}$, such that $\mathsf{P}^g_{\boldsymbol{\chi}} \cap \mathsf{P}^h_{\boldsymbol{\chi}} = \emptyset$, and $\overline{\mathsf{P}^g_{\boldsymbol{\chi}} \cup \mathsf{P}^h_{\boldsymbol{\chi}}} = \mathsf{P}_{\boldsymbol{\chi}}$. In order to develop a space-time variational formulation, it is sufficient to test (180) against an appropriate *vector* test function $\hat{\mathbf{W}}(\boldsymbol{\chi}, t)$, and then apply integration by parts to account for the *natural* (or Neumann) boundary conditions and the initial conditions. The Dirichlet boundary conditions are instead directly incorporated into the function spaces representing the solution. The space-time weak form then reads:

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi}, T)\, \hat{\mathbf{W}}(\boldsymbol{\chi}, T^-) \cdot \mathbf{U}(\boldsymbol{\chi}, T^-) - \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi}, 0)\, \hat{\mathbf{W}}(\boldsymbol{\chi}, 0^+) \cdot \mathbf{U}(\boldsymbol{\chi}, 0^-) \\
& - \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \nabla_{[t,\boldsymbol{\chi}]} \hat{\mathbf{W}} : \mathbf{G}_{[t,\boldsymbol{\chi}]} + \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot (\hat{\mathsf{J}}\mathbf{Z}) \\
& + \int_{\mathsf{P}^g_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\left(\hat{J} \mathbf{G} \hat{\boldsymbol{F}}^{-T} \right) \boldsymbol{n}_{\boldsymbol{\chi}} \right) + \int_{\mathsf{P}^h_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \mathbf{h} \\
= \; & \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi}, T)\, \hat{\mathbf{W}}(\boldsymbol{\chi}, T^-) \cdot \mathbf{U}(\boldsymbol{\chi}, T^-) - \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi}, 0)\, \hat{\mathbf{W}}(\boldsymbol{\chi}, 0^+) \cdot \mathbf{U}(\boldsymbol{\chi}, 0^-) \\
& - \left(\int_{\mathsf{Q}_{\boldsymbol{\chi}}} (\partial_t|_{\boldsymbol{\chi}} \hat{\mathbf{W}}) \cdot (\hat{J}\mathbf{U}) + \nabla_{\boldsymbol{\chi}} \hat{\mathbf{W}} : \left(\hat{J} \mathbf{G} \hat{\boldsymbol{F}}^{-T} \right) \right) + \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot (\hat{J}\mathbf{Z}) \\
& + \int_{\mathsf{P}^g_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\left(\hat{J} \mathbf{G} \hat{\boldsymbol{F}}^{-T} \right) \boldsymbol{n}_{\boldsymbol{\chi}} \right) + \int_{\mathsf{P}^h_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \mathbf{h} \,,
\end{aligned}
\tag{188}
$$

where $t^{\pm} = \lim_{\varepsilon \to 0^{\pm}} (t + \varepsilon)$, and $\mathbf{h}$ is the vector of Neumann boundary conditions. Imposing the Dirichlet boundary conditions *strongly* means that the test function space is such that the corresponding entries of the vector $\hat{\mathbf{W}}$ vanish at the Dirichlet boundary, where the boundary value of the solution is enforced. In order to derive a particular space-time discretization, the only additional step to be accomplished is the specific definition of discrete test and trial spaces, over time slabs $\mathsf{Q}_{\boldsymbol{\chi};n} = \Omega_{\boldsymbol{\chi}} \times]t_{n-1}, t_n[$ such that $[0,T] = \overline{\bigcup_{n=1}^{N+1} \mathsf{Q}_{\boldsymbol{\chi};n}}$. This approach yields a time-stepping methodology, as depicted on the right-hand side of Figure 9. Equation (188) can also be pushed forward to the current configuration, obtaining

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{x}}(T)} \mathbf{W}(\boldsymbol{x}, T^-) \cdot \mathbf{U}(\boldsymbol{x}, T^-) - \int_{\Omega_{\boldsymbol{x}}(0)} \mathbf{W}(\boldsymbol{x}, 0^+) \cdot \mathbf{U}(\boldsymbol{x}, 0^-) \\
& - \left(\int_{\mathsf{Q}_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathsf{Q}_{\boldsymbol{\chi}})} (\partial_t|_{\boldsymbol{\chi}} \mathbf{W}) \cdot \mathbf{U} + \nabla_{\boldsymbol{x}} \mathbf{W} : \mathbf{G} \right) + \int_{\mathsf{Q}_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathsf{Q}_{\boldsymbol{\chi}})} \mathbf{W} \cdot \mathbf{Z} \\
& + \int_{\mathsf{P}^g_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathsf{P}^g_{\boldsymbol{\chi}})} \mathbf{W} \cdot (\mathbf{G} \boldsymbol{n}_{\boldsymbol{x}}) + \int_{\mathsf{P}^h_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathsf{P}^h_{\boldsymbol{\chi}})} \mathbf{W} \cdot \mathbf{h} \,,
\end{aligned}
\tag{189}
$$

where $\mathbf{W}(\boldsymbol{x}, t) = \hat{\mathbf{W}}(\hat{\boldsymbol{\varphi}}^{-1}(\boldsymbol{x}, t), t)$.

Remark 19 The proposed space-time approach is *prismatic* in time, that is, the time slab $\mathsf{Q}_{\boldsymbol{\chi}}$ (or, resp., $\mathsf{Q}_{\boldsymbol{x}}$) is *extruded* from the domain $\Omega_{\boldsymbol{\chi}}$ (resp., $\Omega_{\boldsymbol{x}}$). A different methodology is to define a space-time discontinous Galerkin formulation, in which the space-time domain is discretized, for example, using $(n_d + 1)$-dimensional space-time simplices. This approach, advocated in [25] entails completely new time-stepping

algorithms which leverage advancing-front strategies to update the solution in each of the space-time simplices of the discretization.

7.1.3 Euler-Lagrange equations

The Euler-Lagrange equations provide understanding on the nature of the variational formulation. They are obtained assuming that the solution is sufficiently smooth, so that integration by parts can be performed. If this is the case, (188) yields

$$
\begin{aligned}
0 &= \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\partial_t|_{\boldsymbol{\chi}} (\hat{J}\mathbf{U}) + \nabla_{\boldsymbol{x}} \cdot \left(\hat{J}\mathbf{G}\hat{\boldsymbol{F}}^{-T} \right) + \hat{J}\mathbf{Z} \right) \\
&\quad + \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi},0)\hat{\mathbf{W}}(\boldsymbol{\chi},0^+) \cdot [\![\mathbf{U}(\boldsymbol{\chi},0)]\!] - \int_{\mathsf{P}^h_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\hat{J}\mathbf{G}\hat{\boldsymbol{F}}^{-T}\boldsymbol{n}_{\boldsymbol{\chi}} - \mathbf{h} \right) \\
&= \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\nabla_{[t,\boldsymbol{\chi}]} \cdot \mathbf{G}_{[t,\boldsymbol{\chi}]} + \hat{\mathsf{J}}\mathbf{Z} \right) \\
&\quad + \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi},0)\hat{\mathbf{W}}(\boldsymbol{\chi},0^+) \cdot [\![\mathbf{U}(\boldsymbol{\chi},0)]\!] - \int_{\mathsf{P}^h_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\hat{J}\mathbf{G}\hat{\boldsymbol{F}}^{-T}\boldsymbol{n}_{\boldsymbol{\chi}} - \mathbf{h} \right) \ . \qquad (190)
\end{aligned}
$$

Equation (190) can also be written in the current configuration, with a more intuitive interpretation:

$$
\begin{aligned}
0 &= \int_{\mathsf{Q}_{\boldsymbol{x}}} \mathbf{W} \cdot \left(\hat{J}^{-1}\partial_t|_{\boldsymbol{\chi}}(\hat{J}\mathbf{U}) + \nabla_{\boldsymbol{x}} \cdot \mathbf{G} + \mathbf{Z} \right) \\
&\quad + \int_{\Omega_{\boldsymbol{x}}(0)} \mathbf{W}(\boldsymbol{x},0^+) \cdot [\![\mathbf{U}(\boldsymbol{x},0)]\!] - \int_{\mathsf{P}^h_{\boldsymbol{x}}} \mathbf{W} \cdot (\mathbf{G}\boldsymbol{n}_{\boldsymbol{x}} - \mathbf{h}) \ , \qquad (191)
\end{aligned}
$$

which enforces the Navier-Stokes equations on the interior of $\mathsf{Q}_{\boldsymbol{x}}$, Neumann boundary conditions on the boundary $\mathsf{P}^h_{\boldsymbol{x}}$, initial conditions at time $t = 0$, through *causality* of the solution, that is the weak satisfaction of the continuity requirement $[\![\mathbf{U}(\boldsymbol{x},0)]\!] = \mathbf{U}(\boldsymbol{x},0^+) - \mathbf{U}(\boldsymbol{x},0^-) = \mathbf{0}$.

Homework 11 Derive the Euler-Lagrange equations assuming the solution is smooth and integration by parts can be performed.

7.1.4 Conservation properties

It is straightforward to prove the formulation outlined in (188) or (189) embeds *global* conservation properties. In fact, assuming that $\mathbf{Z} = \mathbf{0}$, that only *homogenous* Neumann boundary conditions are imposed, and, by consistency, that the weak formulation can be tested on a shape function whose components are equal to unity over the entire domain, (189) implies

$$
\mathbf{0} = \int_{\Omega_{\boldsymbol{x}}(T)} \mathbf{U}(\boldsymbol{x},T^-) - \int_{\Omega_{\boldsymbol{x}}(0)} \mathbf{U}(\boldsymbol{x},0^-) \ , \qquad (192)
$$

which is a conservation statement between the time instants $t = 0$ and $t = T$, for the vector of conserved variables $\mathbf{U}$. An analogous statement holds when the space-time formulation is applied to each of the discrete time-slabs $\mathsf{Q}_{\boldsymbol{x};n}$.

7.2 Incompressible Navier-Stokes equations

We present two formulations for the incompressible Navier-Stokes equations, both widely applied in numerical computations. The first is a conservative space-time formulation, the second a semi-discrete advective formulation.

7.2.1 Constitutive laws and initial/boundary value problem

The incompressible case requires the additional constraint that the Lagrangian time-derivative of the density ρ vanishes. Using (160) or (161), this implies:

$$\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0 \; . \tag{193}$$

The impact of this additional constraint on the Navier-Stokes equations is significant, since the pressure assumes the role of a *Lagrange multiplier*, enforcing incompressibility of the flow, and no equation of state of the type (174) is needed. In addition, the energy equation *uncouples* from the momentum equation, reducing to a transport equation for the internal energy e (or, alternatively, the temperature θ). In writing the equations, the initial distribution of density is not assumed constant. Therefore a dependency of ρ on $\boldsymbol{x}$ and t appears in the equations which follow, according to the relation $\rho = \rho(\boldsymbol{X}) = \rho(\boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t))$. The initial boundary value problem for the incompressible Navier-Stokes equations reads:

$$\begin{aligned}
\partial_t|_{\boldsymbol{\chi}} (\hat{J}\rho\boldsymbol{v}) + \nabla_{\boldsymbol{\chi}} \cdot \left((\rho\boldsymbol{v} \otimes \boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla_{\boldsymbol{x}}^s \boldsymbol{v})\hat{J}\hat{\boldsymbol{F}}^{-T} \right) + \hat{J}\rho\boldsymbol{b} &= \boldsymbol{0}, \quad \text{in } \Omega_{\boldsymbol{\chi}} \; , && (194)\\
\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} &= \boldsymbol{0}, \quad \text{in } \Omega_{\boldsymbol{\chi}} \; , && (195)\\
\boldsymbol{v} &= \boldsymbol{v}_g \; , \quad \text{on } \partial\Omega_{\boldsymbol{\chi}}^g \; , && (196)\\
(p\boldsymbol{I} - 2\rho\nu\nabla_{\boldsymbol{x}}^s \boldsymbol{v}) \, \mathbf{cof}\hat{\boldsymbol{F}} \; \boldsymbol{n}_{\boldsymbol{\chi}} &= \boldsymbol{h} \; , \quad \text{on } \partial\Omega_{\boldsymbol{\chi}}^{h;+} \; , && (197)\\
(\rho\boldsymbol{v} \otimes \boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla_{\boldsymbol{x}}^s \boldsymbol{v}) \, \mathbf{cof}\hat{\boldsymbol{F}} \; \boldsymbol{n}_{\boldsymbol{\chi}} &= \boldsymbol{h} \; , \quad \text{on } \partial\Omega_{\boldsymbol{\chi}}^{h;-} \; , && (198)\\
\left\{ \begin{array}{c} \boldsymbol{v}(\boldsymbol{\chi}, 0) \\ p(\boldsymbol{\chi}, 0) \end{array} \right\} &= \left\{ \begin{array}{c} \boldsymbol{v}_0 \\ p_0 \end{array} \right\} \; , && (199)
\end{aligned}$$

where $\boldsymbol{h}$ is the n_d-dimensional vector of Neumann boundary conditions, and $\partial\Omega_{\boldsymbol{\chi}}^{h;-}$ and $\partial\Omega_{\boldsymbol{\chi}}^{h;+}$ are the inflow and outflow Neumann boundaries, respectively. That is

$$\begin{aligned}
\partial\Omega_{\boldsymbol{\chi}}^{h;+} &= \{\boldsymbol{\chi} \in \partial\Omega_{\boldsymbol{\chi}}^h \mid \tilde{\boldsymbol{v}} \cdot \boldsymbol{n}_{\boldsymbol{\chi}} \geq 0\} \; , && (200)\\
\partial\Omega_{\boldsymbol{\chi}}^{h;-} &= \Omega_{\boldsymbol{\chi}}^h \setminus \partial\Omega_{\boldsymbol{\chi}}^{h;+} \; . && (201)
\end{aligned}$$

Notice the structure of the Neumann flux boundary condition, for which the convective part of the flux $\rho\boldsymbol{v} \otimes \boldsymbol{c}$ is enforced only at the inflow (198), but not at the outflow (197), to preserve consistency of the

formulation, in the limit of vanishing diffusion [14, 13]. Alternatively, (194)–(199) can be written as

$$
\begin{aligned}
\hat{J}^{-1}\partial_t|_{\boldsymbol{x}}\,(\hat{J}\rho\boldsymbol{v}) + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v}) + \rho\boldsymbol{b} &= \boldsymbol{0}, \quad \text{in } \Omega_{\boldsymbol{x}}\ , && (202)\\
\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} &= \boldsymbol{0}, \quad \text{in } \Omega_{\boldsymbol{x}}\ , && (203)\\
\boldsymbol{v} &= \boldsymbol{v}_g\ , \quad \text{on } \partial\Omega^g_{\boldsymbol{x}}\ , && (204)\\
(p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v})\,\boldsymbol{n}_{\boldsymbol{x}} &= \boldsymbol{h}\ , \quad \text{on } \partial\Omega^{h;+}_{\boldsymbol{x}}\ , && (205)\\
(\rho\boldsymbol{v}\otimes\boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v})\,\boldsymbol{n}_{\boldsymbol{x}} &= \boldsymbol{h}\ , \quad \text{on } \partial\Omega^{h;-}_{\boldsymbol{x}}\ , && (206)\\
\left\{\begin{array}{c}\boldsymbol{v}(\boldsymbol{x},0)\\ p(\boldsymbol{x},0)\end{array}\right\} &= \left\{\begin{array}{c}\boldsymbol{v}_0\\ p_0\end{array}\right\}\ , && (207)
\end{aligned}
$$

where

$$
\begin{aligned}
\partial\Omega^{h;+}_{\boldsymbol{x}} &= \{\boldsymbol{x}\in\partial\Omega^h_{\boldsymbol{x}} \mid \boldsymbol{c}\cdot\boldsymbol{n}_{\boldsymbol{x}} \geq 0\}\ , && (208)\\
\partial\Omega^{h;-}_{\boldsymbol{x}} &= \Omega^h_{\boldsymbol{x}} \setminus \partial\Omega^{h;+}_{\boldsymbol{x}}\ . && (209)
\end{aligned}
$$

7.2.2 Conservative space-time variational formulation

The space-time variational formulation of the incompressible Navier-Stokes can be derived using the momentum equation (194) (or, resp., (202)), in combination with the incompressibility constraint (193):

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi},T)\hat{\boldsymbol{w}}(\boldsymbol{\chi},T^-)\cdot\big(\rho(\boldsymbol{\chi},T^-)\boldsymbol{v}(\boldsymbol{\chi},T^-)\big) - \hat{J}(\boldsymbol{\chi},0)\hat{\boldsymbol{w}}(\boldsymbol{\chi},0^+)\cdot\big(\rho(\boldsymbol{\chi},0^-)\boldsymbol{v}(\boldsymbol{\chi},0^-)\big)\\
& - \left(\int_{\mathsf{Q}_{\boldsymbol{\chi}}} (\partial_t|_{\boldsymbol{\chi}}\,\hat{\boldsymbol{w}})\cdot(\hat{J}\rho\boldsymbol{v}) + \nabla_{\boldsymbol{\chi}}\hat{\boldsymbol{w}} : \Big((\rho\boldsymbol{v}\otimes\boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v})\hat{J}\hat{\boldsymbol{F}}^{-T}\Big)\right) + \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \hat{\boldsymbol{w}}\cdot(\hat{J}\rho\boldsymbol{b})\\
& + \int_{\mathsf{Q}_{\boldsymbol{\chi}}} \hat{w}_p\ \nabla_{\boldsymbol{\chi}}\cdot\Big(\hat{J}\hat{\boldsymbol{F}}^{-1}\boldsymbol{v}\Big) + \int_{\mathsf{P}^{h;+}_{\boldsymbol{\chi}}} \hat{\boldsymbol{w}}\cdot\Big(\Big(\hat{J}(\rho\boldsymbol{v}\otimes\boldsymbol{c})\hat{\boldsymbol{F}}^{-T}\Big)\boldsymbol{n}_{\boldsymbol{\chi}}\Big) + \int_{\mathsf{P}^h_{\boldsymbol{\chi}}} \hat{\boldsymbol{w}}\cdot\boldsymbol{h}\ , \qquad (210)
\end{aligned}
$$

where $\hat{\boldsymbol{w}}$ is the test function vector for the momentum equation and $\hat{w}_p$ is the test function for the incompressibility constraint. By $\mathsf{P}^{h;+}_{\boldsymbol{\chi}}$ we have indicated the outflow Neumann boundary, that is

$$
\mathsf{P}^{h;+}_{\boldsymbol{\chi}} = \{(t,\boldsymbol{\chi}) \in \mathsf{P}^h_{\boldsymbol{\chi}} \mid \tilde{\boldsymbol{v}}\cdot\boldsymbol{n}_{\boldsymbol{\chi}} \geq 0\}\ , \qquad (211)
$$

which entails the definition of a set $\mathsf{P}^{h;+}_{\boldsymbol{x}} = \hat{\boldsymbol{\varphi}}(\mathsf{P}^{h;+}_{\boldsymbol{\chi}})$, namely

$$
\mathsf{P}^{h;+}_{\boldsymbol{x}} = \{(t,\boldsymbol{x}) \in \mathsf{P}^h_{\boldsymbol{x}} \mid \boldsymbol{c}\cdot\boldsymbol{n}_{\boldsymbol{x}} \geq 0\}\ . \qquad (212)
$$

The inflow boundary is defined as $\mathsf{P}^{h;-}_{\boldsymbol{\chi}} = \mathsf{P}^h_{\boldsymbol{\chi}} \setminus \mathsf{P}^{h;+}_{\boldsymbol{\chi}}$ (resp., $\mathsf{P}^{h;-}_{\boldsymbol{x}} = \mathsf{P}^h_{\boldsymbol{x}} \setminus \mathsf{P}^{h;+}_{\boldsymbol{x}}$). Pushing forward to the current configuration gives a better understanding of the formulation:

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{x}}(T)} \boldsymbol{w}(\boldsymbol{x},T^-)\cdot\big(\rho(\boldsymbol{x},T^-)\boldsymbol{v}(\boldsymbol{x},T^-)\big) - \int_{\Omega_{\boldsymbol{x}}(0)} \boldsymbol{w}(\boldsymbol{x},0^+)\cdot\big(\rho(\boldsymbol{x},0^-)\boldsymbol{v}(\boldsymbol{x},0^-)\big)\\
& - \left(\int_{\mathsf{Q}_{\boldsymbol{x}}} (\partial_t|_{\boldsymbol{\chi}}\,\boldsymbol{w})\cdot(\rho\boldsymbol{v}) + \nabla_{\boldsymbol{x}}\boldsymbol{w} : (\rho\boldsymbol{v}\otimes\boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v})\right) + \int_{\mathsf{Q}_{\boldsymbol{x}}} \boldsymbol{w}\cdot(\rho\boldsymbol{b})\\
& + \int_{\mathsf{Q}_{\boldsymbol{x}}} w_p\ \nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} + \int_{\mathsf{P}^{h;+}_{\boldsymbol{x}}} \boldsymbol{w}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{c})\,\boldsymbol{n}_{\boldsymbol{x}} + \int_{\mathsf{P}^h_{\boldsymbol{x}}} \boldsymbol{w}\cdot\boldsymbol{h}\ , \qquad (213)
\end{aligned}
$$

where $\boldsymbol{w} = \hat{\boldsymbol{w}}(\hat{\boldsymbol{\varphi}}^{-1}(\boldsymbol{x},t),t)$, and $w_p = \hat{w}_p(\hat{\boldsymbol{\varphi}}^{-1}(\boldsymbol{x},t),t)$.

7.2.3 Euler-Lagrange equations

The Euler-Lagrange equations derived from (213) yield:

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{x}}(0)} \boldsymbol{w}(\boldsymbol{x},0^+)\cdot [\![\rho(\boldsymbol{x},0)\boldsymbol{v}(\boldsymbol{x},0)]\!] \\
& \int_{\mathsf{Q}_{\boldsymbol{x}}} \boldsymbol{w}\cdot\Big(\hat{J}^{-1}\partial_t|_{\boldsymbol{\chi}}\,(\hat{J}\rho\boldsymbol{v}) + \nabla_{\boldsymbol{x}}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v}) + \rho\boldsymbol{b}\Big) + \int_{\mathsf{Q}_{\boldsymbol{x}}} w_p\;\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} \\
& -\int_{\mathsf{P}^{h;-}_{\boldsymbol{x}}} \boldsymbol{w}\cdot((\rho\boldsymbol{v}\otimes\boldsymbol{c} + p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v})\,\boldsymbol{n}_{\boldsymbol{x}} - \boldsymbol{h}) \\
& -\int_{\mathsf{P}^{h;+}_{\boldsymbol{x}}} \boldsymbol{w}\cdot((p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v})\,\boldsymbol{n}_{\boldsymbol{x}} - \boldsymbol{h})\;.
\end{aligned}
\tag{214}
$$

Therefore, *causality* of the solution is enforced at time $t = 0$, the Navier-Stokes equations and the incompressibility constraint are enforced on $\mathsf{Q}_{\boldsymbol{x}}$, and Neumann conditions are imposed on $\mathsf{P}^h_{\boldsymbol{x}}$.

7.2.4 Advective semi-discrete variational formulation

The incompressible Navier-Stokes equations are very often implemented in advective form, for which equation (202) is manipulated to remove the mass conservation equation, and reduces to

$$
\rho\,\partial_t\boldsymbol{v}\,|_{\boldsymbol{\chi}} + \rho(\nabla_{\boldsymbol{\chi}}\boldsymbol{v})\boldsymbol{c} + \nabla_{\boldsymbol{x}}p - \nabla_{\boldsymbol{x}}\cdot(2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v}) + \rho\boldsymbol{b} = \boldsymbol{0}\;. \tag{215}
$$

A space-time formulation is not applicable in this case, and one resorts to the following semi-discrete approach,

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{x}}} \boldsymbol{w}\cdot(\rho\partial_t|_{\boldsymbol{\chi}}\,\boldsymbol{v} + \rho(\nabla_{\boldsymbol{x}}\boldsymbol{v})\boldsymbol{c}) - \nabla_{\boldsymbol{x}}\boldsymbol{w} : (p\boldsymbol{I} - 2\rho\nu\nabla^s_{\boldsymbol{x}}\boldsymbol{v}) + \boldsymbol{w}\cdot(\rho\boldsymbol{b}) \\
& + \int_{\Omega_{\boldsymbol{x}}} w_p\;\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} + \int_{\partial\Omega^{h;+}_{\boldsymbol{x}}} \boldsymbol{w}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{c})\,\boldsymbol{n}_{\boldsymbol{x}} + \int_{\partial\Omega^h_{\boldsymbol{x}}} \boldsymbol{w}\cdot\boldsymbol{h}\;.
\end{aligned}
\tag{216}
$$

8 Geometric conservation laws and computational implications

Geometric conservation laws (GCLs) represent a very interesting aspect of arbitrary Lagrangian-Eulerian computational methods, only very recently investigated (see, e.g., [18, 8, 9, 22]). The analysis of geometric conservation laws shows that the an inappropriate discretization of the arbitrary motion of the mesh may affect the quality of the solution in even very simple test cases. The statement of the GCL property, adopting the definition suggested in [18, 8, 9], is very simple:

Definition 1 A numerical scheme posed on an arbitraary moving domain satisfies the GCL if it is able to exactly reproduce a constant solution.

The space-time formulations introduced for the compressible and incompressible Navier-Stokes equations become very useful in elucidating the discussion. In the case of the incompressible Navier-Stokes equations, also the advective formulation introduced in (216) is considered, and the reader can refer to [18, 8, 9, 22] for a more general discussion on semi-discrete time integrators.

Remark
One point worth of notice is that *the GCL condition is not a necessary nor sufficient condition for stability* [8, 3, 22, 9]. However, numerical computations performed in [18] show that the accuracy of the solution is greatly improved, as high frequency, fine-scale, spurious oscillations are removed from the solution by GCL-abiding numerical schemes. Additional benefits can be gained in mechanical systems involving the coupling of a fluid and a structural domain, where instabilities can arise due to inconsistent geometry representation.

8.1 Geometric conservation laws for space-time conservative formulations

To understand the issue of geometric conservation laws, one can start from the compressible Navier-Stokes case. This is how historically GCL were first developed (see, [18], and references therein). Let us consider (188) written on a time slab $\mathsf{Q}_{\boldsymbol{\chi};n}$, and assume that the solution $\mathbf{U}$ to (188) is constant in space and time, namely $\mathbf{U}(\boldsymbol{x},t) = \mathbf{U}_0$. For this to be the case, one has to enforce $\mathbf{Z} = \mathbf{0}$ and $\mathbf{h} = \hat{J}\mathbf{G}(\mathbf{U}_0)\hat{\boldsymbol{F}}^{-T}\boldsymbol{n}_{\boldsymbol{\chi}}$, so that (188) yields:

$$
\begin{aligned}
0 \;=\; & \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi},T)\;\hat{\mathbf{W}}(\boldsymbol{\chi},T^-)\cdot\mathbf{U}_0 - \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi},0)\;\hat{\mathbf{W}}(\boldsymbol{\chi},0^+)\cdot\mathbf{U}_0 \\
& - \left(\int_{\mathsf{Q}_{\boldsymbol{\chi}}} (\partial_t|_{\boldsymbol{\chi}}\hat{\mathbf{W}})\cdot(\hat{J}\mathbf{U}_0) + \nabla_{\boldsymbol{\chi}}\hat{\mathbf{W}} : \left(\hat{J}(\mathbf{U}_0\otimes(\boldsymbol{v}_0-\hat{\boldsymbol{v}}) + \mathbf{G}_{\mathsf{L}}(\mathbf{U}_0))\hat{\boldsymbol{F}}^{-T}\right)\right) \\
& + \int_{\mathsf{P}_{\boldsymbol{\chi}}} \hat{\mathbf{W}}\cdot\left(\left(\hat{J}(\mathbf{U}_0\otimes(\boldsymbol{v}_0-\hat{\boldsymbol{v}}) + \mathbf{G}_{\mathsf{L}}(\mathbf{U}_0))\hat{\boldsymbol{F}}^{-T}\right)\boldsymbol{n}_{\boldsymbol{\chi}}\right)\;,
\end{aligned}
\tag{217}
$$

where $\mathbf{U}_0\otimes\boldsymbol{v}_0 + \mathbf{G}_{\mathsf{L}}(\mathbf{U}_0) = \mathbf{G}_0$ is a constant matrix. Pushing forward to the current configuration:

$$
\int_{\mathsf{Q}_{\boldsymbol{\chi};n}} \nabla_{\boldsymbol{\chi}}\hat{\mathbf{W}} : \left(\mathbf{G}_0\mathbf{cof}\,\hat{\boldsymbol{F}}\right) \;=\; \int_{\mathsf{Q}_{\boldsymbol{\chi};n}} \hat{J}\left(\nabla_{\boldsymbol{\chi}}\hat{\mathbf{W}}\;\hat{\boldsymbol{F}}^{-1}\right) : \mathbf{G}_0 = \int_{\mathsf{Q}_{\boldsymbol{x};n}} \nabla_{\boldsymbol{x}}\mathbf{W} : \mathbf{G}_0\;, \tag{218}
$$

$$
\int_{\mathsf{P}_{\boldsymbol{\chi}}} \hat{\mathbf{W}}\cdot\left(\hat{J}\mathbf{G}_0\hat{\boldsymbol{F}}^{-T}\boldsymbol{n}_{\boldsymbol{\chi}}\right) \;=\; \int_{\mathsf{P}_{\boldsymbol{\chi}}} \mathbf{W}\cdot(\mathbf{G}_0\boldsymbol{n}_{\boldsymbol{x}})\;. \tag{219}
$$

Hence:

$$
\begin{aligned}
& -\int_{\mathsf{Q}_{\boldsymbol{\chi};n}} \nabla_{\boldsymbol{\chi}} \hat{\mathbf{W}} : \left(\mathbf{G}_0 \mathbf{cof} \hat{\boldsymbol{F}}\right) + \int_{\mathsf{P}_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\hat{J} \mathbf{G}_0 \hat{\boldsymbol{F}}^{-T} \boldsymbol{n}_{\boldsymbol{\chi}}\right) \\
= & -\int_{\mathsf{Q}_{\boldsymbol{x};n}} \nabla_{\boldsymbol{x}} \mathbf{W} : \mathbf{G}_0 + \int_{\mathsf{P}_{\boldsymbol{\chi}}} \mathbf{W} \cdot (\mathbf{G}_0 \boldsymbol{n}_{\boldsymbol{x}}) \\
= & +\int_{\mathsf{Q}_{\boldsymbol{x};n}} \mathbf{W} \cdot (\nabla_{\boldsymbol{x}} \cdot \mathbf{G}_0) - \int_{\mathsf{P}_{\boldsymbol{\chi}}} \mathbf{W} \cdot (\mathbf{G}_0 \boldsymbol{n}_{\boldsymbol{x}}) + \int_{\mathsf{P}_{\boldsymbol{\chi}}} \mathbf{W} \cdot (\mathbf{G}_0 \boldsymbol{n}_{\boldsymbol{x}}) \\
= & \ 0 \ ,
\end{aligned} \tag{220}
$$

since $\mathbf{G}_0$ is constant. Then (217) yields

$$
\begin{aligned}
0 \ = \ & \mathbf{U}_0 \cdot \left(\int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi}, t_{n+1}) \ \hat{\mathbf{W}}(\boldsymbol{\chi}, t_{n+1}^-) - \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\boldsymbol{\chi}, t_n) \ \hat{\mathbf{W}}(\boldsymbol{\chi}, t_n^+) - \int_{\mathsf{Q}_{\boldsymbol{\chi};n}} (\partial_t|_{\boldsymbol{\chi}} \hat{\mathbf{W}}) \hat{J} \right) \\
& + \int_{\mathsf{Q}_{\boldsymbol{\chi};n}} \nabla_{\boldsymbol{\chi}} \hat{\mathbf{W}} : \left(\mathbf{U}_0 \hat{\boldsymbol{v}}^T \mathbf{cof} \hat{\boldsymbol{F}}\right) - \int_{\mathsf{P}_{\boldsymbol{\chi}}} \hat{\mathbf{W}} \cdot \left(\mathbf{U}_0 \hat{\boldsymbol{v}}^T \mathbf{cof} \hat{\boldsymbol{F}}\right) \boldsymbol{n}_{\boldsymbol{\chi}} \ .
\end{aligned} \tag{221}
$$

Now taking a vector $\mathbf{U}_0$ for which one entry is equal to unity, and all other entries are zero, and indicating by $\hat{\psi}$ the corresponding entry for the test function vector $\hat{\mathbf{W}}$, equation (221) yields

$$
\begin{aligned}
0 \ = \ & \int_{\Omega_{\boldsymbol{\chi}}} \hat{\psi}(\boldsymbol{\chi}, t_{n+1}^-) \ \hat{J}(\boldsymbol{\chi}, t_{n+1}) - \int_{\Omega_{\boldsymbol{\chi}}} \hat{\psi}(\boldsymbol{\chi}, t_n^+) \ \hat{J}(\boldsymbol{\chi}, t_n) \ - \int_{\mathsf{Q}_{\boldsymbol{\chi};n}} (\partial_t|_{\boldsymbol{\chi}} \hat{\psi}) \hat{J} \\
& + \int_{\mathsf{Q}_{\boldsymbol{\chi};n}} \left(\mathbf{cof} \hat{\boldsymbol{F}} (\nabla_{\boldsymbol{\chi}} \hat{\psi})\right) \cdot \hat{\boldsymbol{v}} - \int_{\mathsf{P}_{\boldsymbol{\chi};n}} \hat{\psi} \ \hat{\boldsymbol{v}} \cdot \left(\mathbf{cof} \hat{\boldsymbol{F}} \boldsymbol{n}_{\boldsymbol{\chi}}\right) \ .
\end{aligned} \tag{222}
$$

Since $\psi(\boldsymbol{x}, t) = \hat{\psi}(\hat{\boldsymbol{\varphi}}^{-1}(\boldsymbol{x}, t), t)$ and $\mathbf{cof} \hat{\boldsymbol{F}} (\nabla_{\boldsymbol{\chi}} \hat{\psi}) = \nabla_{\boldsymbol{x}} \psi$, (222) can also be written as

$$
0 = \int_{\Omega_{\boldsymbol{x}}(t_{n+1})} \psi(\boldsymbol{x}, t_{n+1}^-) - \int_{\Omega_{\boldsymbol{x}}(t_n)} \psi(\boldsymbol{x}, t_n^+) - \int_{\mathsf{Q}_{\boldsymbol{x};n}} (\partial_t|_{\boldsymbol{\chi}} \psi) + \int_{\mathsf{Q}_{\boldsymbol{x};n}} (\nabla_{\boldsymbol{x}} \psi) \cdot \hat{\boldsymbol{v}} - \int_{\mathsf{P}_{\boldsymbol{x};n}} \psi \ \hat{\boldsymbol{v}} \cdot \boldsymbol{n}_{\boldsymbol{x}} \ , \tag{223}
$$

which, performing a simple integration by parts, represents a weak statement of identity (51), namely,

$$
0 \ = \ \partial_t \hat{J}|_{\boldsymbol{\chi}} - \hat{J} \ \nabla_{\boldsymbol{x}} \cdot \hat{\boldsymbol{v}} \ . \tag{224}
$$

Therefore, in the case of conservative ALE formulations, Definition 1 translates into the requirement that (222) (resp. (223)) must be exactly satisfied. This is clearly the case, if exact integration is adopted in both space and time. However, for practical purposes, approximate numerical integration is usually adopted, and it becomes important to investigate the minimum requirements to exactly satisfy (222). In many practical situations, integration in space can be assumed exact. This is for example the case of piecewise linear continuous finite elements on tetrahedral meshes, but it is not the case for piecewise linear continuous finite elements on hexahedral meshes. We have then the following

Theorem 15 Let the test function $\hat{\psi}$ in (222) and mesh displacement vector $\hat{\boldsymbol{u}}$ be polynomials of degree k and s in time, respectively. Let us assume that the space integration quadrature formulas are exact. Then, an integration formula in time with polynomial exactness $k - 1 + n_d s$ is sufficient to exactly satisfy (222).

Proof:
In time, $\partial_t|_{\boldsymbol{\chi}}\,\hat{\psi}$ is a polynomial of degree $k-1$, $\hat{\boldsymbol{v}}$ is a polynomial of degree $s-1$, $\hat{J}$ is a polynomial of degree $n_d s$, and $\mathbf{cof}\,\hat{\boldsymbol{F}}$ is a polynomial of degree $(n_d-1)s$. The first two integrals in (222) are assumed exact, the third integral requires exactness $k-1+n_d s$ and the last two remaining integrals require exactness $k+(n_d-1)s+s-1=k+n_d s-1$, respectively. Therefore $k-1+n_d s$ is the minimal level of exactness required for the quadrature formula in time. □

Remarks

1. For example, if the space-time formulation proposed in [29, 30], one has $k=0$ and $s=1$, so that one-point and two-point time integration quadratures are needed in two and three space dimensions, respectively. Integration in space is exact, for example, if single point integration quadratures in space are used with piecewise-linear finite elements on triangular or tetrahedral meshes. The proposed example is very similar to the time integrators discussed in [18, 8, 9].

2. Notice also that in the case of the incompressible Navier-Stokes equations in conservation form (210) or (213), equation (222) is again obtained as the GCL requirement, from the momentum equation in weak form.

3. When semi-discrete, conservative formulations are adopted, the discussion is analogous to space-time case with $k=0$, and $s\geq 1$. The reader can refer to [18, 8, 9] and references therein for a more detailed discussion.

8.2 Geometric conservation laws for semi-discrete advective formulations

It is almost trivial to observe that all advective weak formulations satisfy the GLC condition expressed in Definition 1. For example, the advective form of the incompressible Navier-Stokes equations (216) vanishes exactly for a constant solution field, and incorporates (222) automatically, provided exact quadratures in space are adopted. The reader can verify that this is indeed the case, by observing that, when the advective form (139) of the mass conservation equation was derived, the identity (224) was explicitly removed from the conservative form. Analogous considerations hold for the advective form of the momuntum, energy, and entropy equations. As already mentioned, a detailed discussion on semi-discrete time integration schemes for advective weak formulations of the Navier-Stokes equations is presented in [8, 9].

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Appendix

A Solutions to homework problems

Homework 1

Proof:
For the first part of the problem, let us recall that the gradients are contravariat operators, so that $\nabla_{\boldsymbol{X}} \boldsymbol{w} = \boldsymbol{F} \nabla_{\boldsymbol{x}} \boldsymbol{w}$, $\forall \boldsymbol{w}$. In addition, by (30) $\frac{\partial \alpha}{\partial t}\big|_{\boldsymbol{X}} = \frac{\partial \alpha}{\partial t}\big|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \alpha$, $\forall \alpha$. The result follows by arranging in matrix form these results.

The second part is straightforward, using the suggested hint. Consider the linear system $\hat{\mathsf{F}}\mathsf{x} = \mathsf{b}$. In particular, let $\mathsf{x} = [x_0, \boldsymbol{x}^T]^T$ and $\mathsf{b} = [b_0, \boldsymbol{b}^T]^T$. Then,

$$
\begin{aligned}
x_0 &= b_0, && (225)\\
\boldsymbol{v} x_0 + \boldsymbol{F}\boldsymbol{x} &= \boldsymbol{b}, && (226)
\end{aligned}
$$

or

$$
\begin{aligned}
x_0 &= b_0, && (227)\\
\boldsymbol{x} &= -\boldsymbol{F}^{-1}(\boldsymbol{v} b_0 + \boldsymbol{b}). && (228)
\end{aligned}
$$

Arranging in matrix notation the previous result concludes the proof.

Homework 2

The solution is analogous to the solution of Homework 1.

Homework 3

Equation (67) is derived as follows:

$$
\begin{aligned}
\left.\frac{\partial \alpha(\boldsymbol{\chi}, t)}{\partial t}\right|_{\boldsymbol{X}} &= \left.\frac{\partial \alpha(\tilde{\boldsymbol{\varphi}}(\boldsymbol{X}, t), t)}{\partial t}\right|_{\boldsymbol{X}} \\
&= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \alpha \cdot \left.\frac{\partial \tilde{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{X}} \\
&= \left.\frac{\partial \alpha}{\partial t}\right|_{\boldsymbol{\chi}} + \tilde{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{\chi}} \alpha \,. && (229)
\end{aligned}
$$

Similarly, for (68):

$$
\begin{aligned}
\left.\frac{\partial \tilde{J}}{\partial t}\right|_{\boldsymbol{X}} &= \left.\frac{\partial(\det \tilde{\boldsymbol{F}})}{\partial t}\right|_{\boldsymbol{X}} \\
&= \tilde{J}\ \mathrm{tr}\left(\left.\frac{\partial \tilde{\boldsymbol{F}}}{\partial t}\right|_{\boldsymbol{X}} \tilde{\boldsymbol{F}}^{-1}\right) \\
&= \tilde{J}\ \mathrm{tr}\left(\left(\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{X}} \frac{\partial \tilde{\boldsymbol{\varphi}}}{\partial \boldsymbol{X}}\right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\chi}}\right) \\
&= \tilde{J}\ \mathrm{tr}\left(\left(\frac{\partial}{\partial \boldsymbol{X}} \left.\frac{\partial \tilde{\boldsymbol{\varphi}}}{\partial t}\right|_{\boldsymbol{X}}\right) \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\chi}}\right) \\
&= \tilde{J}\ \mathrm{tr}\left(\frac{\partial \tilde{\boldsymbol{v}}}{\partial \boldsymbol{X}} \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{\chi}}\right) \\
&= \tilde{J}\ \nabla_{\boldsymbol{\chi}} \cdot \tilde{\boldsymbol{v}}\ . \qquad (230)
\end{aligned}
$$

Homework 4

The solution is analogous to the solution of Homework 1.

Homework 5 Prove the generalized Reynolds transport theorem by pulling back to the domain $\Omega_{\boldsymbol{X}}$ in two steps, using the map $\hat{\boldsymbol{\varphi}}$ first and then the map $\tilde{\boldsymbol{\varphi}}$

Proof:
$\Omega_{\boldsymbol{x}}$ is a material domain, and deforms according to the material velocity $\boldsymbol{v}$, that is $\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})$. One can also introduce an arbitrary moving domain $\hat{\boldsymbol{\varphi}}(\Omega_{\boldsymbol{\chi}})$ set to coincide with $\Omega_{\boldsymbol{x}}$ at the instant t under consideration, and moving with velocity $\hat{\boldsymbol{v}} \neq \boldsymbol{v}$. When observed from the Lagrangian reference frame, $\Omega_{\boldsymbol{\chi}}$ moves with velocity $\tilde{\boldsymbol{v}}$. Therefore, pulling back the integral over $\Omega_{\boldsymbol{x}}$ to the domain $\Omega_{\boldsymbol{\chi}}$ first, and then to the domain $\Omega_{\boldsymbol{X}}$, and defining $\hat{\alpha} = \hat{J}\alpha$, yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{x}}} \alpha \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{\chi}}} \alpha\hat{J} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{X}}} (\hat{J}\alpha)\tilde{J} \;=\; \int_{\Omega_{\boldsymbol{X}}} \left.\frac{\partial(\hat{\alpha}\tilde{J})}{\partial t}\right|_{\boldsymbol{X}} \;=\; \int_{\Omega_{\boldsymbol{X}}} \dot{\hat{\alpha}}\tilde{J} + \hat{\alpha}\dot{\tilde{J}}\,, \tag{231}$$

Notice that in (231), the time derivative can be moved inside the integral only when the domain of integration is $\Omega_{\boldsymbol{X}}$, since $\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}(\Omega_{\boldsymbol{X}})$ is a material domain. Now, using (67) and (68),

$$\begin{aligned}
\dot{\hat{\alpha}}\tilde{J} + \hat{\alpha}\dot{\tilde{J}} &= \dot{\hat{\alpha}}\tilde{J} + \tilde{J}\,\nabla_{\boldsymbol{\chi}}\cdot\tilde{\boldsymbol{v}} \\
&= \tilde{J}\left(\left.\frac{\partial\hat{\alpha}}{\partial t}\right|_{\boldsymbol{\chi}} + \tilde{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{\chi}}\hat{\alpha} + \nabla_{\boldsymbol{\chi}}\cdot\tilde{\boldsymbol{v}}\right) \\
&= \tilde{J}\left(\left.\frac{\partial\hat{\alpha}}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot(\hat{\alpha}\tilde{\boldsymbol{v}})\right)
\end{aligned} \tag{232}$$

Mapping forward to the domain $\Omega_{\boldsymbol{\chi}}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_{\boldsymbol{\chi}}} \alpha\hat{J} \;=\; \int_{\Omega_{\boldsymbol{\chi}}} \left(\left.\frac{\partial(\hat{J}\alpha)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot(\hat{J}\alpha\;\tilde{\boldsymbol{v}})\right) \tag{233}$$

The Gauss divergence theorem yields (79). □

Homework 6

Proof:
Pulling $\mathcal{C}_{\boldsymbol{x}}$ back to $\mathcal{C}_{\boldsymbol{\chi}}$ through the map $\hat{\boldsymbol{\varphi}}$, recalling the chain rule $\mathrm{d}\boldsymbol{x} = \hat{\boldsymbol{F}}\mathrm{d}\boldsymbol{\chi}$, and using the result, analogous to (88), that $\partial_t\hat{\boldsymbol{F}}|_{\boldsymbol{\chi}} = (\nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}})\hat{\boldsymbol{F}}$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_{\boldsymbol{x}}=\boldsymbol{\varphi}(\mathcal{C}_{\boldsymbol{X}})} \alpha\;\mathrm{d}\boldsymbol{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\mathcal{C}_{\boldsymbol{\chi}}} \alpha\hat{\boldsymbol{F}}\;\mathrm{d}\boldsymbol{\chi} \\
&= \oint_{\mathcal{C}_{\boldsymbol{X}}} \left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{\chi}}\boldsymbol{F} + \alpha\left.\frac{\partial\hat{\boldsymbol{F}}}{\partial t}\right|_{\boldsymbol{\chi}}\right)\mathrm{d}\boldsymbol{\chi} \\
&= \oint_{\mathcal{C}_{\boldsymbol{X}}} \left(\left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{\chi}} + \alpha\nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}}\right)\hat{\boldsymbol{F}}\mathrm{d}\boldsymbol{\chi} \\
&= \oint_{\mathcal{C}_{\boldsymbol{x}}} \left.\frac{\partial\alpha}{\partial t}\right|_{\boldsymbol{\chi}}\;\mathrm{d}\boldsymbol{x} + \oint_{\mathcal{C}_{\boldsymbol{x}}} \alpha(\nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}})\mathrm{d}\boldsymbol{x}\,.
\end{aligned} \tag{234}$$

□

Homework 7 Starting from the space-time balance law (133), and using the space-time Piola identity (75), prove that

$$
\begin{aligned}
0 &= \int_{\mathsf{Q}_{\boldsymbol{\mathcal{X}}}=\tilde{\boldsymbol{\varphi}}(\mathsf{Q}_{\boldsymbol{X}})} \nabla_{[t,\boldsymbol{\mathcal{X}}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{\mathcal{X}}]} - \hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{X}}} \nabla_{[t,\boldsymbol{X}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{X}]} - \mathsf{J}\beta \, . \qquad (235)
\end{aligned}
$$

It is also possible to achieve the proof using the composition $\boldsymbol{\varphi} = \hat{\boldsymbol{\varphi}} \circ \tilde{\boldsymbol{\varphi}}$.
Proof:

$$
\begin{aligned}
0 &= \int_{\mathsf{Q}_{\boldsymbol{\mathcal{X}}}=\tilde{\boldsymbol{\varphi}}(\mathsf{Q}_{\boldsymbol{X}})} \nabla_{[t,\boldsymbol{\mathcal{X}}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{\mathcal{X}}]} - \hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{X}}} \left(\tilde{\mathsf{J}}\tilde{\mathbf{F}}^{-T}\nabla_{[t,\boldsymbol{X}]}\right) \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{\mathcal{X}}]} - \tilde{\mathsf{J}}\hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{X}}} \nabla_{[t,\boldsymbol{X}]} \cdot \left(\tilde{\mathsf{J}}\tilde{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{\mathcal{X}}]}\right) - \tilde{\mathsf{J}}\hat{\mathsf{J}}\beta \\
&= \int_{\mathsf{Q}_{\boldsymbol{X}}} \nabla_{[t,\boldsymbol{X}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{X}]} - \mathsf{J}\beta \, . \qquad (236)
\end{aligned}
$$

The last step in (236) is obtained by observing that

$$
\boldsymbol{\varphi} = \hat{\boldsymbol{\varphi}} \circ \tilde{\boldsymbol{\varphi}} \, . \qquad (237)
$$

Hence, by the chain rule,

$$
\begin{aligned}
\mathbf{F} &= \hat{\mathbf{F}}\tilde{\mathbf{F}} \, , && (238) \\
\mathsf{J} &= \det \mathbf{F} = \det(\hat{\mathbf{F}}\tilde{\mathbf{F}}) = \hat{\mathsf{J}}\tilde{\mathsf{J}} \, , && (239)
\end{aligned}
$$

which imply

$$
\begin{aligned}
\nabla_{[t,\boldsymbol{X}]} \cdot \left(\tilde{\mathsf{J}}\tilde{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{\mathcal{X}}]}\right) &= \nabla_{[t,\boldsymbol{X}]} \cdot \left(\tilde{\mathsf{J}}\tilde{\mathbf{F}}^{-1}\hat{\mathsf{J}}\hat{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{x}]}\right) \\
&= \nabla_{[t,\boldsymbol{X}]} \cdot \left(\tilde{\mathsf{J}}\hat{\mathsf{J}}\tilde{\mathbf{F}}^{-1}\hat{\mathbf{F}}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{x}]}\right) \\
&= \nabla_{[t,\boldsymbol{X}]} \cdot \left(\mathsf{J}\mathbf{F}^{-1}\boldsymbol{\gamma}_{[t,\boldsymbol{x}]}\right) \\
&= \nabla_{[t,\boldsymbol{X}]} \cdot \boldsymbol{\gamma}_{[t,\boldsymbol{X}]} \, . \qquad (240)
\end{aligned}
$$

Homework 8 Show that conservation of angular momentum is equivalent to require the tensor $\boldsymbol{\sigma}$ to be symmetric. Hint: use index notation, and collect the cross product of the position times the momentum equation. Use then the definition of $\hat{F}_{ij}$ and the fact that $\boldsymbol{v} \times \boldsymbol{v} = \mathbf{0}$, $\forall \boldsymbol{v}$.

Proof:

Starting from (145), and using the tensor ε_{ijk}, which equals $+1$ if its indices are a positive permutation of 123, equals -1 if its indices are a negative permutation of 123, and vanishes if two of its indices are identical,

$$
\begin{aligned}
\mathbf{0} &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho\varepsilon_{ijk}x_j v_k)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_p}\left(\hat{J}(\rho\varepsilon_{ijk}x_j v_k c_l - \varepsilon_{ijk}x_j\sigma_{kl})\hat{F}^{-T}_{lp}\right) + \hat{J}\rho\varepsilon_{ijk}x_j b_k \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \varepsilon_{ijk}x_j \left.\frac{\partial(\hat{J}\rho v_k)}{\partial t}\right|_{\boldsymbol{\chi}} + \varepsilon_{ijk}x_j\partial_{\chi_p}\left(\hat{J}(\rho v_k c_l - \sigma_{kl})\hat{F}^{-T}_{lp}\right) + \hat{J}\rho\varepsilon_{ijk}x_j b_k \\
&\qquad + \hat{J}\rho\varepsilon_{ijk}\left.\frac{\partial x_j}{\partial t}\right|_{\boldsymbol{\chi}} v_k + (\partial_{\chi_p}x_j)\hat{J}\varepsilon_{ijk}(\rho v_k c_l - \sigma_{kl})\hat{F}^{-T}_{lp} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \boldsymbol{x} \times \underbrace{\left(\left.\frac{\partial(\hat{J}\rho\boldsymbol{v})}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}}\cdot\left(\hat{J}(\rho\boldsymbol{v}\otimes\boldsymbol{c} - \boldsymbol{\sigma})\hat{\boldsymbol{F}}^{-T}\right) + \hat{J}\rho\boldsymbol{b}\right)}_{=0 \text{ by } (142)} \\
&\qquad + \hat{J}\rho\varepsilon_{ijk}\left.\frac{\partial\hat{\varphi}_j}{\partial t}\right|_{\boldsymbol{\chi}} v_k + \underbrace{\partial_{\chi_p}x_j\hat{F}^{-T}_{lp}}_{=\delta_{jl}}(\hat{J}\varepsilon_{ijk}(\rho v_k c_l - \sigma_{kl})) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho\varepsilon_{ijk}\hat{v}_j v_k + \hat{J}(\rho\varepsilon_{ijk}v_k(v_j - \hat{v}_j) - \varepsilon_{ijk}\sigma_{kj}) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\rho\varepsilon_{ijk}v_k v_j - \varepsilon_{ijk}\sigma_{kj}) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}(\rho\underbrace{\boldsymbol{v}\times\boldsymbol{v}}_{=0} - \varepsilon_{ijk}\sigma_{kj})\,. \qquad (241)
\end{aligned}
$$

Using the localization theorem, in the limit for a domain $\Omega_{\boldsymbol{\chi}}$ of measure reducing to zero, we have the fundamental condition

$$\varepsilon_{ijk}\sigma_{kj} = 0 \;\Leftrightarrow\; \sigma_{jk} = \sigma_{kj} \;\Leftrightarrow\; \boldsymbol{\sigma} \text{ symmetric}\,. \qquad (242)$$

Therefore, we have proven that conservation of angular momentum is satisfied as long as the stress $\boldsymbol{\sigma}$ is a symmetric tensor.

Homework 9 Derive (157).

Proof:

$$
\begin{aligned}
\hat{\mathsf{J}}\mathsf{G}_{[t,\boldsymbol{x}]}\hat{\mathsf{F}}^{-T} &= \hat{J}\left[\mathsf{U} \;\middle|\; \mathsf{U}\boldsymbol{v}^T - \begin{bmatrix} \mathbf{0}^T \\ \boldsymbol{\sigma} \\ \boldsymbol{v}^T\boldsymbol{\sigma} - \boldsymbol{q} \end{bmatrix}\right] \begin{bmatrix} 1 & -\hat{\boldsymbol{v}}^T\hat{\boldsymbol{F}}^{-T} \\ \mathbf{0} & \hat{\boldsymbol{F}}^{-T} \end{bmatrix} \\
&= \left[\hat{J}\mathsf{U} \;\middle|\; \hat{J}\mathsf{U}(\boldsymbol{v}^T - \hat{\boldsymbol{v}}^T)\hat{\boldsymbol{F}}^{-T} - \hat{J}\begin{bmatrix} \mathbf{0}^T \\ \boldsymbol{\sigma} \\ \boldsymbol{v}^T\boldsymbol{\sigma} - \boldsymbol{q} \end{bmatrix}\hat{\boldsymbol{F}}^{-T}\right] \\
&= \left[\hat{J}\mathsf{U} \;\middle|\; \hat{J}\mathsf{U}\boldsymbol{c}^T\hat{\boldsymbol{F}}^{-T} - \hat{J}\begin{bmatrix} \mathbf{0}^T \\ \boldsymbol{\sigma} \\ \boldsymbol{v}^T\boldsymbol{\sigma} - \boldsymbol{q} \end{bmatrix}\hat{\boldsymbol{F}}^{-T}\right] \\
&= \left[\hat{J}\mathsf{U} \;\middle|\; \hat{J}\mathsf{G}\hat{\boldsymbol{F}}^{-T}\right] \\
&= \mathsf{G}_{[t,\boldsymbol{x}]} \,. \qquad (243)
\end{aligned}
$$

Homework 10 Show that

$$
\begin{aligned}
0 &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot (\hat{J}\hat{\boldsymbol{F}}^{-1}(\rho E\boldsymbol{c} - \boldsymbol{\sigma}^T\boldsymbol{v} + \boldsymbol{q})) + \hat{J}\rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho\tilde{\boldsymbol{v}}\nabla_{\boldsymbol{\chi}} e - \left(\hat{J}\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right) : \nabla_{\boldsymbol{\chi}}\boldsymbol{v} + \nabla_{\boldsymbol{\chi}} \cdot (\hat{J}\boldsymbol{F}^{-1}\boldsymbol{q}) + \hat{J}\rho r \,. \qquad (244)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
0 &= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho E)}{\partial t}\right|_{\boldsymbol{\chi}} + \nabla_{\boldsymbol{\chi}} \cdot (\hat{J}\hat{\boldsymbol{F}}^{-1}(\rho E\boldsymbol{c} - \boldsymbol{\sigma}^T\boldsymbol{v} + \boldsymbol{q})) + \hat{J}\rho(\boldsymbol{b}\cdot\boldsymbol{v} + r) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho(e + v_k v_k/2))}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_i}(\hat{J}F_{ij}^{-1}(\rho(e + v_k v_k/2))c_j - \sigma_{kj}v_k + q_j)) + \hat{J}\rho(b_k v_k + r) \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \left.\frac{\partial(\hat{J}\rho e)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_i}\Big(\hat{J}F_{ij}^{-1}(\rho e c_j)\Big) - \hat{J}F_{ij}^{-1}\sigma_{kj}\partial_{\chi_i}v_k + \partial_{\chi_i}(\hat{J}F_{ij}^{-1}q_j) + \hat{J}\rho r \\
&\qquad + v_k \underbrace{\left(\left.\frac{\partial(\hat{J}\rho v_k)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_i}\Big(\hat{J}F_{ij}^{-1}(\rho v_k c_j - \sigma_{kj})\Big) + \hat{J}\rho b_k \right)}_{=0 \text{ by } (142)} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho c_j F_{ij}^{-1}\partial_{\chi_i}e - \hat{J}F_{ij}^{-1}\sigma_{kj}\partial_{\chi_i}v_k + \partial_{\chi_i}(\hat{J}F_{ij}^{-1}q_j) + \hat{J}\rho r \\
&\qquad e \underbrace{\left(\left.\frac{\partial(\hat{J}\rho)}{\partial t}\right|_{\boldsymbol{\chi}} + \partial_{\chi_i}(\hat{J}F_{ij}^{-1}\rho c_j) \right)}_{=0 \text{ by } (139)} \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho(\boldsymbol{F}^{-1}\boldsymbol{c})\cdot\nabla_{\boldsymbol{\chi}} e - \left(\hat{J}\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right) : \nabla_{\boldsymbol{\chi}}\boldsymbol{v} + \nabla_{\boldsymbol{\chi}} \cdot (\hat{J}\boldsymbol{F}^{-1}\boldsymbol{q}) + \hat{J}\rho r \\
&= \int_{\Omega_{\boldsymbol{\chi}}} \hat{J}\rho \left.\frac{\partial e}{\partial t}\right|_{\boldsymbol{\chi}} + \hat{J}\rho\tilde{\boldsymbol{v}}\nabla_{\boldsymbol{\chi}} e - \left(\hat{J}\boldsymbol{\sigma}\boldsymbol{F}^{-T}\right) : \nabla_{\boldsymbol{\chi}}\boldsymbol{v} + \nabla_{\boldsymbol{\chi}} \cdot (\hat{J}\boldsymbol{F}^{-1}\boldsymbol{q}) + \hat{J}\rho r \,. \qquad (245)
\end{aligned}
$$

Homework 11

The solution is straightforward.

DISTRIBUTION:

1 Thomas J. R. Hughes
ICES, U. Texas at Austin
1 University Station C0200
Austin, TX 78712-0027

1 Mikhail Shashkov
Theoretical Division, Group T7
LANL, MS-B284
Los Alamos, New Mexico 87545

1 Mark Christon
CCS-2 Computational Physics and Methods Group
LANL, Box 1663, MS-D413
Los Alamos, New Mexico 87545

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