VARIATIONAL AND MULTISCALE METHODS IN TURBULENCE

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Abstract An overview is presented of variational and multiscale methods used in Large-Eddy Simulations of turbulence. Results for the problem of bypass transition of a boundary layer are presented illustrating the performance of a recently developed method.

Keywords: Consistent variational multiscale method, turbulence modeling, bypass transition, finite difference method, finite volume method.

1. Introduction

Variational multiscale concepts for Large Eddy Simulation (LES) were introduced in Hughes et al., 2000. The basic idea was to use variational projections in place of the traditional filtered equations and to focus modeling on fine-scale equations rather than coarse-scale equations. Avoidance of filters eliminates many difficulties associated with the traditional approach, namely, inhomogeneous non-commutative filters necessary for wall-bounded flows, use of complex filtered quantities in compressible flows, the closure problem, etc. In addition, modeling confined to the fine-scale equations retains numerical consistency in the coarse-scale equations and thus permits full rate-of-convergence of the underlying numerical method in contrast with the usual approach which limits convergence rate due to artificial viscosity effects in the fully resolved scales (O($h^{4/3}$) in the case of Smagorinsky-type models). Initial versions of the variational multiscale method focused on dividing resolved scales into coarse and fine designations, and eddy viscosities, inspired

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by traditional models, were *only* included in the fine scale equations, and acted *only* on the fine scales. This version was studied in Hughes et al., 2001a, Hughes et al., 2001b, Oberai and Hughes, 2002 and found to work very well on homogeneous isotropic flows and fully-developed equilibrium and non-equilibrium turbulent channel flows. Static eddy viscosity models were employed in these studies but superior results were subsequently obtained through the use of dynamic models, as reported in Holmen et al., 2004 and Hughes et al., 2004b. Good numerical results were obtained with the static approach by other of investigators, namely, Collis, 2002, Jeanmart and Winckelmans, 2002 and Ramakrishnan and Collis, 2002, Ramakrishnan and Collis, 2004c, Ramakrishnan and Collis, 2004b, Ramakrishnan and Collis, 2004a. Particular mention should be made of the work of Farhat and Koobus, 2002, and Koobus and Farhat, 2004, who have implemented this procedure in an unstructured mesh, finite volume, compressible flow code, and applied it very successfully to a number of complex test cases and industrial flows. We believe that this initial version of the variational multiscale concept has already demonstrated its viability and practical utility and is, at the very least, competitive with traditional LES turbulence modeling approaches.

Nevertheless, there is still significant room for improvement. The use of traditional eddy viscosities to represent fine-scale dissipation is an inefficient mechanism. Employing an eddy viscosity in the resolved fine scales to represent turbulent dissipation introduces a consistency error which results in the resolved fine scales being "sacrificed" to retain full consistency in the coarse scales. (In our opinion, this is still better than the traditional approach in which consistency in *all* resolved scales is sacrificed to represent turbulent dissipation.) This procedure is felt to be "inefficient" because approximately 7/8 of the resolved scales are typically ascribed to the fine scales. Another shortcoming noted for the initial version of the variational multiscale method is too small an energy transfer to unresolved modes when the discretization is very coarse (see, e.g., Hughes et al., 2004b). This phenomenon is also noted for some traditional models, such as the dynamic Smagorinsky model Hughes et al., 2004b, but, by design is more pronounced for the multiscale version of the dynamic model. The objectives of recent multiscale work have been to capture all scales consistently and to avoid use of eddy viscosities altogether. This holds the promise of much more accurate and efficient LES procedures. In this work, we describe a new variational multiscale formulation which makes considerable progress toward these goals. In what follows, all resolved scales are viewed as coarse scales, which obviates the issue of inefficiency *ab initio*.

We begin by taking the view that the decomposition into coarse and fine scales is exact. For example, in the spectral case, the coarse-scale space consists of all Fourier modes beneath some cut-off wave number and the fine-scale space consists of all remaining Fourier modes. Consequently, the coarse-scale space has finite dimension whereas the fine-scale space is infinite dimensional. The derivation of the coarse- and fine-scale equations proceeds, first, by substituting the split of the exact solution into coarse and fine scales into the Navier-Stokes equations, then, second, by projecting this equation into the coarse- and fine-scale subspaces. The projection into coarse scales is a finite dimensional system for the coarse-scale component of the solution, which depends parametrically on the fine-scale component. In the spectral case, in addition to the usual terms involving the coarse-scale component, only the cross-stress and Reynolds-stress terms involve the fine-scale component. In the case of non-orthogonal bases, even the linear terms give rise to coupling between coarse and fine scales. The coarse-scale component plays an analogous role to the filtered field in the classical approach, but has the advantage of avoiding all problems associated with homogeneity, commutativity, walls, compressibility, etc. The projection into fine scales is an infinite-dimensional system for the fine-scale component of the solution which depends parametrically on the coarse-scale component. We also assume the cut-off wave number is sufficiently large that the philosophy of LES is appropriate. For example, if there is a well-defined inertial sub-range, then we assume the cut-off wave number resides somewhere within it. This assumption enables us to further assume that the energy content in the fine scales is small compared with the coarse scales. This turns out to be *crucial* in our efforts to analytically represent the solution of the fine-scale equations. The strategy is to obtain approximate analytical expressions for the fine scales then substitute them into the coarse-scale equations which are, in turn, solved numerically. If the scale decomposition is performed in space and time, the *only* approximation in the procedure is the representation of the fine-scale solution. To provide a framework for the fine-scale approximation, we assume an infinite perturbation series expansion to treat the fine-scale nonlinear term in the fine-scale equation. By virtue of the smallness of the fine scales, this expansion is expected to converge rapidly under the circumstances described in many cases of practical interest. The remaining part of the fine-scale Navier-Stokes system is the *linearized* operator which is formally inverted through the use of a matrix Green's function. The combination of a perturbation series and Green's function provides an exact formal solution of the fine-scale Navier-Stokes equations. The driving force in these equations is the Navier-Stokes system residual computed from the coarse scales. This expresses the intuitively obvious fact that if the coarse scales constitute a good approximation to the solution of the problem, the coarse-scale residual will be small and the resulting fine-scale solution will be small as well. This is the case we have in mind and it provides a rational basis for assuming the perturbation series converges rapidly. Note that one cannot use such an argument on the original problem because in this case the perturbation series would almost definitely fail to converge. (If we could have used this argument, we would have solved the Navier-Stokes equations analytically! Unfortunately, it does not work.) The formal solution of the fine-scale equations suggests various approximations may be employed in practical problem solving. We are tempted to use the word "modeling" because approximate analytical representations of the fine scales constitute the only approximation and hence may be thought of as the "modeling" component of the present approach but we want to emphasize that it is very different from classical modeling ideas which are dominated by the *addition* of *ad hoc* eddy viscosities. We will present numerical results that demonstrate these eddy-viscosity terms are unnecessary in the present circumstances. There are two aspects to the approximation of the fine scales: 1) Approximation of the matrix Green's function for the linearized Navier-Stokes system; and 2) approximation of the nonlinearities represented by the perturbation series. The first and obvious thought for the latter aspect, nonlinearity, is to simply truncate the perturbation series. This idea is investigated, as well as another promising idea, in conjunction with some simple approximations of the Green's function. It turns out there is considerable experience in local scaling approximations of the Green's function based on the theory of *stabilized methods* Hughes, 1995, Hughes et al., 1998, Hughes et al., 2004a. The Green's function is typically approximated by locally defined algebraic operators (i.e., the "τ's" of stabilized methods) multiplied by local values of the coarse-scale residual. With this approximation of the solution of the linearized operator, nonlinearities can be easily accounted for in perturbation series fashion. Another approach that accounts for nonlinearities in the fine-scale equations is to introduce a nonlinear algebraic scaling of the Navier-Stokes equations. The resulting local nonlinear algebraic system can be analytically solved. It possesses the reasonable analytical property that if the coarse-scale residual is small, it converges to the linearized solution.

These newer variational multiscale ideas, and the older variants, were implemented in a finite volume code that has enjoyed widespread use in turbulence simulations (see Pierce and Moin, 2001). Following along the lines of Jacobs and Durbin, 2000, Jacobs and Durbin, 2001, who

L_x	$620\ \delta_0$
L_y	$40\ \delta_0$
L_x	$30\ \delta_0$
θ_0	$0.1336\ \delta_0$
Re_{δ_0}	795

Table 1. Bypass-transition parameters

Designation	N_x	N_y	N_z
Fine	2048	180	192
Medium	1024	90	96
Coarse	512	48	48

Table 2. Resolution of bypass-transition meshes.

performed DNS investigations of bypass transition of a boundary layer, we examine this difficult problem from the point of view of the variational multiscale and classical LES. Our aim was to solve this problem as an LES and demonstrate the efficacy of the new ideas in the process.

In our work we endeavor to show the effectiveness, or deficiencies, of LES approaches by studying them over a range of resolutions, from coarse to fine. In our studies of bypass transition we went as far as DNS in the fine-scale end of the spectrum, and approximately 1/8 DNS resolution in each spatial direction. The coarsest LES mesh then involves about 1/256 the number of equations as the DNS mesh and approximately 1/4,096 of the computational effort. We found, independent of the LES method, that in order to accurately simulate bypass transition, the decay of input homogeneous, isotropic, free-stream turbulence must be the same for all meshes. A procedure was developed in which we were able to simulate consistent energy decay with distance of the free-stream turbulence across the range of meshes considered. We then compared the methods to represent bypass transition. We found that the "1/8 DNS mesh" was incapable of representing either the laminar region of the boundary layer or the free-stream turbulence evolution due to too few points in the wall normal direction. We found that all methods gave essentially the same solution at the DNS level, whereas the new variational multiscale formulation was able to attain relatively mesh independent solutions without parameter adjustment for the 1/4, 1/2 and full DNS mesh cases. The 1/4 DNS involves 1/64 the number of mesh points as the DNS case and 1/256 the computational effort. We believe that the new method offers a promising new path for turbulence research in LES. However, it obviously needs testing on a wider variety of flows and implementation in a variety of numerical frameworks, such as, spectral, finite difference and finite element, before definitive conclusions can be drawn. In our experience, the particular numerical discretization method has an enormous impact on the results, and its influence is often underestimated by practitioners evaluating models.

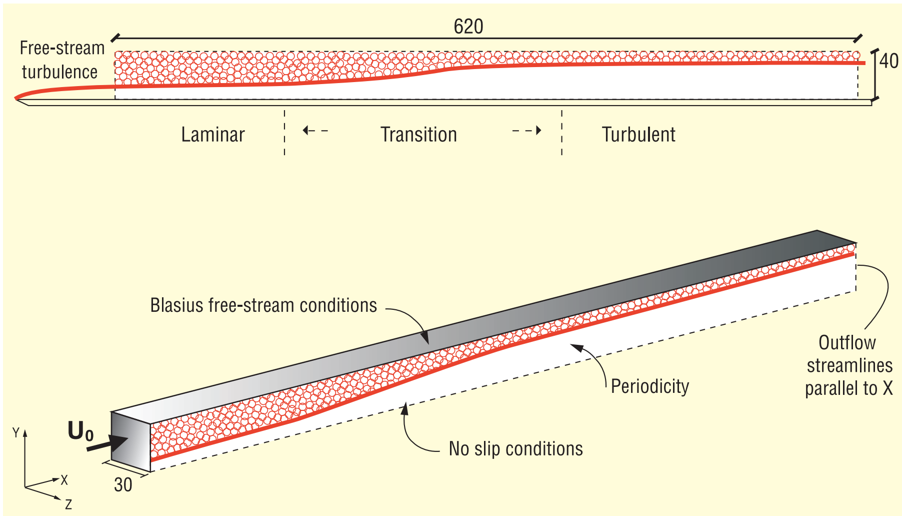


Figure 1. Bypass transition. Problem description.

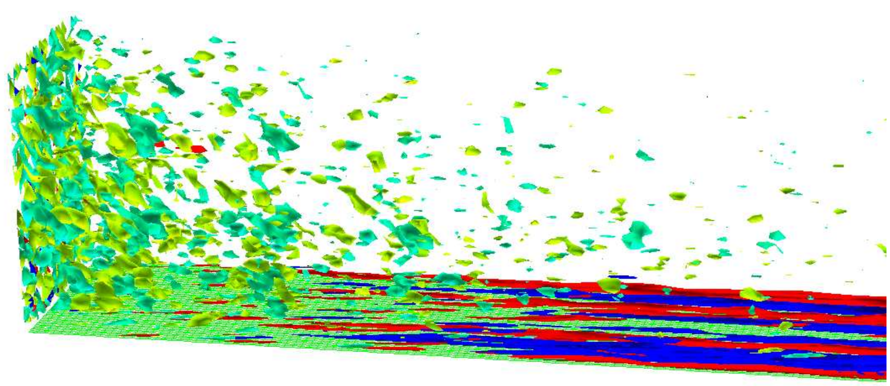


Figure 2. Bypass transition. Decay of free-stream turbulence.

2. Bypass Transition

The problem description is presented in figure 1. Parameters used in the calculations are defined in table 1. δ_0 is the boundary layer thickness at the inlet, L_x, L_y, and L_z are the lengths of the domain in the streamwise, wall-normal, and spanwise directions, respectively, θ_0 is the

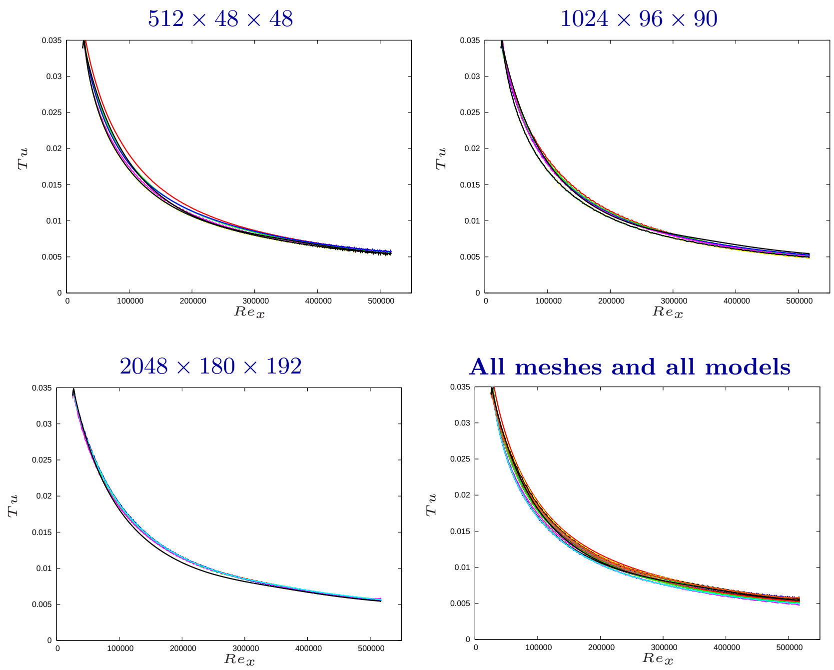


Figure 3. Bypass transition. Decay of free-stream turbulence. Turbulent kinetic energy versus streamwise position.

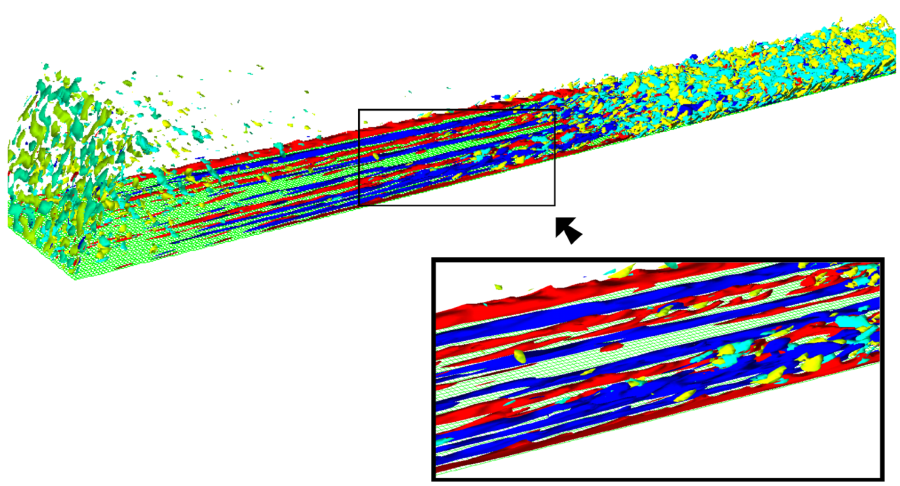
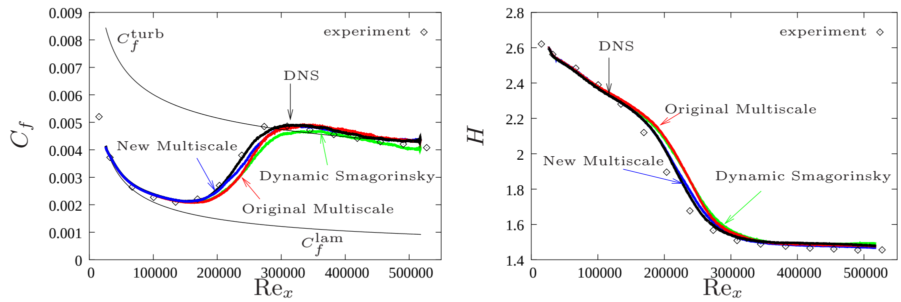


Figure 4. Bypass transition. Velocity fluctuation isosurfaces. The red and blue streaks are streamwise velocity fluctuations. The decay of free-stream turbulence is shown on the left and the fully-developed turbulent boundary layer is seen on the right.

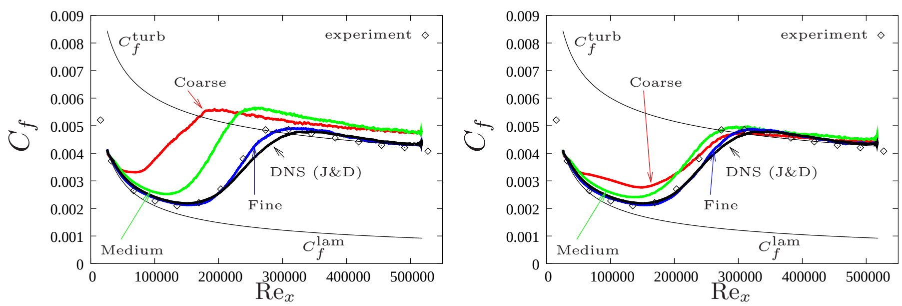


(a) Skin-friction coefficient

(b) Shape factor

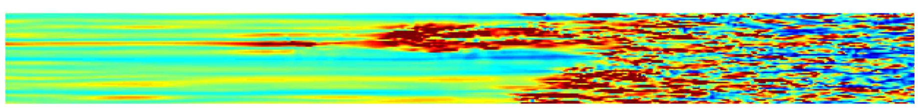
Figure 5. Bypass transition. Numerical results for the $2048 \times 180 \times 192$ mesh.

momentum thickness at the inlet, and $\text{Re}_{\delta_0} (= U_0\delta_0/\nu)$ is the Reynolds number based on the free-stream velocity and length-scale δ_0. The methods tested were incorporated in the program developed by Pierce and Moin, 2001, which employs a second-order central difference scheme on a staggered mesh, an explicit-implicit second-order time-stepping algorithm (Akselvoll and Moin, 1995), and an approximate factorization technique which decouples the velocity and pressure equations (Le and Moin, 1994). The meshes employed in the calculations are described in table 2. The fine mesh corresponds to the DNS mesh case 1 of Jacobs and Durbin, 2000, Jacobs and Durbin, 2001. Synthetic homogeneous isotropic turbulent fluctuations are generated at the inlet in a manner similar to that described in Jacobs and Durbin, 2000, Jacobs and Durbin, 2001. The decay of the free-stream turbulence is illustrated in figure 2. It is essential that the free-stream turbulence, quantified by the turbulent kinetic energy, Tu, decays in a consistent fashion across all meshes and methods. Otherwise, there is no chance of accurately capturing the transition process which is characterized by the formation of "turbulent spots". The procedure employed to achieve this end requires a somewhat lengthy description (omitted here) but results are shown in figure 2 in which Tu is plotted versus Re_x, the Reynolds number based on streamwise position from the inlet. It is important to accurately represent the interaction between the free-stream turbulence and the longitudinal streaks in the laminar boundary layer. The streaks are shown in figure 4 in the form of isosurfaces of streamwise velocity fluctuations. Also shown are the free-stream turbulence (to the left, in the form of isosurfaces of spanwise fluctuations), the fully-developed turbulent boundary layer (to the right), and, in the detail, spot initiation. Time- and span-averaged

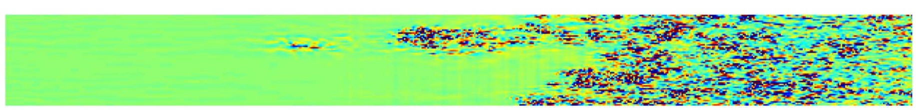


(a) No model (b) New multiscale

Figure 6. Bypass transition. Skin-friction coefficient for coarse, medium, and fine meshes.



(a) Streamwise



(b) Spanwise

Figure 7. Bypass transition. Velocity fluctuations.

results are shown in figure 5 for the fine mesh ($2048 \times 180 \times 192$). Results are presented for the DNS case (no model), dynamic Smagorinsky, original (dynamic) multiscale, and the new residual-based multiscale models. Skin-friction coefficient is presented in figure 5(a), and shape factor, the ratio of the displacement thickness to the momentum thickness, is presented in figure 5(b). At this level of resolution, all the models are very close to each other and there is good agreement with the experimental results of Roach and Brierley, 1992. In figure 6, skin-friction coefficient results are presented for the no-model case and the new multiscale method for coarse, medium, and fine meshes. In the no-model case, figure 6(a), the coarse and medium meshes produce premature transitions and do not achieve correct turbulent correlations in the fully-developed region to the right. On the other hand, the new multiscale method, figure 6(b), attains accurate turbulent correlations in the fully-developed region for all meshes, and the transition region is also fairly accurately captured across meshes. There is discrepancy in the laminar region for the coarsest mesh because, to achieve the correct evolution of Tu, high levels of Tu needed to be specified at the inlet which, in combination with the coarse resolution, perturbed the laminar boundary layer, resulting in departure from the laminar correlation. In figure 7, streamwise and spanwise velocity fluctuation isosurfaces are shown for the DNS case (fine mesh, no model). The plan view corresponds to $y^+ \approx 2$, where plus units are calculated based on the maximum wall shear stress from the DNS. The formation of turbulent spots is clearly apparent.

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