LEAST SQUARES FINITE ELEMENT METHODS FOR VISCOUS, INCOMPRESSIBLE FLOWS

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Abstract

This paper is concerned with finite element methods of least-squares type for the approximate numerical solution of incompressible, viscous flow problems. Our main focus is on issues that are critical for the success of the finite element methods, such as decomposition of the Navier-Stokes equations into equivalent first-order systems, mathematical prerequisites for the optimality of the methods, and the use of mesh-dependent norms. In conclusion we present a novel application of least-squares principles involving an optimal boundary control problem for fluid flows.

1 INTRODUCTION

In this paper we examine the use of least-squares variational principles for the numerical solution of the incompressible, steady state Navier-Stokes equations. In the last few years the corresponding *least-squares finite element methods* have been receiving increasing attention in both the engineering and mathematics communities; see, e.g., Aziz et al. (1985)-Bochev (1997), Jiang et al. (1990) - Lefebvre et al. (1992). This interest has been largely motivated by the significant analytic and computational advantages offered by least-squares principles in the algorithmic design that are not present in, e.g., mixed Galerkin discretizations. Specifically, in the context of the Navier-Stokes equations these advantages are as follows:

- methods are not subject to the inf-sup (LBB) stability

condition:
$\Longrightarrow$ a single approximating space can be used for both the velocity and the pressure;
$\Longrightarrow$ implementation of the algorithms is simplified and more efficient;

- used in conjunction with a Newton linearization least-squares lead to symmetric, positive definite linear systems, at least in the neighborhood of a solution:
  $\Longrightarrow$ efficient iterative methods, e.g., conjugate gradients, can be used in the solution process;
  $\Longrightarrow$ algorithms can be implemented without assembly of the discretization matrix;

- essential boundary conditions can be enforced in a weak, variational sense:
  $\Longrightarrow$ approximating spaces are not subject to the essential boundary conditions;
  $\Longrightarrow$ simplified implementation for domains with complex geometries.

However, it is now well-understood that successful utilization of these advantages requires a thorough consideration of the settings for the least-squares method. In many settings such as the primitive variable formulation, least-squares methods suffer from two serious problems. The first one is that conforming discretizations require the use of continuously differentiable finite element functions. The second one is that the condition number of the discrete equations is often proportional to h^{-4}, where h denotes a measure of the grid size. As a result, most of the recent research; see, e.g., Bochev and Gunzburger (1993b) - Bochev (1997), Jiang et al. (1990) - Lefebvre et al. (1992), has focused on applications of the least-squares methodology to the Navier-Stokes equation written in an equivalent *first-order* form. Besides the fact that the use of first-order systems helps to avoid the above two problems, it also has the significant added advantage of allowing direct approximations of physically meaningful quantities such as vorticity and stress. In the next section we give a necessarily brief outline of three basic first-order decomposition approaches that have been used in the context of least-squares methods. Then we describe the corresponding finite element methods and their respective properties.

2 FIRST-ORDER SYSTEMS

Let Ω be a bounded, open domain in $\mathbf{R}^n$, $n = 2, 3$. We consider the steady state incompressible Navier-Stokes equations given by

$$\begin{aligned} -\nu\triangle\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega & (1)\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega & (2)\\ \mathbf{u} &= 0 \quad \text{on } \Gamma\,, & (3)\end{aligned}$$

where $\mathbf{u}$, p, ν and $\mathbf{f}$ denote velocity, pressure, kinematic viscosity (the inverse of the Reynolds number Re), and a given body force, respectively. Transformation of (1)-(2) into an equivalent first-order system requires a choice of new dependent variables, and addition of one or more compatibility conditions. Customarily, the new dependent variables involve derivatives of the velocity field. Three of the most common choices can be described in terms of the nonsymmetric velocity gradient tensor $\nabla\mathbf{u}$ as follows.

Vorticity formulation. The new variables are given by the skew-symmetric part of the velocity gradient. This choice corresponds to the use of the vorticity

$$\boldsymbol{\omega} = \nabla\times\mathbf{u} \tag{4}$$

as a new dependent variable. In view of (2) and the vector identities

$$\nabla\times\nabla\times\mathbf{u} = -\triangle\mathbf{u} + \nabla\nabla\cdot\mathbf{u}\,, \tag{5}$$

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\times\nabla\times\mathbf{u}\,, \tag{6}$$

one can replace (1) by

$$\nu\nabla\times\boldsymbol{\omega} + \boldsymbol{\omega}\times\mathbf{u} + \nabla r = \mathbf{f} \quad \text{in } \Omega\,, \tag{7}$$

where $r = p + 1/2|\mathbf{u}|^2$ denotes the total head. This yields the *velocity-vorticity-pressure* Navier-Stokes system given by (7), (2), and (4). In 2D and 3D this system has 4 and 7 unknown fields, respectively. In the latter case it also must be augmented by the compatibility condition $\nabla\cdot\boldsymbol{\omega} = 0$; see Chang and Gunzburger (1987), and Chang and Jiang (1990).

Stress formulation. The new variables are now given by the symmetric part of $\nabla\mathbf{u}$ (see Bochev and Gunzburger (1995)):

$$T = \sqrt{2\nu}\,\epsilon(\mathbf{u}) \equiv \sqrt{2\nu}\,\frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right)\,. \tag{8}$$

In view of (2) and the identity

$$\nabla \cdot T = \sqrt{2\nu}\left(\triangle \mathbf{u} + \nabla(\nabla \cdot \mathbf{u})\right),$$

equation (1) can be replaced by

$$\sqrt{2\nu}\,\nabla \cdot T - \nabla p + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{f} \quad \text{in } \Omega\,. \tag{9}$$

The corresponding *velocity-pressure-stress* first-order system is given by (9), (2), and (8). The number of unknown fields for this system is 6 and 10 in 2D and 3D, respectively.

Gradient formulation. Here, all components of the velocity gradient are used as new dependent variables (see, e.g., Cai et al. (1997) and Bochev et al. (1996)), i.e., we set

$$\underline{\mathbf{U}} = \nabla \mathbf{u}\,. \tag{10}$$

Then one can rewrite (1) as follows:

$$-\nu(\nabla \cdot \underline{\mathbf{U}})^t + \underline{\mathbf{U}} \cdot \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega\,. \tag{11}$$

In Cai et al. (1997) the new variable $\underline{\mathbf{U}}$ is called *velocity flux*, thus the first-order system (11), (2), and (10) is usually referred to as the *velocity flux* Navier-Stokes equations. The number of unknowns increases to 7 in 2D and to 13 in 3D. This system is seldom considered without two additional constraints given by

$$\nabla(tr\underline{\mathbf{U}}) = \mathbf{0} \quad \text{in } \Omega \tag{12}$$
$$\nabla \times \underline{\mathbf{U}} = \underline{\mathbf{0}} \quad \text{in } \Omega\,. \tag{13}$$

The need for these two constraints will be explained in the next section.

3 LEAST-SQUARES METHODS

In this section we consider the development of least-squares methods for the problem (1)-(3). Our main goals are first, to present a general framework that can be used for this purpose, and second, to discuss how this framework can be adapted to various settings for the least-squares methods.

To describe the main ideas let U denote a set of dependent variables corresponding to a first-order Navier-Stokes system. We denote the associated first-order Stokes operator by $\mathcal{L}$, and let $\mathcal{R}U$ denote an admissible boundary condition for this operator. We express symbolically the associated first-order Navier-Stokes equations as

$$\begin{aligned} \mathcal{L}U + \mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U) &= \mathbf{0} \quad \text{in } \Omega \\ \mathcal{R}U &= \mathbf{0} \quad \text{on } \Gamma\,, \end{aligned} \tag{14}$$

where $\mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U)$ accounts for the nonlinear term and the body force. We assume that $\mathcal{L}$ and $\mathcal{R}$ are well-posed in the sense of ADN elliptic theory (see Agmon, Douglis and Nirenberg (1964)), i.e., that $\mathcal{L}$ is uniformly elliptic and that there exist function spaces $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Y}(\Gamma)$ such that

$$\|U\|_{\mathbf{X}} \le C(\|\mathcal{L}U\|_{\mathbf{Y}} + \|\mathcal{R}U\|_{\mathbf{Y}(\Gamma)}). \tag{15}$$

Then, with the system (14) we associate a least-squares functional given by

$$\mathcal{J}(U) = \frac{1}{2}\left(\|\mathcal{L}U + \mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U)\|^2_{\mathbf{Y}} + \|\mathcal{R}U\|^2_{\mathbf{Y}(\Gamma)}\right). \tag{16}$$

In the following for simplicity we shall assume that the space $\mathbf{X}$ is constrained by the boundary conditions in (14). As a result, the boundary residual in (16) can be omitted.

It is not difficult to see that a solution of (14) minimizes (16), and vice-versa. This observation forms the foundation of the least-squares approach, i.e., solution of (14) is sought by means of minimizing the functional (16). Minimizers of (16) are subject to a necessary condition (Euler-Lagrange equation) given by the problem:

seek $U \in \mathbf{X}$ *such that*

$$\left(\mathcal{L}U + \mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U), \mathcal{L}V + D_U\mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U) \cdot V\right)_{\mathbf{Y}} = 0 \tag{17}$$

for all $V \in \mathbf{X}$,

where $D_U\mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U)$ denotes the Frèchet derivative of $\mathcal{N}$ with respect to the third argument. As a result, solution of (14) is accomplished by solving the nonlinear variational problem (17). To define a least-squares method for (14) the problem (17) is discretized using a finite element subspace $\mathbf{X}^h$ of $\mathbf{X}$. This yields a nonlinear system of algebraic equations given by:

seek $U^h \in \mathbf{X}^h$ *such that*

$$\Big(\mathcal{L}U^h + \mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U^h), \tag{18}$$

$$\mathcal{L}V^h + D_U\mathcal{N}(\frac{1}{\nu}, \mathbf{f}, U^h) \cdot V^h\Big)_{\mathbf{Y}} = 0$$

for all $V^h \in \mathbf{X}^h$.

Although the explicit form of (18) may be quite formidable, this problem has some very attractive computational properties. In contrast to mixed Galerkin discretizations, the nonlinear variational problem (18) corresponds to minimization, rather than to a saddle-point problem. As a result, in a neighborhood of a solution the Jacobian matrix of (18) is necessarily symmetric and positive definite. Furthermore, since the attraction ball of Newton's method is nontrivial, using a properly implemented continuation with respect to the Reynolds number one can devise algorithms that, even for large values of Re, are guaranteed to encounter only symmetric and positive definite linear systems at each Newton step. These systems can be solved using robust and efficient iterative methods, such as preconditioned conjugate gradients; see, e.g., Bochev and Gunzburger (1993b), Jiang et al. (1993), Jiang and Povinelli (1990), and Lefebvre et al. (1992).

Majority of the available research literature has focused on efficient implementations of methods formulated along these lines, and obtaining solutions for a variety of model and practical problems. Less attention has been paid to rigorous mathematical analyses, including error estimates and accuracy issues. Such analysis can be found in Bochev and Gunzburger (1994) - Bochev (1997). Among the main results established there is the fundamental role of the a priori estimate (15), i.e., the well-posedness of the associated Stokes problem, for the existence of optimal discretization error estimates in the least-squares method. Moreover, there exist counterexamples that demonstrate loss of convergence order when (15) is not valid (some of these examples are presented in the next section).

Mathematical analysis of (18) involves two stages. At the first stage (15) is used to establish optimal error estimates for a least-squares method for the associated *linear* first-order Stokes problem. On the second stage, the abstract nonlinear theory of Brezzi et al. (1980) is utilized to establish the error estimates for the least-squares approximation of the Navier-Stokes equations; see, e.g., Bochev et al. (1996) and Bochev (1997). Under the assumption that the latter have a regular branch of solutions one can show that the accuracy of the approximate solution for the nonlinear problem is of the same order as the accuracy of the approximation for the linear Stokes problem.

In what follows we discuss three broad categories of least-squares methods that result from application of this framework. These categories are distinguished primarily by the type of norms employed in the functional (16), e.g., L^2-norms, weighted L^2-norms, and discrete negative norms.

4 BASIC L^2 METHODS

A seemingly natural choice of the spaces $\mathbf{Y}$ and $\mathbf{X}$ for the first-order system (14) is given by products of L^2 and H^1-spaces, respectively. Then, all equation residuals in (16) are measured in the norm of L^2. Such least-squares functionals are quite attractive from computational point of view. First, conforming methods can be implemented using standard finite element spaces. Second, resulting algebraic systems have condition numbers of order h^{-2}. Third, if an a priori estimate of the form (15) is valid, one can establish optimal error estimates in the norm of H^1 for *all variables*, as well as optimal convergence of multiplicative and additive multigrid methods; see Cai et al. (1996). Unfortunately, validity of (15) in such spaces cannot be always guaranteed. For example, a basic L^2 least-squares functional for the velocity-vorticity-pressure Navier-Stokes problem is given by

$$\begin{aligned}\mathcal{J}(\boldsymbol{\omega}, \mathbf{u}, r) = \frac{1}{2}\big(&\|\nu\nabla\times\boldsymbol{\omega} + \nabla r + \boldsymbol{\omega}\times\mathbf{u} - \mathbf{f}\|_0^2 \\ &+ \|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\nabla\cdot\mathbf{u}\|_0^2\big)\,. \qquad (19)\end{aligned}$$

Least-squares methods based on (19) were first developed in Chang and Jiang (1990), Jiang and Povinelli (1990), Jiang and Sonnad (1991), Lefebvre et al (1992), and Bochev and Gunzburger (1993a-1993b). In this context (15) specializes to

$$\begin{aligned}\|\boldsymbol{\omega}\|_1 + \|r\|_1 + \|\mathbf{u}\|_1 \le C(&\|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|_0 \\ &+ \|\nu\nabla\times\boldsymbol{\omega} + \nabla r\|_0 + \|\nabla\cdot\mathbf{u}\|_0)\,. \qquad (20)\end{aligned}$$

However, analysis based on ADN theory; see Bochev and Gunzburger (1994), shows that (20) is valid only for non-standard "no-slip" and "no-penetration" type boundary conditions (see also Jiang et al. (1994)), whereas for the velocity boundary condition (3) the relevant estimate is given by

$$\|\boldsymbol{\omega}\|_1 + \|r\|_1 + \|\mathbf{u}\|_2 \leq C(\|\nabla \times \mathbf{u} - \boldsymbol{\omega}\|_1 + \|\nu\nabla \times \boldsymbol{\omega} + \nabla r\|_0 + \|\nabla \cdot \mathbf{u}\|_1) \,. \tag{21}$$

Table 1. Convergence rates for linear approximations

	H^1 error rates		
Function	BC1	BC2	Optimal
$\mathbf{u}$	0.12	0.93	**1.00**
ω	0.17	0.91	**1.00**
r	0.17	0.96	**1.00**

Table 2. Convergence rates for quadratic approximations - functional (23)

	Error rates			
Function	L^2	Optimal	H^1	Optimal
$\mathbf{u}$	3.05	**3.00**	1.99	**2.00**
$\underline{\mathbf{U}}$	3.10	**3.00**	2.01	**2.00**
p	3.34	**3.00**	2.10	**2.00**

As a result, at the first stage of the mathematical analysis optimality of the basic L^2 method for the Stokes equations can only be established for such nonstandard boundary conditions. Moreover, this suboptimality is not a purely theoretical fact, but it can also be observed computationally; see, e.g., Bochev and Gunzburger (1994), Deang and Gunzburger (1997). For an illustration of this fact in Table 1 we present computational results due to Deang and Gunzburger (1997). This table compares convergence rates for the Stokes problem obtained with a basic L^2 least-squares method implemented using piecewise linear elements and two different boundary conditions. The (BC1) column in Table 1 corresponds to the rates obtained with the velocity boundary condition for which the relevant a priori estimate (20) is not valid. The (BC2) column corresponds to the convergence rates computed with a non-standard boundary condition given by $\mathbf{u} \cdot \mathbf{n} = U_n$ and $p = P$ on Γ, for which the estimate (20) is valid (see Bochev and Gunzburger (1994)). While the rates obtained with the nonstandard condition (BC2) are in good agreement with the optimal theoretical rates, data in Table 1 indicates that the basic L^2 functional is not optimal for the Stokes problem with the velocity boundary condition.

Despite the poor convergence rates for the Stokes problem, in engineering practice, use of the basic functional (19) for the Navier-Stokes equations with the velocity boundary condition has resulted in very high quality computational results (see, e.g., Bochev and Gunzburger (1993b), Jiang et al. (1993), Lefebvre et al. (1992)). In the next section we offer one possible explanation of this paradox, using as a starting point a different approach for the formulation of the least-squares functional. Let us also mention that computations with (19) have likewise demonstrated very good agreement with benchmark studies reported in the literature; see Figure 1. This figure illustrates driven cavity flow results obtained with (19). The curves in Figure 1 represent velocity

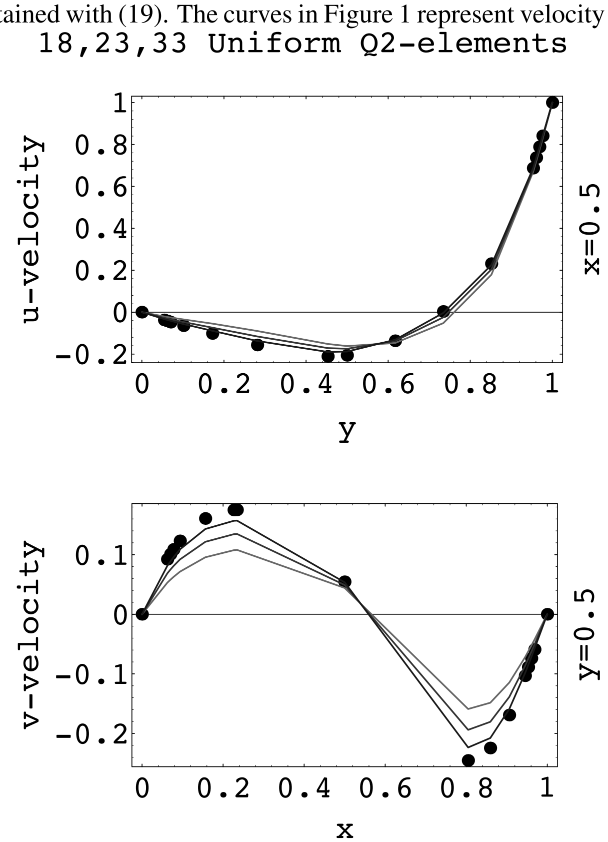

Figure 1. Velocity profiles computed with (19) vs. benchmark results of Ghia et al. (1982) (solid dots). Driven Cavity flow at $Re = 100$.

profiles through the geometrical center of the cavity computed using uniform triangulations of 18, 23, and 33 biquadratic elements.

An estimate similar to (21) is valid for the velocity-pressure-stress first-order system, regardless of the choice of the boundary conditions. As a result, a basic L^2 functional for the associated Stokes problem is not optimal.

An example of a first-order formulation that leads to a theoretically optimal basic L^2 functional is given by the velocity-flux system (11), (2), and (10), augmented with the two additional constraints (12)-(13). The relevant a priori estimate for the augmented system is given by (see Cai et al. (1997))

$$\begin{aligned} \|\underline{\mathbf{U}}\|_1 + \|\mathbf{u}\|_1 + \|p\|_1 \le C(&\|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|_0 \\ + \;& \| -\nu(\nabla \cdot \underline{\mathbf{U}})^t + \underline{\mathbf{U}}^t \mathbf{u} + \nabla p - \mathbf{f}\|_0 \\ + \;& \|\nabla \cdot \mathbf{u}\|_0 + \|\nabla (\mathrm{tr}\underline{\mathbf{U}})\|_0 + \|\nabla \times \underline{\mathbf{U}}\|_0) \,. \end{aligned} \tag{22}$$

The two constraints (12)-(13) are essential for (22) to be valid, i.e., the H^1-norms of the variables cannot be bounded without the last two terms in (22); see Cai et al. (1997). As a result, a basic least-squares functional of the form

$$\begin{aligned} \mathcal{J}(\underline{\mathbf{U}}, \mathbf{u}, p) &= \| -\nu(\nabla \cdot \underline{\mathbf{U}})^t + \underline{\mathbf{U}}^t \mathbf{u} + \nabla p - \mathbf{f}\|_0^2 \\ &+ \|\nabla \cdot \mathbf{u}\|_0^2 + \|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|_0^2 \\ &+ \|\nabla (\mathrm{tr}\underline{\mathbf{U}})\|_0^2 + \|\nabla \times \underline{\mathbf{U}}\|_0^2 \end{aligned} \tag{23}$$

Table 3. Convergence rates for quadratic-linear approximations

	H^1 error rates		
Function	BC1	BC1W	Optimal
$\mathbf{u}$	0.14	1.91	**2.00**
$\boldsymbol{\omega}$	0.16	1.22	**1.00**
r	0.16	1.21	**1.00**

yields optimal discretization errors, see Table 2 and Bochev et al. (1996).

5 WEIGHTED L^2 METHODS

The estimate (21) indicates that, for the velocity boundary condition, an optimal functional should use the residual norms $\|\nabla \times \mathbf{u} - \boldsymbol{\omega}\|_1$ and $\|\nabla \cdot \mathbf{u}\|_1$, instead of the L^2-norms that appear in (19). This norms are impractical in the sense that they require discretization by finite element spaces that are *continuously differentiable* across the element edges. The main idea of the weighted least-squares approach is to replace such norms by properly scaled L^2-norms. The appropriate scaling factor can be determined using, e.g., the inverse inequality $\|q^h\|_1 \le Ch^{-1}\|q^h\|_0$; see Ciarlet (1978). This inequality suggests that, for finite element functions, one can "simulate" the norm $\|q^h\|_1$ by $h^{-1}\|q^h\|_0$, i.e., the two H^1 residuals can be substituted by $h^{-1}\|\nabla \times \mathbf{u} - \boldsymbol{\omega}\|_0$ and $h^{-1}\|\nabla \cdot \mathbf{u}\|_0$, respectively. Analysis of the corresponding *weighted least-squares* method for the velocity-vorticity-pressure Stokes equations can be found in Bochev and Gunzburger (1994). This analysis suggests that one may use polynomials of one degree lower for the pressure and vorticity than one uses for the velocity, and that with the velocity boundary condition the weights are necessary for the validity of optimal error estimates. For example, the use of piecewise quadratic polynomials for the velocity and piecewise linear polynomials for the vorticity and the pressure gives the estimate

$$\|\mathbf{u} - \mathbf{u}^h\|_1 + \|r - r^h\|_0 + \|\boldsymbol{\omega} - \boldsymbol{\omega}^h\|_0 = O(h^2)$$

Using the inverse inequality one can also show that

$$\|r - r^h\|_1 + \|\boldsymbol{\omega} - \boldsymbol{\omega}^h\|_1 = O(h).$$

These estimates, as well as the importance of the weights for achieving optimal convergence rates, are illustrated in Table 3 (see Deang and Gunzburger (1997)). In this table columns (BC1W) and (BC1) correspond to results obtained with and without the weights in the functional, respectively.

These and other preliminary theoretical and computational results indicate the following choice of a least-squares functional for the velocity-vorticity-pressure Navier-Stokes equations:

$$\begin{aligned} \mathcal{J}(\boldsymbol{\omega}, \mathbf{u}, r) &= \frac{1}{2}\big(\nu^{-2}\|\nu\nabla \times \boldsymbol{\omega} + \nabla r + \boldsymbol{\omega} \times \mathbf{u} - \mathbf{f}\|_0^2 \\ &+ h^{-2}\|\nabla \times \mathbf{u} - \boldsymbol{\omega}\|_0^2 + h^{-2}\|\nabla \cdot \mathbf{u}\|_0^2\big) \end{aligned} \tag{24}$$

Note that the vorticity transport equation in (24) is weighted by the square of the Reynolds number. The need for such a weight comes from theoretical considerations, however, it can be used to explain the paradox that computations with the unweighted functional (19) yield good results while analyses seem to indicate that one should use the functional (24). Indeed, it is often the case that one chooses h to depend on ν, at least locally, e.g., in boundary layers. This is particularly true for moderate and high values of Re, where efficient numerical solution of the Navier-Stokes equations usually requires the use of highly nonuniform grids to resolve boundary layers. If, for example, one chooses $h = O(\nu)$, then the functionals (19) and (24) differ only by an unimportant constant scale factor, i.e., computationally these functionals are equivalent.

6 NEGATIVE NORM METHODS

Our last example involves a method that represents a relatively recent development in the least-squares approach. To illustrate the main ideas we use again the velocity-vorticity-pressure first-order system. Using the ADN theory one can show validity of the following a priori estimate for the associated first-order Stokes operator along with the velocity boundary condition:

$$\begin{aligned} \|\boldsymbol{\omega}\|_0 + \|r\|_0 + \|\mathbf{u}\|_1 \le C(&\|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|_0 \\ &+\|\nu\nabla\times\boldsymbol{\omega} + \nabla r\|_{-1} + \|\nabla\cdot\mathbf{u}\|_0)\,. \end{aligned} \qquad (25)$$

Note that as in (21) vorticity and pressure cannot have the same differentiability order as the velocity field. According to (25) an optimal least-squares method can be devised using a functional given by

$$\begin{aligned} \mathcal{J}(\boldsymbol{\omega},\mathbf{u},r) \quad = \quad & \frac{1}{2}\left(\nu^{-2}\|\nu\nabla\times\boldsymbol{\omega} + \nabla r + \boldsymbol{\omega}\times\mathbf{u} - \mathbf{f}\|_{-1}^2\right. \\ & \left. + \|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\nabla\cdot\mathbf{u}\|_0^2\right)\,. \end{aligned} \qquad (26)$$

Because the H^{-1} norm is not computable, the functional (26) is not any more practical than, e.g., a functional involving the terms $\|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|_1$ and $\|\nabla\cdot\mathbf{u}\|_1$. The key to a practical method, suggested in Bramble et al. (1994), is to replace $\|\cdot\|_{-1}$ by a discrete equivalent norm given by $\|\cdot\|_{-h}^2 = ((h^2 I + B^h)\cdot,\cdot)_0$, where B^h denotes a preconditioner for the Laplace operator $\triangle$. Note that with the choice $B^h \equiv 0$ we obtain the weighted functional (24) scaled by the common (and unimportant for the minimization) factor h^2. If $h = O(\nu)$ this functional is again computationally equivalent to the unweighted functional (19).

7 METHODS FOR OPTIMAL CONTROL

In this section we present a novel application of least-squares methods that involves an optimal boundary control problem for the driven cavity flow; see Bochev and Bedivan (1997). The model problem that is considered here is due to Desai and Ito (1994). Customarily, the velocity boundary condition for the driven cavity flow ($\mathbf{f} = \mathbf{0}$) specifies $\mathbf{u} = (1,0)$ on the top surface, and zero otherwise. If the bottom surface is also allowed to move in the same direction, i.e., if $\mathbf{u} = (u_B, 0)$ on this surface, then computational experiments indicate that the flow tends to separate into two regions, see Figure 2. Thus, one can argue that there exists a horizontal line Γ_S, such that $\mathbf{u}\cdot\mathbf{n}_S = 0$ along Γ_S. Based on this observa-

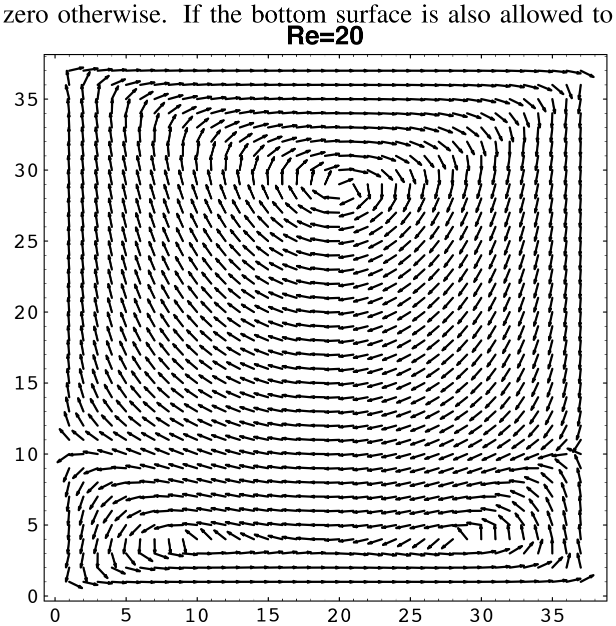

Figure 2. Flow separation for $u_B = 0.1$**,** $u_T = 1.0$

tion Desai and Ito (1994) proposed the following optimal control problem:

> Given the bottom velocity $\mathbf{u}_B = (u_B, 0)$, find the top velocity $\mathbf{u}_T = (u_T, 0)$ such that separation of the flow occurs at a desired horizontal line location Γ_S.

The cost functional associated with this problem is given by

$$J(\mathbf{u}, p, \mathbf{u}_T) = \int_{\Gamma_S} |\mathbf{u} \cdot \mathbf{n}_S|^2 = \int_{\Gamma_S} |v|^2 \, d\Gamma \,, \qquad (27)$$

where v denotes the second component of the velocity field. In what follows we shall develop the least-squares optimal control method for this problem in the context of the velocity-vorticity-pressure Navier-Stokes equations. The main idea is to penalize a least-squares functional for the state equations using a properly weighted cost functional (27). The new objective functional is then minimized over a suitable set of admissible states.

Below we develop the least-squares method for a distributed control form of the model problem. For this purpose, let $U_0 = (\boldsymbol{\omega}_0, \mathbf{u}_0, r_0)$ denote a solution of the velocity-vorticity-pressure Stokes problem with boundary condition given by $\mathbf{u} = (u_B, 0)$ on the bottom surface, and zero otherwise. Similarly, let $U_1 = (\boldsymbol{\omega}_1, \mathbf{u}_1, r_1)$ denote solution of the same equations with $\mathbf{u} = (1, 0)$ on the top surface, and zero otherwise. Then, the optimal solution is sought in the form $U_l = (\boldsymbol{\omega}_l, \mathbf{u}_l, r_l)$ where

$$\boldsymbol{\omega}_l = \boldsymbol{\omega} + \boldsymbol{\omega}_0 + l\boldsymbol{\omega}_1; \quad \mathbf{u}_l = \mathbf{u} + \mathbf{u}_0 + l\mathbf{u}_1; \quad r_l = r + r_0 + lr_1,$$

Optimal flow

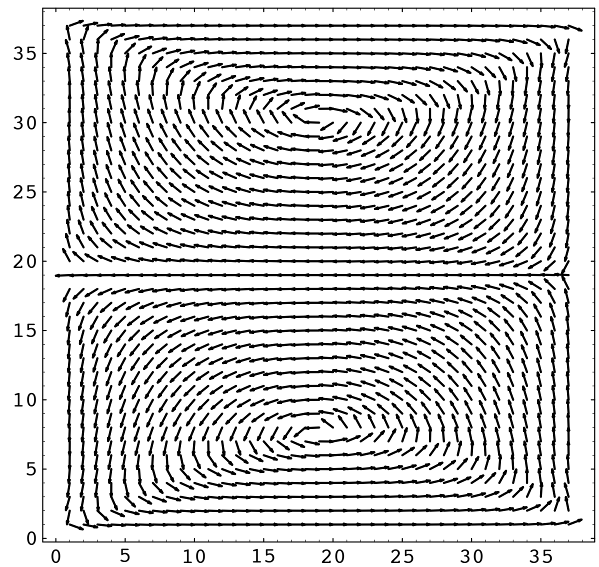

Figure 3. Optimal flow for $u_B = 0.1$ and Γ_S - horizontal line through the cavity center.

and $U = (\boldsymbol{\omega}, \mathbf{u}, r)$ are unknown functions, such that $\mathbf{u} = 0$ on Γ, i.e., the control is introduced via the scalar l. As a result, the distributed control problem is given by

minimize

$$\int_{\Gamma_S} |v_l|^2 d\Gamma = \int_{\Gamma_S} |v + v_0 + lv_l|^2 d\Gamma$$

subject to

$$\begin{aligned} \nu\nabla \times \boldsymbol{\omega} + \boldsymbol{\omega}_l \times \mathbf{u}_l + \nabla r &= \mathbf{0} \;\text{ in}\, \Omega \\ \nabla \times \mathbf{u} - \boldsymbol{\omega} &= 0 \;\text{ in}\, \Omega \\ \nabla \cdot \mathbf{u} &= 0 \;\text{ in}\, \Omega \\ \mathbf{u} &= \mathbf{0} \;\text{ on}\, \Gamma \,. \end{aligned}$$

The state equations in the above control problem were obtained by evaluating the velocity-vorticity-pressure Navier-Stokes equations at $\boldsymbol{\omega}_l$, $\mathbf{u}_l$ and r_l. Then we have used the fact that $\boldsymbol{\omega}_i$, $\mathbf{u}_i$ and r_i solve a Stokes problem with a homogeneous right hand side. With the distributed control problem we associate a penalized least-squares functional given by

$$\begin{aligned} J(\boldsymbol{\omega}, \mathbf{u}, r, l) &= \|\nu\nabla\times\boldsymbol{\omega} + \nabla r + \boldsymbol{\omega}_l \times \mathbf{u}_l\|_0^2 \\ &+ \|\nabla\times\mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\nabla\cdot\mathbf{u}\|_0^2 \\ &+ \alpha\int_{\Gamma_S} |v_l|^2 d\Gamma\,. \end{aligned} \tag{28}$$

Let $\mathbf{X}^h$ denote a finite element space for the approximation of the unknown functions $(\boldsymbol{\omega}, \mathbf{u}, r)$ and let $[0, L]$, $L > 0$,

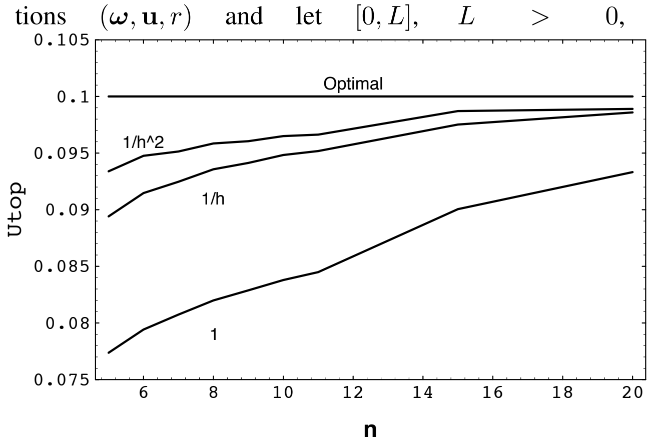

Figure 4. u_T approximations computed with α equal to 1, $1/h$ and $1/h^2$

denote the admissible set for the control parameter l. Then, we have the following discrete least-squares optimization problem:

seek $U^h = (\boldsymbol{\omega}^h, \mathbf{u}^h, r^h) \in \mathbf{X}^h$ *and* $l \in [0, L]$ *such that*

$$J(\boldsymbol{\omega}^h, \mathbf{u}^h, r^h, l) \le J(\boldsymbol{\xi}^h, \mathbf{v}^h, q^h, k)$$

for all $V^h \in \mathbf{X}^h$*, and* $k \in [0, L]$.

The necessary condition for this problem can be obtained by setting the first variation of (28) to zero. This yields a nonlinear algebraic system of equations for the unknown discrete functions $(\boldsymbol{\omega}^h, \mathbf{u}^h, p^h)$ and the unknown control value l. This system is solved using Newton linearization combined with assembly-free preconditioned conjugate gradients solver. The weight α in (28) has great importance for the quality of the computed optimal control value u_T. This weight can be determined using a scaling argument. More precisely, if $\mathbf{u}$ is in $H^1(\Omega)$, then its restriction on Γ_S is in $H^{1/2}(\Gamma_S)$. For discrete functions the appropriate scaling factor that relates the trace norm on $H^{1/2}(\Gamma_S)$ with an L^2 norm is given by $1/\sqrt{h}$. As a result, we choose $\alpha = 1/h$.

To illustrate this method computationally we consider the optimal control problem with the following parameters: $Re = 10$, $u_B = 0.1$, and Γ_S is the horizontal line through the center of the cavity. Thus, the expected optimal value of the top velocity is also equal to 0.1. Corresponding numerical results are presented in Figures 3-4. Figure 3 contains the optimal flow computed with the least-squares control method using triangulation of 19 by 19 biquadratic elements. Figure 4 illustrates how the choice of the weight α affects the computed optimal value. Computations for this experiment were carried over a sequence of triangulations, starting from 4 by 4 biquadratic elements. From this figure we can see that $\alpha = 1/h$ provides better asymptotic rate than $\alpha = 1/h^2$ and better approximations than $\alpha = 1$.

The Dirichlet control used in the model problem has limited scope for high values of the Reynolds number; thus our main goal

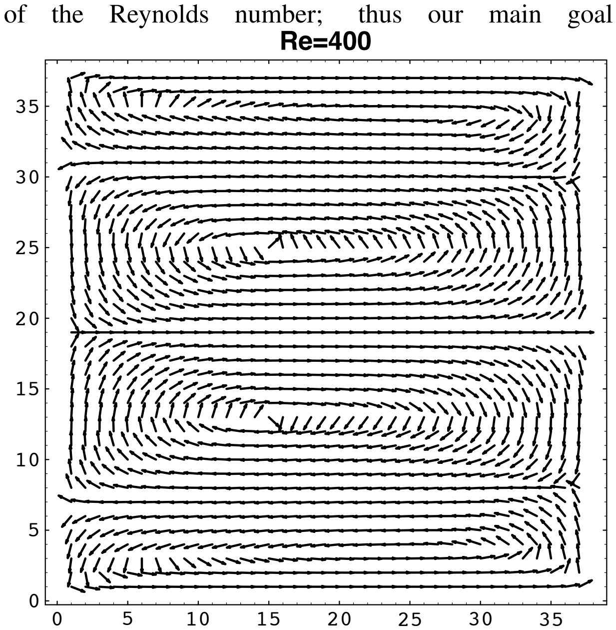

Figure 5. Multiple separation lines in the driven cavity flow for $u_B = u_T = 0.1$

was to illustrate how least-squares ideas can be applied for the solution of optimal control problems. In fact, for $Re \geq 200$ separation in the driven cavity flow occurs along multiple lines, i.e., the optimal control problem does not have a unique solution. This is illustrated by the numerical results presented in Figure 5. The driven cavity flow in this figure has been computed with $Re = 400$ and using boundary conditions given by $\mathbf{u} = (0.1, 0)$ on both the bottom and the top surfaces, and zero otherwise. Similar results were obtained for values of the Reynolds number up to 1000.

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