

LEAST-SQUARES METHODS FOR THE NAVIER-STOKES EQUATIONS: TRENDS AND DEVELOPMENTS

P. BOCHEV *

Abstract. We consider issues related to the design and analysis of least-squares methods for the incompressible Navier-Stokes equations. An abstract framework which allows to treat a large class of methods is outlined and illustrated by means of several specific examples of least-squares methods.

Key words. Navier-Stokes equations, least-squares principle, finite element methods.

AMS subject classifications. 76D05, 76D07, 65F10, 65F30

1. Introduction. Let Ω be a bounded, open domain in \mathcal{R}^n , $n = 2, 3$. We shall be concerned with finite element methods of least-squares type for the incompressible, steady state Navier-Stokes equations

$$\begin{aligned} (1) \quad & -\nu\Delta\mathbf{u} + \mathbf{u} \cdot \text{grad } \mathbf{u} + \text{grad } p = \mathbf{f} \quad \text{in } \Omega \\ (2) \quad & \text{div } \mathbf{u} = 0 \quad \text{in } \Omega \\ (3) \quad & \mathbf{u} = 0 \quad \text{on } \Gamma. \end{aligned}$$

As usual, \mathbf{u} , p , ν and \mathbf{f} denote velocity, pressure, kinematic viscosity (the inverse of the Reynolds number Re), and a given body force, respectively. The Stokes problem associated with (1)-(2) is obtained by neglecting the convective term $\mathbf{u} \cdot \text{grad } \mathbf{u}$ in (1):

$$(4) \quad -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega$$

Let us recall that in the mixed Galerkin method (see, e.g., [14], [15], and [17]) a weak form of (1)-(3), given by: seek $\mathbf{u} \in [H_0^1(\Omega)]^n$ and $p \in L_0^2(\Omega)$ such that

$$\begin{aligned} (5) \quad & \int_{\Omega} \text{grad } \mathbf{u} \cdot \text{grad } \mathbf{v} \, d\Omega - \int_{\Omega} p \, \text{div } \mathbf{v} \, d\Omega + \int_{\Omega} \mathbf{u} \cdot \text{grad } \mathbf{u} \cdot \mathbf{v} \, d\Omega = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\Omega \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^n \\ (6) \quad & \int_{\Omega} q \, \text{div } \mathbf{u} \, d\Omega = 0 \quad \forall q \in L_0^2(\Omega) \end{aligned}$$

is discretized using a pair of finite element spaces (V^h, P^h) for the approximation of \mathbf{u} and p , respectively. This yields a nonlinear system of algebraic equations of the form

$$(7) \quad \begin{bmatrix} A + G(Re, \mathbf{u}^h) & B \\ B^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^h \\ p^h \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}.$$

The use of the weak problem (5)-(6) gives rise to several complications in the algorithmic design of methods for (1)-(3). These complications are primarily caused by the saddle-point optimization nature of (5)-(6) and its discrete counterpart (7). The first difficulty arises at the discretization level. It is now well-known that the pair (V^h, P^h) must satisfy the LBB condition; see [1], if stable and optimally accurate approximations are desired. Let us note that conformity of the pair (V^h, P^h) does not by itself guarantee that the LBB condition is satisfied. For instance, conforming pairs such as $(P_1 - P_0)$ and $(Q_1 - Q_0)$ are unstable; see [15]. The second difficulty arises at the solution level. The main obstacle here is again the saddle-point character of (7). Furthermore, if one uses, e.g., Newton linearization to solve (7) then, as the Reynolds number increases, the block $A + D_{\mathbf{u}}G(Re, \mathbf{u}_{n-1}^h)$ becomes both highly nonsymmetric and indefinite. Despite the significant progress made in the recent years, such problems are still hard to solve.

The adverse effects of the LBB condition on the algorithmic design have prompted considerable efforts aimed at circumventing this condition. One approach is to retain the weak problem (5)-(6) and to introduce

*Department of Mathematics, Box 19408, University of Texas at Arlington, Arlington, TX 76019-0408, (bochev@utam.uta.edu). This work was sponsored by the National Science Foundation under grant number DMS-9705793.

stabilizing terms by modifying (6) or (2). Such are, for example, *penalty*, *stabilized Galerkin* and *Galerkin-least-squares* methods. An example of a *penalty* procedure is given by the method of [2] where (6) is modified according to

$$\int_{\Omega} q^h \nabla \cdot \mathbf{u}^h d\Omega - \sum_{\Delta} h^2 \int_{\Delta} \nabla p^h \cdot \nabla q^h d\Omega = 0.$$

This method introduces a penalty error of $O(h)$ independent of the choice of the discretization space. Note that the penalty term may also be interpreted as adding $h^2 \Delta p$ to (2). An example of a *Galerkin-least-squares* procedure is provided by the method of [16] in which (2) is modified according to

$$\int_{\Omega} q^h \nabla \cdot \mathbf{u}^h d\Omega - \sum_{\Delta} h^2 \int_{\Delta} (-\Delta \mathbf{u}^h + \nabla p^h - \mathbf{f}) \cdot \nabla q^h d\Omega = 0.$$

Note that this is not a *penalty* formulation since the additional term vanishes on the solution, i.e., it does not cause loss of accuracy as in penalty methods.

In this paper we shall be concerned with a different approach which circumvents the LBB condition with the help of least-squares variational principles. In contrast to (5)-(6) weak problems are now defined as necessary conditions for a problem dependent least-squares functional. As a result, the associated discrete problems are in general symmetric and positive definite, at least in a neighborhood of the exact solution.

2. A least-squares framework for the Navier-Stokes equations. Unlike the mixed Galerkin method, application of least-squares principles to (1)-(2) can lead to substantially different methods. The specific outcome depends on several factors among which the most important are

- the possibility to use various equivalent forms of the Navier-Stokes equations;
- the choice of norms in the least-squares functional;
- the choice of discretization spaces;
- the treatment of essential boundary conditions, e.g., using interpolants or boundary residuals.

This abundance of choices has to be carefully examined in order to obtain methods that are both optimally accurate and practical. Nevertheless, as we demonstrate below, it is possible to consolidate analysis and development of a wide range of least-squares methods for (1)-(2) into an unified abstract framework based upon application of the implicit function theorem. The origins of this framework can be found in the abstract nonlinear theory of Brezzi, Rappaz and Raviart; see [14].

Below we present a necessarily brief summary of the least-squares framework for (1)-(2) and then proceed with several examples that illustrate specific applications of this framework. In what follows we use \mathcal{L} and \mathcal{R} to denote a Stokes operator corresponding to a particular form of (1)-(2) and an admissible boundary operator, respectively. The set of dependent variables associated with the particular form of the Navier-Stokes equations is denoted by U . We express these equations symbolically as

$$(8) \quad \begin{aligned} \mathcal{L}(U) + \mathcal{N}(\lambda, U) &= F \quad \text{in } \Omega; \\ \mathcal{R}(U) &= G \quad \text{on } \Gamma, \end{aligned}$$

where $\mathcal{N}(\lambda, U)$ accounts for the nonlinear term and $\lambda \equiv Re$. In (8) it is tacitly assumed that momentum equation is scaled by Re . Development of least-squares methods for (8) will be accomplished in two stages. At the first stage a least-squares principle is used to introduce a nonlinear variational problem posed in some functional spaces. At the second stage this, or a related problem, is discretized resulting in a nonlinear system of algebraic equations. Concerning the problem (8) we assume that

H.1 there exist two Hilbert scales \mathbf{X}_q and $\mathbf{Y}_q \times \mathbf{Y}_{q\Gamma}$ for the solution and the data, respectively such that the associated Stokes problem is well-posed;

H.2 the Navier-Stokes equations (8) have a nonsingular branch of solutions $\{\lambda, U(\lambda)\}$, such that for some fixed q , $U(\lambda) \in \mathbf{X}_q$ for all $\lambda \in \Lambda$. Here Λ denotes a compact interval in \mathcal{R} .

The first hypothesis amounts to the validity of an a priori estimate given by

$$(9) \quad \|U\|_{\mathbf{X}_q} \leq C \left(\|\mathcal{L}(U)\|_{\mathbf{Y}_q} + \|\mathcal{R}(U)\|_{\mathbf{Y}_{q\Gamma}} \right),$$

while the second hypothesis implies that problem (8), linearized at $U(\lambda)$, has a unique solution for any $\lambda \in \Lambda$. Each hypothesis has critical importance for the development of the least-squares method. From (9) it follows that a *norm equivalent* functional for (8) is given by

$$(10) \quad J_q(U; F, G) = \frac{1}{2} \left(\|\mathcal{L}(U) + \mathcal{N}(\lambda, U) - F\|_{\mathbf{Y}_q}^2 + \|\mathcal{R}(U) - G\|_{\mathbf{Y}_{q\Gamma}}^2 \right).$$

The least-squares principle for (8) then is to seek a minimizer for (10) out of \mathbf{X}_q . Standard techniques from calculus of variations can be used to show that the Euler-Lagrange equation for (10) is given by the nonlinear variational problem: *seek* $U \in \mathbf{X}_q$ *such that*

$$(11) \quad \left(\mathcal{L}(U) + \mathcal{N}(\lambda, U) - F, \mathcal{L}(V) + \mathcal{N}'(\lambda, U) \cdot V \right)_{\mathbf{Y}_q} + \left(\mathcal{R}(U) - G, \mathcal{R}(V) \right)_{\mathbf{Y}_{q\Gamma}} = 0, \quad \forall V \in \mathbf{X}_q,$$

where $\mathcal{N}'(\lambda, U)$ denotes the derivative of \mathcal{N} with respect to the second argument. If

$$\begin{aligned} B_S(U, V) &= (\mathcal{L}(U), \mathcal{L}(V))_{\mathbf{Y}_q} + (\mathcal{R}(U), \mathcal{R}(V))_{\mathbf{Y}_{q\Gamma}} \\ B_G(U, V) &= (\mathcal{N}(\lambda, U), \mathcal{L}(V))_{\mathbf{Y}_q} + (\mathcal{L}(U) + \mathcal{N}(\lambda, U), \mathcal{N}'(\lambda, U) \cdot V)_{\mathbf{Y}_q} \\ Q(U; V) &= B_S(U, V) + B_G(U, V) \\ \mathcal{F}(V) &= (F, \mathcal{L}(V) + \mathcal{N}'(\lambda, U) \cdot V)_{\mathbf{Y}_q} + (G, \mathcal{R}(V))_{\mathbf{Y}_{q\Gamma}}, \end{aligned}$$

then (11) can be written compactly as: $Q(U; V) = \mathcal{F}(V)$, $\forall V \in \mathbf{X}_q$. Let us show that this problem is equivalent to an abstract nonlinear equation of the form

$$(12) \quad \text{seek } U : \Lambda \times \mathbf{X} \mapsto \mathbf{X} \text{ such that } F(\lambda, U) \equiv U + T \cdot G(\lambda, U) = 0 \quad \forall \lambda \in \Lambda,$$

where $\Lambda \subset \mathcal{R}$ is compact interval, \mathbf{X} and \mathbf{Y} are Banach spaces, $T : \mathbf{Y} \mapsto \mathbf{X}$ is a linear operator in $L(\mathbf{Y}, \mathbf{X})$, and $G : \Lambda \times \mathbf{X} \mapsto \mathbf{Y}$ is a C^2 mapping. For this purpose we let \mathbf{X} and \mathbf{Y} be the two Hilbert scales from H.1, and identify T with a least-squares solution operator for \mathcal{L} , i.e.,

$$T \cdot U = \mathbf{g} \in \mathbf{X}_q^* \text{ if and only if } B_S(U, V) = (\mathbf{g}, U)_{\mathbf{X}_q} \text{ for all } V \in \mathbf{X}_q.$$

Similarly, the operator G is identified with the nonlinear term in (11), i.e.,

$$G(\lambda, U) = \mathbf{g} \text{ if and only if } B_G(U, V) = (\mathbf{g}, V) \text{ for all } V \in \mathbf{X}_q.$$

With these identifications it is not difficult to see that (11) is indeed equivalent to (12). The two hypotheses H.1 and H.2 are critical for the well-posedness of (11). Thanks to (9) (which is a consequence of H.1) it follows that $B_S(\cdot, \cdot)$ is coercive on $\mathbf{X}_q \times \mathbf{X}_q$ and, as a result, T is a well-defined operator. The second hypothesis, on the other hand, guarantees that if $U(\lambda)$ is a nonsingular solution of (8), then the Fréchet derivative $D_U F(\lambda, U(\lambda))$ is an isomorphism of \mathbf{X}_q .

At this point we have arrived at a well-posed nonlinear variational equation. The next step is to use this equation to define a discrete problem which can actually be solved on a computer. The most straightforward way, of course, is to consider conforming discretization of (11). This is also the most convenient way from the point of view of the analysis of resulting methods. However, sometimes it may lead to an impractical method. In such a case it is necessary to consider other options in which (11) or (10) are not used directly in the discretization process but rather serve as templates for the discrete problem. Typically this involves replacing the primary least-squares functional (10) by a discrete counterpart which retains most of the properties of (10) when restricted to finite element spaces. Then the discrete problem is defined as a necessary condition for this new functional. The specific details of this process can vary significantly from case to case and it is much more instructive to look at concrete examples of finite element methods. Such examples will be discussed in the next section, while in the remainder of this section we concentrate on a framework only for conforming discretizations.

To define a conforming method for (8) let us fix the scale parameter q and introduce a finite dimensional subspace \mathbf{X}_q^h of \mathbf{X}_q . Then, a discrete problem is obtained in the usual manner by restricting (11) to the space \mathbf{X}_q^h , i.e., it is given by: *seek* $U^h \in \mathbf{X}_q^h$ *such that*

$$(13) \quad Q(U^h; V^h) = \mathcal{F}(V^h), \quad \forall V^h \in \mathbf{X}_q^h.$$

The structure of the discrete problem (13) is very similar to that of (11). Indeed, let T^h denote an approximation of T defined by

$$T^h \cdot U^h = \mathbf{g} \in \mathbf{X}_q^* \text{ if and only if } B_S(U^h, V^h) = (\mathbf{g}, U^h)_{\mathbf{X}_q} \text{ for all } V^h \in \mathbf{X}_q^h.$$

Then, it is easy to see that (13) is equivalent to

$$(14) \quad F^h(\lambda, U^h) \equiv U^h + T^h \cdot G(\lambda, U^h) = \mathbf{0}.$$

There are two issues that must be addressed at this point. First, one must show that (13) is a well-posed problem and that U^h is close to the solution of (8). Second, (13) is a nonlinear system of algebraic equations that must be solved iteratively. In both cases, hypotheses H.1 and H.2 are again of critical importance.

Consider first the well-posedness of (13) and the error estimates. Since $\mathbf{X}_q^h \subset \mathbf{X}_q$ form $B_S(\cdot, \cdot)$ is coercive on $\mathbf{X}_q^h \times \mathbf{X}_q^h$ and, therefore, T^h is well-defined. Under appropriate regularity assumptions one can show that

$$(15) \quad \lim_{h \rightarrow 0} \|(T - T^h)\mathbf{g}\|_{\mathbf{X}_q} = 0.$$

This, in combination with H.2 can be used to show that for h small enough $D_U F^h(\lambda, U(\lambda))$ is an isomorphism of \mathbf{X}_q^h . Now, existence and uniqueness of solutions to (13) follows by virtue of the implicit function theorem. Furthermore, if (15) is valid, one can show that the discretization error of the nonlinear problem (13) is of the same order as the error in the approximation of T by T^h .

Lastly, consider solution of the nonlinear system (13), using, e.g., Newton linearization. This method generates the sequence $\{U_k^h\}$ of Newton iterates by solving the sequence of linear problems

$$(16) \quad (I + T^h \cdot D_U G(\lambda, U_{k-1}^h)) \cdot \Delta U_k^h = -(U_{k-1}^h + T^h \cdot G(\lambda, U_{k-1}^h)).$$

Note that the coefficient matrix $(I + T^h \cdot D_U G(\lambda, U_{k-1}^h))$ in (16) is exactly the Hessian matrix of the least-squares functional (10) and that

$$D_U F(\lambda, U_{k-1}^h) = (I + T^h \cdot D_U G(\lambda, U_{k-1}^h)).$$

Since $D_U F^h(\lambda, U(\lambda))$ is also an isomorphism of \mathbf{X}_q^h it follows that in a neighborhood of the minimizer this matrix is necessarily symmetric and positive definite. As a result, the attraction ball for the Newton's method is non-trivial, and the linear system (16) can be solved by robust and efficient iterative methods.

3. Least-squares methods for the Navier-Stokes equations. In this section we consider several concrete examples of least-squares methods for (1)-(2). All these methods share a common trait: the first stage of their development is based upon a least-squares principle for a *norm-equivalent* functional. Although formally this would be sufficient to ensure that a conforming discretization yields an optimally accurate method, the main question at the second stage becomes whether or not such method is also practical. To deem a method as being practical several criteria must be met:

- the discrete system (13) should not be more difficult to obtain than, e.g., the system (7) in the mixed method;
- discretization should be accomplished using standard, easy to use finite element spaces;
- the discrete problem should have reasonable condition number.

To satisfy the first condition all inner products appearing in $Q(\cdot; \cdot)$ must be computable. This is not the case when $Q(\cdot; \cdot)$ involves, e.g., inner products in fractional or negative order Sobolev spaces. Similarly, if $Q(\cdot; \cdot)$ involves second and/or higher order derivatives, then conforming discretization would require at least C^1 finite element spaces which violates the second condition., Methods considered in this section were chosen primarily because they provide examples that illustrate how these problems arise and how they can be resolved. All these methods use equivalent *first-order* formulations of the Navier-Stokes equations which are derived by introducing new dependent variables in (1)-(2).

3.1. Velocity-vorticity-pressure formulation. Using the vorticity $\boldsymbol{\omega} = \mathbf{curl} \mathbf{u}$ as new dependent variable one obtains the first-order *velocity-vorticity-pressure* formulation of the Navier-Stokes equations

$$(17) \quad \mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \mathbf{grad} p = \mathbf{f} \quad \text{in } \Omega$$

$$(18) \quad \mathbf{curl} \mathbf{u} - \boldsymbol{\omega} = 0 \quad \text{in } \Omega$$

$$(19) \quad \mathbf{div} \mathbf{u} = 0 \quad \text{in } \Omega.$$

An admissible boundary operator for this system will be denoted by $\mathcal{R}(\boldsymbol{\omega}, p, \mathbf{u})$. Here we have that $U = (\boldsymbol{\omega}, p, \mathbf{u})$ and \mathcal{L} corresponds to the velocity-vorticity-pressure Stokes operator. An interesting feature of this first order system is that there are two different combinations of Hilbert scales in which the boundary value problem for the Stokes operator is well-posed; see [4]. For some nonstandard boundary conditions, such as pressure-normal velocity, or vorticity-tangential velocity, the relevant Hilbert scales are

$$(20) \quad \mathbf{X}_q = \mathbf{H}^{q+1}(\Omega) \times H^{q+1}(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{q+1}(\Omega)$$

and

$$(21) \quad \mathbf{Y}_q = \mathbf{H}^q(\Omega) \times H^q(\Omega) \times \mathbf{H}^q(\Omega)$$

respectively, while for the velocity boundary condition (3) they are

$$(22) \quad \mathbf{X}_q = \mathbf{H}^{q+1}(\Omega) \times H^{q+1}(\Omega) \cap L_0^2(\Omega) \times \mathbf{H}^{q+2}(\Omega) \cap \mathbf{H}_0^1(\Omega)$$

and

$$(23) \quad \mathbf{Y}_q = \mathbf{H}^q(\Omega) \times H^q(\Omega) \times \mathbf{H}^{q+1}(\Omega),$$

respectively. A specific choice of the scale parameter q now yields a particular form of the a priori estimate (9). Setting, for example, $q = 0$ in (20)-(21) leads to the estimate

$$(24) \quad \|\boldsymbol{\omega}\|_1 + \|p\|_1 + \|\mathbf{u}\|_1 \leq C (\|\mathbf{curl} \boldsymbol{\omega} + \text{grad } p\|_0 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0 + \|\text{div} \mathbf{u}\|_0),$$

which is valid for nonstandard boundary conditions. Operators which are well-posed in such spaces are called H^1 -coercive since the right hand side in (24) defines an equivalent norm on a product of H^1 spaces. Similarly, setting $q = 0$ in (22)-(23) produces an a priori estimate for the velocity boundary condition:

$$(25) \quad \|\boldsymbol{\omega}\|_1 + \|p\|_1 + \|\mathbf{u}\|_2 \leq C (\|\mathbf{curl} \boldsymbol{\omega} + \text{grad } p\|_0 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_1 + \|\text{div} \mathbf{u}\|_1).$$

Let us now consider possible ways to introduce a least-squares functional for the first-order system (17)-(19). For simplicity, assume that the essential boundary conditions $\mathcal{R}(\boldsymbol{\omega}, p, \mathbf{u})$ are satisfied exactly by the approximating spaces. (The case when finite element spaces are not constrained by the boundary conditions will be considered later). Then a least-squares functional for (17)-(19) can be defined in a straightforward manner by summing up L^2 -norms of the equation residuals

$$(26) \quad J(\boldsymbol{\omega}, p, \mathbf{u}) = \frac{1}{2} (\|\mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \text{grad } p - \mathbf{f}\|_0^2 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\text{div} \mathbf{u}\|_0^2).$$

The nonlinear variational problem (11) associated with (26) involves only first-order terms and can be discretized using standard C^0 finite element spaces, i.e., a method based on (26) is practical. Such methods were first considered in [12]-[13], where piecewise linear finite element spaces were used for all variables. However, (24) implies that (26) is norm-equivalent only for nonstandard boundary conditions, i.e., the framework of §2 is not applicable when (26) is considered with the velocity boundary condition (3). The lack of norm-equivalence is not merely a technical inconvenience. There are computational examples which show that (26) actually yields suboptimal convergence rates; see [8] and [11]. According to (25), a norm-equivalent functional for the velocity boundary condition is given by

$$(27) \quad J(\boldsymbol{\omega}, p, \mathbf{u}) = \frac{1}{2} (\|\mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \text{grad } p - \mathbf{f}\|_0^2 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_1^2 + \|\text{div} \mathbf{u}\|_1^2).$$

Because (27) uses H^1 -norms for the residuals of (18) and (19), the form $Q(\cdot; \cdot)$ in (11) now effectively contains second order terms despite the fact that (17)-(19) is a first-order system. Consequently, conforming discretization of (11) would require the use of C^1 finite element spaces, i.e., the method would be impractical.

One possibility to circumvent this inconvenience is to replace the H^1 -norms in (27) by weighted L^2 norms. This approach has been considered in [3] and [8]. The weighted counterpart of (27) is

$$(28) \quad J(\boldsymbol{\omega}, p, \mathbf{u}) = \frac{1}{2} (\|\mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \text{grad } p - \mathbf{f}\|_0^2 + h^{-2} \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0^2 + h^{-2} \|\text{div} \mathbf{u}\|_0^2).$$

Now it is possible to use standard C^0 finite element spaces since (28) and problem (13) contain at most first-order terms. The use of this functional, however, introduces several complications into the abstract framework of §2. Indeed, at the second, discretization, stage we arrive at a problem which is not a conforming discretization of the Euler-Lagrange equation (11) for the original, norm-equivalent functional (27). As a result, it is more difficult to show that (13) is well-posed and has a unique solution. Nevertheless, it is possible to show that a method based on (28) is optimal provided \mathbf{u} is approximated by finite element spaces of one order higher than the spaces used for $\boldsymbol{\omega}$ and p ; see [3] and [8].

Let us now consider another possibility to avoid the second order terms in (27). The idea is to choose a weaker combination of spaces \mathbf{X}_q and \mathbf{Y}_q for the first-order boundary value problem. If we set $q = -1$ in (22)-(23) then the relevant a priori estimate becomes

$$(29) \quad \|\boldsymbol{\omega}\|_0 + \|p\|_0 + \|\mathbf{u}\|_0 \leq C (\|\nu \mathbf{curl} \boldsymbol{\omega} + \text{grad } p\|_{-1} + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0 + \|\text{div } \mathbf{u}\|_0) .$$

As a result,

$$(30) \quad J(\boldsymbol{\omega}, p, \mathbf{u}) = \frac{1}{2} (\|\mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \text{grad } p - \mathbf{f}\|_{-1}^2 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\text{div } \mathbf{u}\|_0^2) .$$

is another norm-equivalent functional for (17)-(19) with the boundary condition (3). Like in (25) we see that velocity and vorticity cannot have the same differentiability, i.e., these variables remain coupled. However, the form $Q(\cdot; \cdot)$ will not contain any second order terms, i.e., a conforming discretization of (11) is formally possible by means of standard C^0 finite element spaces. Still, functional (30) is not any more practical than (27) because now $Q(\cdot; \cdot)$ involves H^{-1} inner products which are not computable. The key to a practical method, suggested in [9], is to observe that

$$\|f\|_{-1}^2 = ((-\Delta)^{-1} f, f)_0 .$$

As a result, if B^h is a computable approximation of $(-\Delta)^{-1}$, then

$$\|f\|_{-h}^2 = ((h^2 I + B^h) f, f)_0$$

defines a computable discrete equivalent of $\|f\|_{-1}$. Using this norm we can replace (30) by the discrete negative norm functional

$$(31) \quad J(\boldsymbol{\omega}, p, \mathbf{u}) = \frac{1}{2} (\|\mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \text{grad } p - \mathbf{f}\|_{-h}^2 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\text{div } \mathbf{u}\|_0^2) .$$

When (31) is restricted to finite element spaces it exhibits norm-equivalence properties similar to those of the primary functional (30). The Euler-Lagrange equation for (31) now furnishes the desired discrete problem.

As in the case with (27) and (28) this gives rise to a discrete problem which is not a conforming discretization of the Euler-Lagrange equation (11) associated with the original functional (30). The corresponding method has been studied in [5]. One of the major complications in the analysis now stems from the fact that, unlike the conforming case, the use of (31) also introduces an approximation in the nonlinear term. More precisely, a conforming discretization of the Euler-Lagrange equation of the primary functional (30) has the canonical form (12), while the abstract form of the Euler-Lagrange equation for (31) is

$$(32) \quad F_{-h}(\lambda, U^h) \equiv U^h + T_{-h} \cdot G_{-h}(\lambda, U^h) = \mathbf{0} .$$

This makes it more difficult to show that $D_U F_{-h}$ is an isomorphism in a neighborhood of the solution. Although both analysis and implementation of the negative norm method are more complicated than that of (28), this method offers some very significant advantages. First, conditioning of the discrete system is better than in the weighted method. Second, efficient preconditioners for the discrete system can be defined in a natural way; see [9] and [5]. Third, error estimates for the negative norm method can be established under more relaxed regularity assumptions than for the weighted method; see [5].

One of the important advantages of least-squares principles is that they allow one to consider finite element spaces that are not constrained by the essential boundary conditions. There are examples of similar Galerkin methods where boundary conditions are enforced in a weak, variational sense using Lagrange

multipliers. In the context of least-squares methods this can be easily accomplished by including boundary residuals in the least-squares functional. For example,

$$(33) \quad J(\boldsymbol{\omega}, p, \mathbf{u}) = \frac{1}{2} \left(\|\mathbf{curl} \boldsymbol{\omega} + \lambda \boldsymbol{\omega} \times \mathbf{u} + \text{grad } p - \mathbf{f}\|_{-1}^2 + \|\mathbf{curl} \mathbf{u} - \boldsymbol{\omega}\|_0^2 + \|\text{div} \mathbf{u}\|_0^2 + \|\mathbf{u}\|_{1/2, \Gamma}^2 \right).$$

is a norm-equivalent functional which includes a residual of a homogeneous velocity boundary condition. Consequently, it is not necessary to constrain the approximation space by this condition since it is enforced weakly in the minimization process. Boundary residuals usually require the use of norms and inner products in fractional order Sobolev spaces. These are not easily computable and before a practical method can be defined one must choose a way to replace such norms and inner products by computable discrete equivalents. One standard choice is to use weighted L^2 norms.

3.2. Velocity-flux formulations. In this section we consider least-squares methods based upon a different first-order form of the Navier-Stokes equations. The objective is to derive a system which is H^1 -coercive with the velocity boundary condition (3). Then it is possible to define least-squares methods that do not require weights or negative norms in the functional. The system in question was first introduced in [10] for the Stokes equations and then extended to the Navier-Stokes equations in [6]. To derive this first-order system we choose as a new dependent variable the velocity gradient tensor (called *velocity flux* in [10])

$$\underline{\mathbf{U}} = (\text{grad } \mathbf{u})^t.$$

In terms of $\underline{\mathbf{U}}$, the Navier-Stokes problem (1)-(3) is given by (see [6])

$$(34) \quad -\text{div } \underline{\mathbf{U}} + \lambda \underline{\mathbf{U}}^t \mathbf{u} + \text{grad } p = \mathbf{f} \quad \text{in } \Omega$$

$$(35) \quad \text{div } \mathbf{u} = 0 \quad \text{in } \Omega$$

$$(36) \quad \underline{\mathbf{U}} - (\text{grad } \mathbf{u})^t = \mathbf{0} \quad \text{in } \Omega$$

and (3). The *velocity flux* system (34)-(36) is well-posed in Hilbert scales very similar to (22)-(23), i.e., it is not yet H^1 -coercive and the relevant a priori estimate is

$$(37) \quad \|\underline{\mathbf{U}}\|_0 + \|\mathbf{u}\|_1 + \|p\|_0 \leq C \left(\|-\text{div } \underline{\mathbf{U}} + \text{grad } p\|_0 + \|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|_0 + \|\text{div } \mathbf{u}\|_0 \right).$$

The main idea of [10] is that H^1 -coercive system can be obtained by augmenting (34)-(36) by additional constraints. In particular, in view of the definition of the new variable $\underline{\mathbf{U}}$ and the incompressibility constraint (35) we have that

$$(38) \quad \text{grad}(tr \underline{\mathbf{U}}) = \mathbf{0} \quad \text{in } \Omega$$

$$(39) \quad \mathbf{curl} \underline{\mathbf{U}} = \mathbf{0} \quad \text{in } \Omega,$$

and

$$(40) \quad \underline{\mathbf{U}} \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma.$$

Although the resulting system (34)-(36), (38), (39), (3) and (40) is overdetermined, it is consistent and H^1 -coercive. The relevant a priori estimate for the augmented velocity flux Stokes operator is

$$(41) \quad \|\underline{\mathbf{U}}\|_1 + \|\mathbf{u}\|_1 + \|p\|_1 \leq C \left(\|-\text{div } \underline{\mathbf{U}} + \text{grad } p\|_0 + \|\text{div } \mathbf{u}\|_0 + \|\underline{\mathbf{U}} - (\text{grad } \mathbf{u})^t\|_0 + \|\text{grad}(tr \underline{\mathbf{U}})\|_0 + \|\mathbf{curl} \underline{\mathbf{U}}\|_0 \right).$$

As a result,

$$(42) \quad J(\underline{\mathbf{U}}, \mathbf{u}, p) = \frac{1}{2} \left(\|-\text{div } \underline{\mathbf{U}} + \lambda \underline{\mathbf{U}}^t \mathbf{u} + \text{grad } p - \mathbf{f}\|_0^2 + \|\text{div } \mathbf{u}\|_0^2 + \|\underline{\mathbf{U}} - (\text{grad } \mathbf{u})^t\|_0^2 + \|\text{grad}(tr \underline{\mathbf{U}})\|_0^2 + \|\mathbf{curl} \underline{\mathbf{U}}\|_0^2 \right)$$

is a norm-equivalent functional. The variational problem (11) for (42) involves at most first-order terms and can be discretized using standard finite element spaces. Because (13) is defined by conforming discretization, analysis of the resulting method follows the framework outlined in §2; see [6].

We note that instead of using the augmented system one could also define a least-squares method using the system (34)-(36). Of course, since the former system is not H^1 -coercive it is necessary to use either a weighted functional similar to (28), or a negative norm functional similar to (31). A discrete negative norm method for the velocity flux system has been considered in [7]. Such method is of interest primarily because it allows to establish optimal discretization error estimates assuming less regularity of the exact solutions than that required for the H^1 -coercive formulation.

REFERENCES

- [1] F. Brezzi, On existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *RAIRO Model. Math. Anal. Numer.*, **21**, 1974, pp. 129-151.
- [2] F. Brezzi and J. Pitkaranta, On the stabilization of finite element approximations of the Stokes problem, in *Efficient Solutions of elliptic systems*, ed. by W. Hackbush, Vieweg, Braunschweig, pp. 11-19, 1984.
- [3] P. Bochev; *Least-squares finite element methods for the Stokes and Navier-Stokes equations*, Ph.D. Thesis, VPI&SU, Blacksburg, VA, 1994, 161 pages.
- [4] P. Bochev; Analysis of least-squares finite element method for the Navier-Stokes equations, *SIAM J. Num. Anal.* 34/5, pp.1817-1844, 1997
- [5] P. Bochev; Negative norm least-squares methods for the velocity-vorticity-pressure Navier-Stokes equations, to appear in *Numerical Methods for PDE's*.
- [6] P. Bochev, Z. Cai, T.A. Manteuffel and S.F. McCormick; Analysis of velocity-flux least squares methods for the Navier-Stokes equations: Part I, to appear in *SIAM J. Numer. Anal.*.
- [7] P. Bochev, T.A. Manteuffel and S.F. McCormick; Analysis of velocity-flux least squares methods for the Navier-Stokes equations: Part II, submitted to *SIAM J. Numer. Anal.*.
- [8] P. Bochev and M. Gunzburger; Analysis of least-squares methods for the Stokes equations, *Mathematics of Computation*, 63/108,1994, pp. 479-506.
- [9] J. H. Bramble, R. D. Lazarov and J. E. Pasciak; A least squares approach based on a discrete minus one inner product for first order systems, Technical Report 94-32, Mathematical Science Institute, Cornell University, 1994.
- [10] Z.Cai, T.A. Manteuffel, and S. F. McCormick, First-order system least-squares for the Stokes equations, with application to linear elasticity, to appear in *SIAM Numer. Anal.*.
- [11] J. Deang and M. D. Gunzburger, Issues related to least squares finite element methods for the Stokes equations, submitted.
- [12] B.-N. Jiang and C. Chang; Least-squares finite elements for the Stokes problem, *Comput. Meth. Appl. Mech. Engrg.* **78**, 1990, pp. 297-311.
- [13] B.-N. Jiang, T. Lin, and L. Pavinelli; A least-squares finite element method for 3D incompressible Navier-Stokes equations, AIAA Report 93-0338, 1993.
- [14] V. Girault and P.-A. Raviart; *Finite Element Methods for Navier-Stokes Equations*, Springer, Berlin, 1986.
- [15] M. Gunzburger; *Finite Element Methods for Viscous Incompressible Flows*, Academic, Boston, 1989.
- [16] T. Hughes, L. Franca, and M. Ballestra, *A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations*, *Comput. Meths. Appl. Mech. Engrg.*, 59 (1986), pp. 85-99.
- [17] R. Temam; *Navier-Stokes equations, Theory and numerical analysis*, North-Holland, 1977.