ANALYSIS OF VELOCITY-FLUX LEAST-SQUARES PRINCIPLES FOR THE NAVIER-STOKES EQUATIONS: PART II

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Abstract. This paper continues the development of the least-squares methodology for the solution of the incompressible Navier-Stokes equations started in Part I. Here we again use a velocity-flux first-order Navier-Stokes system, but our focus now is on a practical algorithm based on a discrete negative norm.

Key words. Navier-Stokes equations, least-squares principle, negative norm, finite element methods.

AMS subject classifications. 76D05, 76D07, 65F10, 65F30

1. Introduction. In this paper, we consider the reduced velocity-flux Navier-Stokes equations given by

$$-\nu(\nabla^t \underline{\mathbf{U}})^t + \underline{\mathbf{U}}^t \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1.1}$$

$$\nabla^t \mathbf{u} = 0 \quad \text{in } \Omega \tag{1.2}$$

$$\underline{\mathbf{U}} - \nabla \mathbf{u}^t = \underline{\mathbf{0}} \quad \text{in } \Omega \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma\,. \tag{1.4}$$

We recall that the *velocity flux* variable $\underline{\mathbf{U}}$ is defined as

$$\underline{\mathbf{U}} = \nabla \mathbf{u}^t\,,$$

that is, $\underline{\mathbf{U}}$ is a matrix with entries $U_{ij} = \partial u_j / \partial x_i$, $1 \le i, j \le n$. In [2], referred to hereafter as Part I, we studied an abstract least-squares functional for (1.2)-(1.4) of the form

$$\begin{aligned}\mathcal{J}_{-1}(\underline{\mathbf{U}}, \mathbf{u}, p) = {} & |-(\nabla^t \underline{\mathbf{U}})^t + \nabla p + \frac{1}{\nu}\left(\underline{\mathbf{U}}^t \mathbf{u} - \mathbf{f}\right)|_{-1}^2 \\ & + \|\nabla^t \mathbf{u}\|_0^2 + \|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|_0^2\,,\end{aligned} \tag{1.5}$$

where the negative seminorm $|\cdot|_{-1}$ is defined by

$$|f|_{-1} = \sup_{\phi \in H_0^1(\Omega)} \frac{(f, \phi)}{|\phi|_1}, \quad \forall f \in H^{-1}(\Omega)\,. \tag{1.6}$$

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We recall that $|f|_1 = (Sf, f)$, where S denotes the solution operator for the Poisson equation with homogeneous Dirichlet boundary condition; see [2], [3]. Since the *exact* evaluation of this norm is not computationally feasible, direct minimization of (1.5) is not practical. However, theoretical results from Part I will be used here as a vehicle for establishing optimal error estimates for a practical counterpart of (1.5) based on a computable discrete negative norm. Such negative norms have been first proposed in [3]-[4]. In [4] and [5], discrete negative norms were used to develop least-squares methods for a perturbed form of the velocity-vorticity-pressure Stokes equations, which include as a particular case the equations of linear elasticity. A negative norm least-squares method for the Stokes problem in primitive variables was also considered in [4].

2. Summary of results. For convenience, in this section we collect several theoretical and technical results that are relevant to our analysis. Here and henceforth, C is a generic constant that may change meaning with each occurrence, but is independent of the grid parameter h. In some instances, C will depend on a parameter λ that belongs to a compact set $\Lambda \subset \mathbf{R}^+$. In such cases we assume that this dependence is uniform in λ. Throughout this paper, X_d^h with $d \geq 0$ will denote a generic finite element space defined with respect to a uniformly regular triangulation $\mathcal{T}^h$ of Ω. It is assumed that X_d^h has the usual approximation property: for each $u \in H^{d+1}(\Omega)$, there exists $u^h \in X_d^h$ such that

$$\|u - u^h\|_0 + h|u - u^h|_1 \leq Ch^{d+1}\|u\|_{d+1} \,. \tag{2.1}$$

This property, for example, is valid for piecewise polynomial finite element spaces of degree $d \geq 1$; see [8]. Furthermore, we assume that the inverse inequality

$$\|u^h\|_1 \leq Ch^{-1}\|u^h\|_0 \tag{2.2}$$

holds for X_d^h.

2.1. Discrete negative norm. In this subsection, we introduce two discrete norms related to (1.6) and establish certain discrete equivalence properties. Here we follow the development in [3].

Consider the discrete Dirichlet problem

find $u^h \in X_d^h \cap H_0^1(\Omega)$ *such that*

$$\int_\Omega \nabla u^h \cdot \nabla v^h dx = \int_\Omega f v^h dx, \quad \forall v^h \in X_d^h \cap H_0^1(\Omega) \,. \tag{2.3}$$

We let S^h denote the solution operator of (2.3), that is, $S^h f = u^h \in X_d^h \cap H_0^1(\Omega)$ for $f \in H^{-1}(\Omega)$ if and only if u^h solves (2.3). Using S^h, we introduce the discrete negative norm

$$|f|_{-1,h} = (S^h f, f)^{1/2} \,. \tag{2.4}$$

Note that

$$|f|_{-1,h} = \sup_{\phi^h \in X_d^h \cap H_0^1(\Omega)} \frac{(f, \phi^h)}{|\phi^h|_1}, \quad \forall f \in H^{-1}(\Omega) \,, \tag{2.5}$$

that is, norm (2.4) is simply the restriction of the definition of the negative norm (1.6) to X_d^h. This discrete norm is still infeasible in general because of the need to solve (2.3) on the same (fine) grid as the primary problem. Following [3], we consider instead a computable discrete negative norm based on a preconditioner for S^h, that is, a symmetric linear operator $B^h : X_d^h \mapsto X_d^h$ that is spectrally equivalent to S^h in the sense that

$$\frac{1}{C}(S^h v^h, v^h) \leq (B^h v^h, v^h) \leq C(S^h v^h, v^h), \quad \forall v^h \in X_d^h\,. \tag{2.6}$$

For example, B^h may represent one or two symmetric multigrid V-cycles applied to variational problem (2.3) or to optimization problem (2.5). In contrast to the spectral equivalence (2.6), the symmetry of B^h is not essential, but it does simplify the analysis. In order to extend B^h to all of $L^2(\Omega)$, let $Q^h : L^2(\Omega) \mapsto X_d^h$ denote the L^2 orthogonal projection onto X_d^h. We assume that X_d^h is such that Q^h is bounded with respect to the norm in $H_0^1(\Omega)$ (see [3]), i.e., that

$$\|Q^h u\|_1 \leq C\|u\|_1, \quad \forall u \in H_0^1(\Omega)\,. \tag{2.7}$$

We then have the following result concerning Q^h.

LEMMA 2.1. *Assume that (2.1) is valid for X^h. Then*

$$|(I - Q^h)v|_{-1} \leq Ch\|v\|_0, \quad \forall v \in L^2(\Omega)\,. \tag{2.8}$$

For the proof of this result, we refer to [3].

Equivalence relation (2.6) can be extended to all of $L^2(\Omega)$ using Q^h. Indeed, by the definitions of Q^h and S^h, we have that $S^h = S^h Q^h$. Similarly, by the symmetry of B^h, we have that $B^h = B^h Q^h$. Hence, (2.6) now holds for any L^2 function. A computable discrete negative norm can now be defined based on operator B^h:

$$|f|_{-h} = (\mathcal{S}^h f, f)^{1/2} = ((h^2 I + B^h)f, f)^{1/2}, \quad \forall f \in L^2(\Omega)\,. \tag{2.9}$$

Both norms (2.4) and (2.9) can be associated with a discrete negative inner product, but in what follows we are particularly interested in the inner product associated with (2.9):

$$(f, g)_{-h} = ((h^2 I + B^h)f, g) = (f, (h^2 I + B^h)g), \quad f, g \in L^2(\Omega)\,. \tag{2.10}$$

Certain fundamental equivalence properties of the discrete norm (2.9) now follow. In particular, the rather ad hoc term $h^2 I$ that is added to B^h in (2.9) is necessary for the equivalence of this norm and the usual negative norm.

LEMMA 2.2. *For any $u \in L^2(\Omega)$,*

$$\frac{1}{C}|u|_{-1} \leq |u|_{-h} \leq C\left(h\|u\|_0 + |u|_{-1}\right), \quad \forall h > 0\,. \tag{2.11}$$

The proof of this lemma can be found in [3] or [4]. We conclude this section with a summary of technical results that will be used frequently throughout the paper.

LEMMA 2.3. *Assume that inverse inequality (2.2) holds for X_d^h. Then*

$$|\frac{\partial u_h}{\partial x_i}|_{-h} \leq C\|u_h\|_0, \quad \forall u_h \in X_d^h, \tag{2.12}$$

$$|uv|_{-h} \leq C\big(h\|u\|_1 + \|u\|_0\big)\|v\|_1\,, \quad \forall u, v \in H^1(\Omega)\,, \tag{2.13}$$

$$|u_h v|_{-h} \le C\|u_h\|_0\|v\|_1 , \quad \forall u_h \in X_d^h, v \in H^1(\Omega) , \tag{2.14}$$

$$|uv|_{-h} \le C\|u\|_0\|v\|_2 , \quad \forall u \in L^2(\Omega), v \in H^2(\Omega) . \tag{2.15}$$

Proof. To show (2.12), we first estimate the negative norm of the derivative as follows:

$$\begin{aligned} |\frac{\partial u_h}{\partial x_i}|_{-1} &= \sup_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega \frac{\partial u_h}{\partial x_i}\phi dx}{|\phi|_1} \\ &= \sup_{\phi \in H_0^1(\Omega)} \frac{\int_\Omega \frac{\partial \phi}{\partial x_i}u_h dx}{|\phi|_1} \\ &\le \|u_h\|_0 . \end{aligned}$$

Then, using (2.11) and (2.2), we obtain

$$\begin{aligned} |\frac{\partial u_h}{\partial x_i}|_{-h} &\le C\left(h\|\frac{\partial u_h}{\partial x_i}\|_0 + |\frac{\partial u_h}{\partial x_i}|_{-1}\right) \\ &\le C\left(h\|u_h\|_1 + \|u_h\|_0\right) \\ &\le C\|u_h\| , \end{aligned}$$

which proves (2.12). To show (2.13), for $2 \le q \le 6$ we have $H^1(\Omega) \subset L^q(\Omega)$ with compact imbedding; see Lemma 7, Part I. In particular, $H^1(\Omega)$ imbeds compactly into $L^4(\Omega)$ and, therefore,

$$\|uv\|_0 \le C\|u\|_1\|v\|_1 .$$

Thus, (2.11) and the inequality $|uv|_{-1} \le \|u\|_0\|v\|_1$ (see Lemma 7 of Part I) yield

$$\begin{aligned} |uv|_{-h} &\le C\left(h\|uv\|_0 + |uv|_{-1}\right) \\ &\le C\left(h\|u\|_1\|v\|_1 + \|u\|_0\|v\|_1\right) , \end{aligned}$$

which proves (2.13). To show (2.14), we use (2.13) and (2.2):

$$|u^h v|_{-h} \le C(h\|u_h\|_1 + \|u_h\|_0)\|v\|_1 \le C\|u_h\|_0\|v\|_1 .$$

Finally, the proof of (2.15) is similar to that of (2.14), but uses the imbedding of $H^2(\Omega)$ into $C^0(\Omega)$. □

2.2. Hypotheses. In this section, we state the hypotheses that will be assumed throughout the rest of this paper. Let

$$\mathbf{X} = L^2(\Omega)^{n^2} \times H_0^1(\Omega)^n \times L_0^2(\Omega) . \tag{2.16}$$

Let $\mathbf{X}^h$ denote a finite element subspace of $\mathbf{X}$ with approximation properties to be stated below. In general, $\mathbf{X}^h$ can be constructed as a direct product of the spaces X_d^h.

Assume that the set $\{(\lambda, \mathcal{U}(\lambda) \equiv (\underline{\mathbf{U}}(\lambda), \mathbf{u}(\lambda), p(\lambda))) \,|\, \lambda \in \Lambda; \mathcal{U}(\lambda) \in \mathbf{X}\}$ forms a regular branch of solutions of the Navier-Stokes equations, so that, in particular, $(\underline{\mathbf{U}}(\lambda), \mathbf{u}(\lambda), p(\lambda))$ is a nonsingular solution of (1.1)-(1.4) for every $\lambda \in \Lambda$. As in

Part I, the parameter λ is identified with the Reynolds number Re. We recall that nonsingularity of $(\underline{\mathbf{U}}(\lambda), \mathbf{u}(\lambda), p(\lambda))$ implies that the linearized problem

$$-(\nabla^t \hat{\underline{\mathbf{U}}})^t + \lambda(\underline{\mathbf{U}}(\lambda)^t \hat{\mathbf{u}} + \hat{\underline{\mathbf{U}}}^t \mathbf{u}(\lambda)) + \nabla \hat{p} = \mathbf{f}^* \tag{2.17}$$

$$\hat{\underline{\mathbf{U}}} - \nabla \hat{\mathbf{u}}^t = \underline{\mathbf{0}} \tag{2.18}$$

$$\nabla^t \hat{\mathbf{u}} = 0 \tag{2.19}$$

along with the boundary condition (1.4) has a unique (weak) solution $(\hat{\underline{\mathbf{U}}}, \hat{\mathbf{u}}, \hat{p}) \in L^2(\Omega)^{n^2} \times H_0^1(\Omega)^n \times L_0^2(\Omega)$ for each $\mathbf{f}^* \in H^{-1}(\Omega)^n$, and that the mapping $\mathbf{f}^* \mapsto (\hat{\underline{\mathbf{U}}}, \hat{\mathbf{u}}, \hat{p})$ is continuous for all $\lambda \in \Lambda$. As a result, the a priori estimate

$$\begin{aligned} \|\hat{\underline{\mathbf{U}}}\|_0 + \|\hat{\mathbf{u}}\|_1 + \|\hat{p}\|_0 \le C\Big(&| - (\nabla^t \hat{\underline{\mathbf{U}}})^t + \lambda(\underline{\mathbf{U}}(\lambda)^t \hat{\mathbf{u}} + \hat{\underline{\mathbf{U}}}^t \mathbf{u}(\lambda)) + \nabla \hat{p}|_{-1} \\ &+ \|\nabla^t \hat{\mathbf{u}}\|_0 + \|\hat{\underline{\mathbf{U}}} - \nabla \hat{\mathbf{u}}^t\|_0\Big), \end{aligned} \tag{2.20}$$

is valid for all $\lambda \in \Lambda$. We also recall the a priori estimate established in [6] for the velocity-flux Stokes problem:

$$\|\underline{\mathbf{U}}\|_0 + \|\mathbf{u}\|_1 + \|p\|_0 \le C\Big(| - (\nabla^t \underline{\mathbf{U}})^t + \nabla p|_{-1} + \|\nabla^t \mathbf{u}\|_0 + \|\underline{\mathbf{U}} - \nabla \mathbf{u}^t\|_0\Big). \tag{2.21}$$

We also need some additional hypotheses on the regular branch $(\underline{\mathbf{U}}(\lambda), \mathbf{u}(\lambda), p(\lambda)))$ and the approximating space $\mathbf{X}^h$:

H-1. The regular branch $\{(\lambda, (\underline{\mathbf{U}}(\lambda), \mathbf{u}(\lambda), p(\lambda))) \,|\, \lambda \in \Lambda\}$ of solutions of the Navier-Stokes equations is such that $\Lambda \ni \lambda \mapsto \mathcal{U}(\lambda) = (\underline{\mathbf{U}}(\lambda), \mathbf{u}(\lambda), p(\lambda))$ is a continuous map from Λ to

$$\mathbf{W}_m = H^m(\Omega)^{n^2} \times [H^{m+1}(\Omega) \cap H_0^1(\Omega)]^n \times [H^m(\Omega) \cap L_0^2(\Omega)], \tag{2.22}$$

for some integer $m \ge 1$.

H-2. There exists an integer $d \ge 1$ such that, for every $\mathcal{U} = (\underline{\mathbf{U}}, \mathbf{u}, p) \in \mathbf{W}_d$, one can find $\mathcal{U}_h = (\underline{\mathbf{U}}^h, \mathbf{u}^h, p^h) \in \mathbf{X}^h$ such that

$$\|\mathcal{U} - \mathcal{U}_h\|_{\mathbf{X}} \le Ch^d \|\mathcal{U}\|_{\mathbf{W}_d}. \tag{2.23}$$

H-3. There exists a continuous function $\lambda \mapsto \mathcal{U}_h(\lambda) \equiv (\underline{\mathbf{U}}_h(\lambda), \mathbf{u}_h(\lambda), p_h(\lambda))$ such that

$$\|\mathcal{U}(\lambda) - \mathcal{U}_h(\lambda)\|_{\mathbf{X}} \le Ch^{\tilde{d}} \|\mathcal{U}(\lambda)\|_{\mathbf{W}_{\tilde{d}}}, \tag{2.24}$$

where $\tilde{d} = \min\{d, m\}$.

Existence of $(\underline{\mathbf{U}}_h(\lambda), \mathbf{u}_h(\lambda), p_h(\lambda))$ in condition **H-3** is assured by Theorem 3, Part I. Condition **H-2** is a reasonable assumption on the finite element space $\mathbf{X}^h$, and it can be satisfied by choosing $\mathbf{X}^h$ to be a direct product of continuous piecewise polynomial finite element spaces. For example, for the choice

$$\mathbf{X}^h = [X_1^h]^{n^2} \times [X_2^h \cap H_0^1(\Omega)]^n \times [X_1^h \cap L_0^2(\Omega)], \tag{2.25}$$

estimate (2.23) holds with $d = 2$. For the convenience of implementation, the space $\mathbf{X}^h$ can also be defined using equal order interpolation, that is, using the same scalar finite element space X_d^h for the approximation of all solution components. It is also possible to consider the use of discontinuous piecewise polynomial spaces for $\underline{\mathbf{U}}$ and p, but this requires accounting for the jumps at the element edges by adding extra terms to the least-squares functional; see [4].

3. Discrete negative norm least-squares. In this section, we introduce a discrete equivalent of the negative norm functional (1.5) considered in Part I. In contrast to the earlier abstract functional, our new functional admits a computationally feasible least-squares method for the numerical solution of the Navier-Stokes equations. Nevertheless, results that have been established in Part I will be critical for the analyses of the discrete negative norm method treated here.

In what follows, we consider methods based on the minimization of the least-squares functional

$$
\begin{aligned}
(3.1)\quad & \mathcal{J}_{-h}(\underline{\mathbf{U}}_h, \mathbf{u}_h, p_h) \\
&= |-(\nabla^t \underline{\mathbf{U}}_h)^t + \nabla p_h + \lambda \left(\underline{\mathbf{U}}_h^t \mathbf{u}_h - \mathbf{f}\right)|_{-h}^2 + \|\nabla^t \mathbf{u}_h\|_0^2 + \|\underline{\mathbf{U}}_h - \nabla \mathbf{u}_h^t\|_0^2 ,
\end{aligned}
$$

over the finite dimensional space $\mathbf{X}^h$. The least-squares principle for (3.1) is given by:

find $\mathcal{U}_h = (\underline{\mathbf{U}}_h, \mathbf{u}_h, p_h) \in \mathbf{X}^h$ *such that*

$$
(3.2)\qquad \mathcal{J}_{-h}(\underline{\mathbf{U}}_h, \mathbf{u}_h, p_h) \le \mathcal{J}_{-h}(\underline{\mathbf{V}}_h, \mathbf{v}_h, q_h), \quad \forall \mathcal{V}_h \in \mathbf{X}^h .
$$

Next, we introduce several variational forms that will be needed in the sequel. The first is given by

$$
\begin{aligned}
(3.3)\qquad \mathcal{B}_{-h}(\mathcal{U}_h, \mathcal{V}_h) = \Big(&-(\nabla^t \underline{\mathbf{U}}_h)^t + \nabla p_h + \lambda \left(\underline{\mathbf{U}}_h^t \mathbf{u}_h - \mathbf{f}\right), \\
&-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda \left(\underline{\mathbf{U}}_h^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h\right)\Big)_{-h} \\
&+ (\nabla^t \mathbf{u}_h, \nabla^t \mathbf{v}_h) + (\underline{\mathbf{U}}_h - \nabla \mathbf{u}_h^t, \underline{\mathbf{V}}_h - \nabla \mathbf{v}_h^t) .
\end{aligned}
$$

Form (3.3) corresponds to the Euler-Lagrange equation for (3.1): a necessary condition for $\mathcal{U}_h$ to satisfy least-squares principle (3.2) is that $\mathcal{U}_h$ solve

find $\mathcal{U}_h = (\underline{\mathbf{U}}_h, \mathbf{u}_h, p_h) \in \mathbf{X}^h$ *such that*

$$
(3.4)\qquad \mathcal{B}_{-h}(\mathcal{U}_h, \mathcal{V}_h) = 0, \quad \forall \mathcal{V}_h \in \mathbf{X}^h .
$$

Once a basis for $\mathbf{X}^h$ is chosen, then (3.4) corresponds to a nonlinear system of algebraic equations, which we denote symbolically by $F_{-h}(\lambda, \mathcal{U}_h) = 0$.

The second form corresponds to a linearization of $\mathcal{B}_{-h}(\cdot,\cdot)$ at $\tilde{\mathcal{U}}$:

$$
\begin{aligned}
(3.5)\qquad & \mathcal{D}\mathcal{B}_{-h}[\tilde{\mathcal{U}}](\mathcal{U}, \mathcal{V}) \\
&= \left(-(\nabla^t \tilde{\underline{\mathbf{U}}})^t + \nabla \tilde{p} + \lambda \left(\tilde{\underline{\mathbf{U}}}^t \tilde{\mathbf{u}} - \mathbf{f}\right), \lambda \left(\underline{\mathbf{U}}^t \mathbf{v} + \underline{\mathbf{V}}^t \mathbf{u}\right)\right)_{-h} \\
&+ \Big(-(\nabla^t \underline{\mathbf{U}})^t + \nabla p + \lambda \left(\tilde{\underline{\mathbf{U}}}^t \mathbf{u} + \underline{\mathbf{U}}^t \tilde{\mathbf{u}}\right), \\
&\qquad -(\nabla^t \underline{\mathbf{V}})^t + \nabla q + \lambda \left(\tilde{\underline{\mathbf{U}}}^t \mathbf{v} + \underline{\mathbf{V}}^t \tilde{\mathbf{u}}\right)\Big)_{-h} \\
&+ (\nabla^t \mathbf{u}, \nabla^t \mathbf{v}) + (\underline{\mathbf{U}} - \nabla \mathbf{u}^t, \underline{\mathbf{V}} - \nabla \mathbf{v}^t) .
\end{aligned}
$$

Our next form corresponds to the linear (Stokes) part of (3.3):

$$
\begin{aligned}
(3.6)\qquad \mathcal{B}_{-h}^S(\mathcal{U}, \mathcal{V}) &= \left(-(\nabla^t \underline{\mathbf{U}})^t + \nabla p, -(\nabla^t \underline{\mathbf{V}})^t + \nabla q\right)_{-h} \\
&+ (\nabla^t \mathbf{u}, \nabla^t \mathbf{v}) + (\underline{\mathbf{U}} - \nabla \mathbf{u}^t, \underline{\mathbf{V}} - \nabla \mathbf{v}^t) .
\end{aligned}
$$

Similarly, the nonlinear part of (3.3) is given by

$$
\begin{aligned}
(3.7)\qquad \mathcal{B}_{-h}^G(\mathcal{U}, \mathcal{V}) &= \left(-(\nabla^t \underline{\mathbf{U}})^t + \nabla p + \lambda \left(\underline{\mathbf{U}}^t \mathbf{u} - \mathbf{f}\right), \lambda \left(\underline{\mathbf{U}}^t \mathbf{v} + \underline{\mathbf{V}}^t \mathbf{u}\right)\right)_{-h} \\
&+ \left(\lambda \left(\underline{\mathbf{U}}^t \mathbf{u} - \mathbf{f}\right), -(\nabla^t \underline{\mathbf{V}})^t + \nabla q\right)_{-h} ,
\end{aligned}
$$

and linearization of (3.7) at $\tilde{\mathcal{U}}$ is given by

$$\begin{aligned}
(3.8) \qquad & \mathcal{D}\mathcal{B}^G_{-h}[\tilde{\mathcal{U}}](\mathcal{U},\mathcal{V}) \\
&= \left(-(\nabla^t\underline{\tilde{\mathbf{U}}})^t + \nabla\tilde{p} + \lambda\left(\underline{\tilde{\mathbf{U}}}^t\tilde{\mathbf{u}} - \mathbf{f}\right), \lambda\left(\underline{\mathbf{U}}^t\mathbf{v} + \underline{\mathbf{V}}^t\mathbf{u}\right)\right)_{-h} \\
&+ \left(-(\nabla^t\underline{\mathbf{U}})^t + \nabla p + \lambda\left(\underline{\tilde{\mathbf{U}}}^t\mathbf{u} + \underline{\mathbf{U}}^t\tilde{\mathbf{u}}\right), \lambda\left(\underline{\tilde{\mathbf{U}}}^t\mathbf{v} + \underline{\mathbf{V}}^t\tilde{\mathbf{u}}\right)\right)_{-h} \\
&+ \left(\lambda\left(\underline{\tilde{\mathbf{U}}}^t\mathbf{u} + \underline{\mathbf{U}}^t\tilde{\mathbf{u}}\right), -(\nabla^t\underline{\mathbf{V}})^t + \nabla q\right)_{-h}.
\end{aligned}$$

With the above definitions, we have the identities

$$(3.9) \qquad \mathcal{B}_{-h}(\mathcal{U},\mathcal{V}) = \mathcal{B}^S_{-h}(\mathcal{U},\mathcal{V}) + \mathcal{B}^G_{-h}(\mathcal{U},\mathcal{V})$$

and

$$(3.10) \qquad \mathcal{D}\mathcal{B}_{-h}[\tilde{\mathcal{U}}](\mathcal{U},\mathcal{V}) = \mathcal{B}^S_{-h}(\mathcal{U},\mathcal{V}) + \mathcal{D}\mathcal{B}^G_{-h}[\tilde{\mathcal{U}}](\mathcal{U},\mathcal{V})\,.$$

3.1. Canonical form of the nonlinear problem. As in Part I, nonlinear problem (3.4) can be cast in a canonical form given in terms of a linear (Stokes) operator and a nonlinear operator. In the present context, we make the following identifications for these operators:

$T_{-h} : \mathbf{Y} \mapsto \mathbf{X}^h$ defined by $\mathcal{U}_h = T_{-h}\mathbf{g}$ for $\mathbf{g} \in \mathbf{Y}$ if and only if

$$(3.11) \qquad \mathcal{B}^S_{-h}(\mathcal{U}_h,\mathcal{V}_h) = (\mathbf{g},\mathcal{V}_h), \quad \forall \mathcal{V}_h \in \mathbf{X}^h,$$

and

$G_{-h} : \Lambda \times \mathbf{X}^h \to \mathbf{Y}$ defined by $\mathbf{g} = G_{-h}(\lambda,\mathcal{U}_h)$ for $\mathcal{U}_h \in \mathbf{X}^h$ if and only if

$$(3.12) \qquad \mathcal{B}^G_{-h}(\mathcal{U}_h,\mathcal{V}_h) = (\mathbf{g},\mathcal{V}_h), \quad \forall \mathcal{V}_h \in \mathbf{X}^h\,,$$

where

$$(3.13) \qquad \mathbf{Y} = L^2(\Omega)^{n^2} \times H^{-1}(\Omega)^n \times L^2(\Omega)\,.$$

Then, problem (3.4) takes the form

$$(3.14) \qquad F_{-h}(\lambda,\mathcal{U}_h) \equiv \mathcal{U}_h + T_{-h}\cdot G_{-h}(\lambda,\mathcal{U}_h) = 0\,.$$

In what follows, we will frequently appeal to variational representations of various identities involving operator $F_{-h}(\lambda,\tilde{\mathcal{U}})$ and its Fréchet derivative $D_{\mathcal{U}}F_{-h}(\lambda,\tilde{\mathcal{U}})$ given in terms of forms (3.3), (3.7), (3.5), and (3.6). In particular, we have the following variational identification of the evaluation of F_{-h} at a given function $\hat{\mathcal{U}}_h$:

$$(3.15) \qquad \mathcal{U}^*_h = F_{-h}(\lambda,\hat{\mathcal{U}}_h) \quad \text{iff} \quad \mathcal{B}^S_{-h}(\mathcal{U}^*_h,\mathcal{V}_h) = \mathcal{B}_{-h}(\hat{\mathcal{U}}_h,\mathcal{V}_h),\ \forall \mathcal{V}_h \in \mathbf{X}^h\,,$$

i.e., the function $\mathcal{U}^*_h$ is obtained by solving a variational problem for the Stokes form (3.6) with right-hand side given by form (3.3) evaluated at $\hat{\mathcal{U}}_h$. To obtain a variational equation for Fréchet derivative $D_{\mathcal{U}}F_{-h}(\lambda,\tilde{\mathcal{U}})$, note that

$$D_{\mathcal{U}}F_{-h}(\lambda,\tilde{\mathcal{U}}) = I + T_{-h}\cdot D_{\mathcal{U}}G_{-h}(\lambda,\tilde{\mathcal{U}})\,.$$

It is not difficult to see that

$D_{\mathcal{U}}G_{-h}(\lambda,\tilde{\mathcal{U}})\cdot : \Lambda \times \mathbf{X}^h \to \mathbf{Y}$ defined by $\mathbf{g} = D_{\mathcal{U}}G_{-h}(\lambda,\tilde{\mathcal{U}})\cdot\hat{\mathcal{U}}_h$ for $\hat{\mathcal{U}}_h \in \mathbf{X}^h$ if and only if

$$\mathcal{DB}^G_{-h}[\tilde{\mathcal{U}}](\hat{\mathcal{U}}_h, \mathcal{V}_h) = (\mathbf{g}, \mathcal{V}_h), \quad \forall \mathcal{V}_h \in \mathbf{X}^h \,. \tag{3.16}$$

As a result, given an element $\hat{\mathcal{U}}_h \in \mathbf{X}^h$, we have that

$$\mathcal{U}^*_h = D_{\mathcal{U}}F_{-h}(\lambda,\tilde{\mathcal{U}}) \cdot \hat{\mathcal{U}}_h \quad \text{iff} \quad \mathcal{B}^S_{-h}(\mathcal{U}^*_h, \mathcal{V}_h) = \mathcal{DB}_{-h}[\tilde{\mathcal{U}}](\hat{\mathcal{U}}_h, \mathcal{V}_h), \ \forall \mathcal{V}_h \in \mathbf{X}^h \,, \tag{3.17}$$

i.e., $\mathcal{U}^*_h$ is obtained by solving a variational problem for Stokes form (3.6), but with right-hand side given by $\mathcal{DB}_{-h}[\tilde{\mathcal{U}}](\hat{\mathcal{U}}_h, \cdot)$. Finally, if $\tilde{\mathcal{U}}$ is such that $D_{\mathcal{U}}F_{-h}(\lambda,\tilde{\mathcal{U}})^{-1}$ exists, we have

$$\hat{\mathcal{U}}^h = D_{\mathcal{U}}F_{-h}(\lambda,\tilde{\mathcal{U}})^{-1}\mathcal{U}^h_* \quad \text{iff} \quad \mathcal{DB}_{-h}[\tilde{\mathcal{U}}](\hat{\mathcal{U}}_h, \mathcal{V}_h) = \mathcal{B}^S_{-h}(\mathcal{U}^*_h, \mathcal{V}_h), \ \forall \mathcal{V}_h \in \mathbf{X}^h \,. \tag{3.18}$$

4. Discretization error estimates. The main goals of this section are to show that discrete nonlinear problem (3.4) is well-posed in the sense that it has a regular branch of solutions, provided the Navier-Stokes equations have such a branch, and to establish discretization error estimates for the solution of (3.4). Our analysis is carried out under the hypotheses **H-1** - **H-3** stated in §2.2. In order to keep notation simple, in what follows, we agree to use $\mathbf{W}$ to denote the space $\mathbf{W}_{\tilde{d}}$ and $\tilde{h}$ to denote $h^{\tilde{d}}$. We recall that $\tilde{d} = \min\{m, d\}$, where m and d are the integers from **H-1** and **H-2**, respectively. Throughout this section, δ is used to denote a quantity that may change in meaning, except that it depends only on the maximum value of $\|\mathcal{U}(\lambda)\|_{\mathbf{W}}$ over $\lambda \in \Lambda$.

4.1. Coercivity and continuity bounds. In this section, we develop several technical results that deal with coercivity properties and continuity estimates for forms (3.3), (3.5), and (3.6). These results will be needed in the sequel to establish existence of regular branches of solutions and corresponding error estimates for problem (3.4).

LEMMA 4.1. *Assume that* $\mathcal{U}_h(\lambda) \in \mathbf{X}^h$ *satisfies* **H-3**. *Then*

$$|-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda \left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\right)|_{-h} \le \delta \tilde{h} \,. \tag{4.1}$$

Proof. The triangle inequality and the fact that $\mathcal{U}(\lambda)$ solves Navier-Stokes velocity-flux equations (1.1)-(1.4) yield

$$\begin{aligned}
&|-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda \left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\right)|_{-h} \\
&\qquad \le |(\nabla^t(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t|_{-h} + |\nabla(p_h(\lambda) - p(\lambda))|_{-h} \\
&\qquad\quad + \lambda|\underline{\mathbf{U}}_h(\lambda)^t\mathbf{u}_h(\lambda) - \underline{\mathbf{U}}(\lambda)^t\mathbf{u}(\lambda)|_{-h} \,.
\end{aligned}$$

To estimate the first term, we use upper bound (2.11) in Lemma 2.2, Lemma 7 of Part I, and **H-3**:

$$\begin{aligned}
&|\nabla^t(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))|_{-h} \\
&\le C\Big(h\|\nabla^t(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))\|_0 + |\nabla^t(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))|_{-1}\Big) \\
&\le C\Big(h\|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_1 + \|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_0\Big) \\
&\le C\tilde{h}\|\mathcal{U}(\lambda)\|_{\mathbf{W}} \,.
\end{aligned}$$

Similarly,

$$|\nabla(p_h(\lambda) - p(\lambda))|_{-h} \le C\tilde{h}\|\mathcal{U}(\lambda)\|_{\mathbf{W}}\,.$$

For the last term, we use the triangle inequality, (2.14)-(2.15), and **H-3**:

$$\begin{aligned}
&\lambda|\underline{\mathbf{U}}_h(\lambda)^t\mathbf{u}_h(\lambda) - \underline{\mathbf{U}}(\lambda)^t\mathbf{u}(\lambda)|_{-h}\\
&\quad\le \lambda\Big(|\underline{\mathbf{U}}_h(\lambda)^t(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda))|_{-h} + |(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t\mathbf{u}(\lambda)|_{-h}\Big)\\
&\quad\le \lambda\Big(\|\underline{\mathbf{U}}_h(\lambda)\|_0\|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)\|_1 + \|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_0\|\mathbf{u}(\lambda)\|_2\Big)\\
&\quad\le \lambda\tilde{h}\|\mathcal{U}(\lambda)\|_{\mathbf{W}}\Big(\|\underline{\mathbf{U}}_h(\lambda)\|_0 + \|\mathbf{u}(\lambda)\|_2\Big)\,.
\end{aligned}$$

Estimate (4.1) now follows by observing from (2.24) that $\|\underline{\mathbf{U}}_h(\lambda)\|_0$ can be bounded by $\|\underline{\mathbf{U}}(\lambda)\|_{\tilde{d}}$. ∎

LEMMA 4.2. *Form $\mathcal{B}^S_{-h}(\cdot,\cdot)$ is continuous and coercive on $\mathbf{X}^h \times \mathbf{X}^h$.*

Proof. We obtain coercivity using definition (3.6), the lower bound in (2.11), and estimate (2.21):

$$\begin{aligned}
\mathcal{B}^S_{-h}(\mathcal{U}_h,\mathcal{U}_h) &= |-(\nabla^t\underline{\mathbf{U}}_h)^t + \nabla p_h|^2_{-h} + \|\nabla^t\mathbf{u}_h\|_0^2 + \|\underline{\mathbf{U}}_h - \nabla\mathbf{u}_h^t\|_0^2\\
&\ge \frac{1}{C}\Big(|-(\nabla^t\underline{\mathbf{U}}_h)^t + \nabla p_h|^2_{-1} + \|\nabla^t\mathbf{u}_h\|_0^2 + \|\underline{\mathbf{U}}_h - \nabla\mathbf{u}_h\|_0^2\Big)\\
&\ge \frac{1}{C}\Big(\|\underline{\mathbf{U}}_h\|_0^2 + \|\mathbf{u}_h\|_1^2 + \|p_h\|_0^2\Big)\,.
\end{aligned}$$

To show continuity of $\mathcal{B}^S_{-h}(\cdot,\cdot)$, we use first the Cauchy and triangle inequalities:

$$\begin{aligned}
\mathcal{B}^S_{-h}(\mathcal{U}^h,\mathcal{V}^h) &\le |-(\nabla^t\underline{\mathbf{U}}_h)^t + \nabla p_h|_{-h}|-(\nabla^t\underline{\mathbf{V}}_h)^t + \nabla q_h|_{-h}\\
&\quad+\|\nabla^t\mathbf{u}_h\|_0\|\nabla^t\mathbf{v}_h\|_0 + \|\underline{\mathbf{U}}_h - \nabla\mathbf{u}_h^t\|_0\|\underline{\mathbf{V}}_h - \nabla\mathbf{v}_h^t\|_0\\
&\le \big(\|(\nabla^t\underline{\mathbf{U}}_h)|_{-h} + |\nabla p_h|_{-h}\big)\big(\|(\nabla^t\underline{\mathbf{V}}_h)|_{-h} + |\nabla q_h|_{-h}\big)\\
&\quad+\|\mathbf{u}_h\|_1\|\mathbf{v}_h\|_1 + \big(\|\underline{\mathbf{U}}_h\|_0 + \|\mathbf{u}_h\|_1\big)\big(\|\underline{\mathbf{V}}_h\|_0 + \|\mathbf{v}_h\|_1\big)\,.
\end{aligned}$$

From (2.12) in Lemma 2.3, it follows that

$$|(\nabla^t\underline{\mathbf{U}}_h)^t|_{-h} \le C\|\underline{\mathbf{U}}_h\|_0$$

and

$$|\nabla p_h|_{-h} \le C\|p_h\|_0\,,$$

with similar bounds for the terms with $\underline{\mathbf{V}}_h$ and q_h. As a result,

$$\begin{aligned}
\mathcal{B}^S_{-h}(\mathcal{U}^h,\mathcal{V}^h) &\le C\Big(\|\underline{\mathbf{U}}_h\|_0 + \|p_h\|_0\Big)\Big(\|\underline{\mathbf{V}}_h\|_0 + \|q_h\|_0\Big) +\\
&\quad+\|\mathbf{u}_h\|_1\|\mathbf{v}_h\|_1 + (\|\underline{\mathbf{U}}_h\|_0 + \|\mathbf{u}_h\|_1)\,(\|\underline{\mathbf{V}}_h\|_0 + \|\mathbf{v}_h\|_1)\\
&\le C\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}}\,.
\end{aligned}$$

∎

LEMMA 4.3. *Assume that $\mathcal{U}_h(\lambda)$ satisfies* **H-3**. *Then, there exists an $h_0 > 0$ such that, for any $h \le h_0$, we have*

$$\frac{1}{C}\|\mathcal{V}_h\|^2_{\mathbf{X}} \le \mathcal{D}\mathcal{B}_{-h}[\mathcal{U}_h(\lambda)](\mathcal{V}_h,\mathcal{V}_h), \quad \forall\mathcal{V}_h \in \mathbf{X}^h\,, \tag{4.2}$$

i.e., form $\mathcal{DB}_{-h}[\mathcal{U}_h(\lambda)](\cdot,\cdot)$ is coercive on $\mathbf{X}^h \times \mathbf{X}^h$ for all sufficiently small h.

Proof. We want to show that, for every $\mathcal{V}_h \in \mathbf{X}^h$,

$$
\begin{aligned}
(4.3) \qquad & \mathcal{DB}_{-h}[\mathcal{U}_h(\lambda)](\mathcal{V}_h,\mathcal{V}_h) \\
&= \Big(-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda\big(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\big), 2\lambda \underline{\mathbf{V}}_h^t \mathbf{v}_h\Big)_{-h} \\
&+ |-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)\right)|_{-h}^2 \\
&+ \|\nabla^t \mathbf{v}_h\|_0^2 + \|\underline{\mathbf{V}}_h - \nabla \mathbf{v}_h{}^t\|_0^2 \\
&\ge C\big(\|\underline{\mathbf{V}}_h\|_0^2 + \|\mathbf{v}_h\|_1^2 + \|q_h\|_0^2\big)\,.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, (4.1) in Lemma 4.1, and (2.14) in Lemma 2.3, the first term in (4.3) is bounded above as follows:

$$
\begin{aligned}
(4.4) \qquad & \Big(-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\right), 2\lambda \underline{\mathbf{V}}_h^t \mathbf{v}_h\Big)_{-h} \\
&\le 2\lambda|-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\right)|_{-h} |\underline{\mathbf{V}}_h^t \mathbf{v}_h|_{-h} \\
&\le \delta\tilde{h}\|\underline{\mathbf{V}}_h\|_0 \|\mathbf{v}_h\|_1 \\
&\le \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}^2\,.
\end{aligned}
$$

Using the triangle inequality, for the second term in (4.3) we obtain

$$
\begin{aligned}
&|-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)\right)|_{-h} \\
&\qquad \ge |-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}(\lambda)\right)|_{-h} \\
&\qquad\qquad -\lambda|(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda))|_{-h}\,.
\end{aligned}
$$

Next, using again the triangle inequality, (2.13), (2.14), and (2.24) yield

$$
\begin{aligned}
&|(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda))|_{-h} \\
&\le |(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t \mathbf{v}_h|_{-h} + |\underline{\mathbf{V}}_h^t(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda))|_{-h} \\
&\le C(h\|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_1 + \|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_0)\|\mathbf{v}_h\|_1 + \|\underline{\mathbf{V}}_h\|_0 \|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)\|_1 \\
&\le \delta\tilde{h}\Big(\|\underline{\mathbf{V}}_h\|_0 + \|\mathbf{v}_h\|_1\Big) \le \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}\,.
\end{aligned}
$$

Thus, the second term in (4.3) is bounded below as follows:

$$
\begin{aligned}
(4.5) \quad & |-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)\right)|_{-h}^2 \\
&\ge |-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}(\lambda)\right)|_{-h}^2 - \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}^2\,.
\end{aligned}
$$

Estimates (4.4)-(4.5) together with the lower bound in (2.11) and a priori estimate (2.20) yield

$$
\begin{aligned}
&\mathcal{DB}_{-h}[\mathcal{U}_h(\lambda)](\mathcal{V}_h,\mathcal{V}_h) \\
&\ge |-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}(\lambda)\right)|_{-h}^2 + \|\nabla^t \mathbf{v}_h\|_0^2 + \\
&\quad \|\underline{\mathbf{V}}_h - \nabla \mathbf{v}_h{}^t\|_0^2 - \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}^2 \\
&\ge \left(\|\underline{\mathbf{V}}_h\|_0^2 + \|\mathbf{v}_h\|_1^2 + \|q_h\|_0^2\right) - \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}^2\,.
\end{aligned}
$$

According to **H-1**, the constant δ is uniformly bounded for all $\lambda \in \Lambda$. Thus, bound (4.2) is valid for all $h \le h_0 = 1/(2\delta)$. □

LEMMA 4.4. *Assume that* $\mathcal{U}_h(\lambda)$ *satisfies* **H-3**. *Then*

$$\mathcal{B}_{-h}(\mathcal{U}_h(\lambda), \mathcal{V}_h) \le \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}, \quad \forall \mathcal{V}_h \in \mathbf{X}^h\,. \tag{4.6}$$

Proof. With $\mathcal{U}(\lambda)$ solving the the velocity-flux Navier-Stokes equations, using the Cauchy-Schwarz and triangle inequalities together with (2.24), we estimate the linear terms in (3.3) as follows:

$$\begin{aligned}
&\Big(\underline{\mathbf{U}}_h(\lambda) - \nabla \mathbf{u}_h(\lambda)^t, \underline{\mathbf{V}}_h - \nabla {\mathbf{v}_h}^t\Big) \\
&= \Big((\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)) - \nabla(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda))^t, \underline{\mathbf{V}}_h - \nabla {\mathbf{v}_h}^t\Big) \\
&\le \Big(\|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_0 + \|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)\|_1\Big)\Big(\|\underline{\mathbf{V}}_h\|_0 + \|\mathbf{v}_h\|_1\Big) \\
&\le \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
\Big(\nabla^t \mathbf{u}_h(\lambda), \nabla^t \mathbf{v}_h\Big) &= \Big(\nabla^t(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)), \nabla^t \mathbf{v}_h\Big) \\
&\le \|\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda)\|_1 \|\mathbf{v}_h\|_1 \\
&\le \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}\,.
\end{aligned} \tag{4.8}$$

To estimate the remaining first term in (3.3), we proceed as in the proof of Lemma 4.3:

$$\begin{aligned}
&\big(-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\right), \\
&\qquad\qquad -(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)\right)\big)_{-h} \\
&\le |-(\nabla^t \underline{\mathbf{U}}_h(\lambda))^t + \nabla p_h(\lambda) + \lambda\left(\underline{\mathbf{U}}_h(\lambda)^t \mathbf{u}_h(\lambda) - \mathbf{f}\right)|_{-h} \\
&\quad \Big(|-(\nabla^t \underline{\mathbf{V}}_h)^t + \nabla q_h|_{-h} + \lambda|\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)|_{-h}\Big) \\
&\le \delta\tilde{h}\left(|(\nabla^t \underline{\mathbf{V}}_h)^t|_{-h} + |\nabla q_h|_{-h} + \lambda|\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)|_{-h}\right),
\end{aligned} \tag{4.9}$$

where the last inequality used (4.1). Next, we bound the first and second terms on the right-hand side of (4.9) using (2.12):

$$|(\nabla^t \underline{\mathbf{V}}_h)^t|_{-h} \le C\|\underline{\mathbf{V}}_h\|_0 \tag{4.10}$$

and

$$|\nabla q_h|_{-h} \le C\|q_h\|_0. \tag{4.11}$$

For the remaining term in (4.9), we first use the triangle inequality:

$$\begin{aligned}
&|\underline{\mathbf{U}}_h(\lambda)^t \mathbf{v}_h + \underline{\mathbf{V}}_h^t \mathbf{u}_h(\lambda)|_{-h} \\
&\le |(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t \mathbf{v}_h|_{-h} + |{\underline{\mathbf{V}}_h}^t(\mathbf{u}_h(\lambda) - \mathbf{u}(\lambda))|_{-h} \\
&+ |\underline{\mathbf{U}}(\lambda)^t \mathbf{v}_h|_{-h} + |\underline{\mathbf{V}}_h^t \mathbf{u}(\lambda)|_{-h}\,.
\end{aligned} \tag{4.12}$$

Then, as in the proof of Lemma 4.2, (2.13) and (2.24) yield

$$\begin{aligned}
&|(\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda))^t \mathbf{v}_h|_{-h} \\
&\le C\big(h\|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_1 + \|\underline{\mathbf{U}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_0\big)\|\mathbf{v}_h\|_1 \le \delta\tilde{h}\|\mathbf{v}_h\|_1\,.
\end{aligned} \tag{4.13}$$

Similarly, (2.14) and (2.24) yield

$$|\underline{\mathbf{V}}_h{}^t(\mathbf{u}_h(\lambda)-\mathbf{u}(\lambda))|_{-h} \le C\|\underline{\mathbf{V}}_h\|_0\|\mathbf{u}_h(\lambda)-\mathbf{u}(\lambda)\|_1 \le \delta\tilde{h}\|\underline{\mathbf{V}}_h\|_0\,. \tag{4.14}$$

Finally, (2.13) yields

$$|\underline{\mathbf{U}}(\lambda)\mathbf{v}_h|_{-h} \le C(h\|\underline{\mathbf{U}}(\lambda)\|_1+\|\underline{\mathbf{U}}(\lambda)\|_0)\|\mathbf{v}_h\|_1 \le \delta\|\mathbf{v}_h\|_1 \tag{4.15}$$

and (2.15) yields

$$|\underline{\mathbf{V}}_h\mathbf{u}(\lambda)|_{-h} \le C\|\underline{\mathbf{V}}_h\|_0\|\mathbf{u}(\lambda)\|_2 \le \delta\|\underline{\mathbf{V}}_h\|_0\,. \tag{4.16}$$

Now, (4.9)-(4.16) combine to yield

$$\begin{aligned}
&\big(-(\nabla^t\underline{\mathbf{U}}_h(\lambda))^t+\nabla p_h(\lambda)+\lambda\left(\underline{\mathbf{U}}_h(\lambda)^t\mathbf{u}_h(\lambda)-\mathbf{f}\right),\\
&\qquad\qquad -(\nabla^t\underline{\mathbf{V}}_h)^t+\nabla q_h+\lambda\left(\underline{\mathbf{U}}_h(\lambda)^t\mathbf{v}_h+\underline{\mathbf{V}}_h^t\mathbf{u}_h(\lambda)\right)\big)_{-h}\\
&\le \delta\tilde{h}\Big((1+\delta(1+\tilde{h}))\|\underline{\mathbf{V}}_h\|_0+\|q_h\|_0+\delta(1+\tilde{h})\|\mathbf{v}_h\|_1\Big)\\
&\le \delta\tilde{h}\|\mathcal{V}_h\|_{\mathbf{X}}\,.
\end{aligned} \tag{4.17}$$

Bound (4.6) now follows from (4.7), (4.8), and (4.17). ∎

LEMMA 4.5. *Let* $\tilde{\mathcal{U}}_h^1$, $\tilde{\mathcal{U}}_h^2$, *and* $\mathcal{U}_h$ *be arbitrary elements of* $\mathbf{X}^h$. *Then*

$$\begin{aligned}
&|\mathcal{DB}_{-h}[\tilde{\mathcal{U}}_h^1](\mathcal{U}_h,\mathcal{V}_h)-\mathcal{DB}_{-h}[\tilde{\mathcal{U}}_h^2](\mathcal{U}_h,\mathcal{V}_h)|\\
&\quad\le C\|\tilde{\mathcal{U}}_h^1-\tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}}\left(1+\|\tilde{\mathcal{U}}_h^1\|_{\mathbf{X}}+\|\tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\right)
\end{aligned} \tag{4.18}$$

Proof. First note that (3.10) implies

$$\begin{aligned}
&\mathcal{DB}_{-h}[\tilde{\mathcal{U}}_h^1](\mathcal{U}_h,\mathcal{V}_h)-\mathcal{DB}_{-h}[\tilde{\mathcal{U}}_h^2](\mathcal{U}_h,\mathcal{V}_h)\\
&=\mathcal{DB}^G_{-h}[\tilde{\mathcal{U}}_h^1](\mathcal{U}_h,\mathcal{V}_h)-\mathcal{DB}^G_{-h}[\tilde{\mathcal{U}}_h^2](\mathcal{U}_h,\mathcal{V}_h)\\
&=\lambda\left(\nabla^t(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2))^t+\nabla(\tilde{p}_h^1-\tilde{p}_h^2)+\lambda\left((\underline{\tilde{\mathbf{U}}}_h^1)^t\tilde{\mathbf{u}}_h^1-(\underline{\tilde{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2\right),\right.\\
&\qquad\qquad \left.\underline{\mathbf{U}}_h^t\mathbf{v}_h+\underline{\mathbf{V}}_h^t\mathbf{u}_h\right)_{-h}\\
&+\lambda\left(-(\nabla^t\underline{\mathbf{U}}_h)^t+\nabla p_h,(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{v}_h+\underline{\mathbf{V}}_h^t(\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2)\right)_{-h}\\
&+\lambda^2\left(\left((\underline{\tilde{\mathbf{U}}}_h^1)^t\mathbf{u}_h+(\underline{\mathbf{U}}_h)^t\tilde{\mathbf{u}}_h^1,(\underline{\tilde{\mathbf{U}}}_h^1)^t\mathbf{v}_h+\underline{\mathbf{V}}_h^t\tilde{\mathbf{u}}_h^1\right)_{-h}\right.\\
&\qquad\qquad \left.-\left((\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{u}^h+(\underline{\mathbf{U}}_h)^t\tilde{\mathbf{u}}_h^2,(\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{v}_h+\underline{\mathbf{V}}_h^t\tilde{\mathbf{u}}_h^2\right)_{-h}\right)\\
&+\lambda\left((\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{u}_h+(\underline{\mathbf{U}}_h)^t(\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2),-(\nabla^t\underline{\mathbf{V}}_h)^t+\nabla q_h\right)_h\,.
\end{aligned} \tag{4.19}$$

The first term in (4.19) can be estimated using the Cauchy-Schwarz and triangle inequalities and (2.14):

$$\begin{aligned}
&\lambda\left(\nabla^t(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2))^t+\nabla(\tilde{p}_h^1-\tilde{p}_2^h)+\lambda\left((\underline{\tilde{\mathbf{U}}}_h^1)^t\tilde{\mathbf{u}}_h^1-(\underline{\tilde{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2\right),\underline{\mathbf{U}}_h^t\mathbf{v}_h+\underline{\mathbf{V}}_h^t\mathbf{u}^h\right)_{-h}\\
&\le\lambda|\nabla^t(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2))^t+\nabla(\tilde{p}_h^1-\tilde{p}_2^h)+\lambda\left((\underline{\tilde{\mathbf{U}}}_h^1)^t\tilde{\mathbf{u}}_h^1-(\underline{\tilde{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2\right)|_{-h}\\
&\quad\cdot\Big(|(\underline{\mathbf{U}}_h)^t\mathbf{v}_h|_{-h}+|\underline{\mathbf{V}}_h^t\mathbf{u}_h|_{-h}\Big)\\
&\le\lambda\Big(|\nabla^t(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2))^t|_{-h}+|\nabla(\tilde{p}_h^1-\tilde{p}_h^2)|_{-h}+\lambda|(\underline{\tilde{\mathbf{U}}}_h^1)^t\tilde{\mathbf{u}}_h^1-(\underline{\tilde{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2|_{-h}\Big)\\
&\quad\cdot\Big(\|\underline{\mathbf{U}}_h\|_0\|\mathbf{v}_h\|_1+\|\underline{\mathbf{V}}_h\|_0\|\mathbf{u}_h\|_1\Big)\,.
\end{aligned}$$

Now, using (2.12), we have

$$\begin{aligned}&|\nabla^t(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2))^t|_{-h} + |\nabla(\tilde{p}_h^1 - \tilde{p}_h^2)|_{-h}\\ &\le C\Big(\|\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2\|_0 + \|\tilde{p}_h^1 - \tilde{p}_h^2\|_0\Big)\\ &\le C\|\tilde{\mathcal{U}}_h^1 - \tilde{\mathcal{U}}_h^2\|_{\mathbf{X}},\end{aligned}$$

and, using the triangle inequality and (2.14), we have

$$\begin{aligned}\lambda|(\tilde{\underline{\mathbf{U}}}_h^1)^t\tilde{\mathbf{u}}_h^1 - (\tilde{\underline{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2|_{-h} &\le \lambda\big(|(\tilde{\underline{\mathbf{U}}}_h^1)^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2)|_{-h} + |(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2|_{-h}\big)\\ &\le \lambda C\big(\|\tilde{\underline{\mathbf{U}}}_h^1\|_0\|\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2\|_1 + \|\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2\|_0\|\tilde{\mathbf{u}}_h^2\|_1\big)\\ &\le \lambda C\|\tilde{\mathcal{U}}_h^1 - \tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\big(\|\tilde{\mathcal{U}}_h^1\|_{\mathbf{X}} + \|\tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&|\nabla^t(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2))^t + \nabla(\tilde{p}_h^1 - \tilde{p}_h^2) + \lambda\left((\tilde{\underline{\mathbf{U}}}_h^1)^t\tilde{\mathbf{u}}_h^1 - (\tilde{\underline{\mathbf{U}}}_h^2)^t\tilde{\mathbf{u}}_h^2\right)|_{-h}\\ &\qquad\le \lambda C\|\tilde{\mathcal{U}}_h^1 - \tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\big(\|\tilde{\mathcal{U}}_h^1\|_{\mathbf{X}} + \|\tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\big).\end{aligned}$$

Since

$$\|\underline{\mathbf{U}}_h\|_0\|\mathbf{v}_h\|_1 + \|\underline{\mathbf{V}}_h\|_0\|\mathbf{u}_h\|_1 \le C\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}},$$

we find that the first term in (4.19) is bounded by

$$\lambda C\|\tilde{\mathcal{U}}_h^1 - \tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}}\big(\|\tilde{\mathcal{U}}_h^1\|_{\mathbf{X}} + \|\tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\big).$$

The second term in (4.19) can be estimated from the Cauchy-Schwarz and triangle inequalities, (2.12), and (2.14) as follows:

$$\begin{aligned}&\lambda\left(-(\nabla^t\underline{\mathbf{U}}_h)^t + \nabla p_h, (\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{v}_h + \underline{\mathbf{V}}_h^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2)\right)_{-h}\\ &\le \lambda|-(\nabla^t\underline{\mathbf{U}}_h)^t + \nabla p_h|_{-h}\big(|(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{v}_h|_{-h} + |\underline{\mathbf{V}}_h^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2)|_{-h}\big)\\ &\le \lambda C\left(\|\underline{\mathbf{U}}_h\|_0 + \|p_h\|_0\right)\left(\|(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)\|_0\|\mathbf{v}_h\|_1 + \|\underline{\mathbf{V}}_h\|_0\|\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2\|_1\right)\\ &\le \lambda C\|\tilde{\mathcal{U}}_h^1 - \tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}}.\end{aligned}$$

For the third term, we repeatedly use the triangle inequality and (2.14):

$$\begin{aligned}&\left((\tilde{\underline{\mathbf{U}}}_h^1)^t\mathbf{u}_h + \underline{\mathbf{U}}_h^t\tilde{\mathbf{u}}_h^1, (\tilde{\underline{\mathbf{U}}}_h^1)^t\mathbf{v}_h + \underline{\mathbf{V}}_h\tilde{\mathbf{u}}_h^1\right)_{-h} - \left((\tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{u}_h + \underline{\mathbf{U}}_h^t\tilde{\mathbf{u}}_h^2, (\tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{v}_h + \underline{\mathbf{V}}_h\tilde{\mathbf{u}}_h^2\right)_{-h}\\ &= \left((\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{u}_h + \underline{\mathbf{U}}_h^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2), (\tilde{\underline{\mathbf{U}}}_h^1)^t\mathbf{v}_h + \underline{\mathbf{V}}_h\tilde{\mathbf{u}}_h^1\right)_{-h}\\ &+ \left((\tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{u}_h + \underline{\mathbf{U}}_h^t\tilde{\mathbf{u}}_h^2, (\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{v}_h + \underline{\mathbf{V}}_h^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2)\right)_{-h}\\ &\le \left(|(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{u}_h|_{-h} + |\underline{\mathbf{U}}_h^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2)|_{-h}\right)\left(|(\tilde{\underline{\mathbf{U}}}_h^1)^t\mathbf{v}_h|_{-h} + |\underline{\mathbf{V}}_h{}^t\tilde{\mathbf{u}}_h^1|_{-h}\right)\\ &+ \left(|(\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{v}_h|_{-h} + |\underline{\mathbf{V}}_h^t(\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2)|_{-h}\right)\left(|(\tilde{\underline{\mathbf{U}}}_h^2)^t\mathbf{u}_h|_{-h} + |\underline{\mathbf{U}}_h^t\tilde{\mathbf{u}}_h^2|_{-h}\right)\\ &\le C\left(\|\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2\|_0\|\mathbf{u}_h\|_1 + \|\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2\|_1\|\underline{\mathbf{U}}_h\|_0\right)\left(\|\tilde{\underline{\mathbf{U}}}_h^1\|_0\|\mathbf{v}_h\|_1 + \|\underline{\mathbf{V}}_h\|_0\|\tilde{\mathbf{u}}_h^1\|_1\right)\\ &+ \left(\|\tilde{\underline{\mathbf{U}}}_h^1 - \tilde{\underline{\mathbf{U}}}_h^2\|_0\|\mathbf{v}_h\|_1 + \|\tilde{\mathbf{u}}_h^1 - \tilde{\mathbf{u}}_h^2\|_1\|\mathcal{V}_h\|_0\right)\left(\|\tilde{\underline{\mathbf{U}}}_h^2\|_0\|\mathbf{u}_h\|_1 + \|\underline{\mathbf{U}}_h\|_0\|\tilde{\mathbf{u}}_h^2\|_1\right)\\ &\le C\|\tilde{\mathcal{U}}_h^1 - \tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}}\big(\|\mathcal{U}_h^1\|_{\mathbf{X}} + \|\mathcal{U}_h^2\|_{\mathbf{X}}\big).\end{aligned}$$

The last term in (4.19) is bounded using the Cauchy-Schwarz inequality, (2.12), and (2.14):

$$
\begin{aligned}
&\lambda\Big((\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{u}_h+\underline{\mathbf{U}}_h^t(\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2),-(\nabla^t\underline{\mathbf{V}}_h)^t+\nabla q_h\Big)_{-h}\\
&\le\lambda|(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{u}_h+\underline{\mathbf{U}}_h^t(\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2)|_{-h}|-(\nabla^t\underline{\mathbf{V}}_h)^t+\nabla q_h|_{-h}\\
&\le\lambda\Big(|(\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2)^t\mathbf{u}_h|_{-h}+|\underline{\mathbf{U}}_h^t(\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2)|_{-h}\Big)\Big(|-(\nabla^t\underline{\mathbf{V}}_h)^t|_{-h}+|\nabla q_h|_{-h}\Big)\\
&\le\lambda C\Big(\|\underline{\tilde{\mathbf{U}}}_h^1-\underline{\tilde{\mathbf{U}}}_h^2\|_0\|\hat{\mathbf{u}}_h\|_1+\|\underline{\mathbf{U}}_h^t\|_0\|\tilde{\mathbf{u}}_h^1-\tilde{\mathbf{u}}_h^2\|_1\Big)\Big(\|\underline{\mathbf{V}}_h\|_0+\|q_h\|_0\Big)\\
&\le\lambda C\|\tilde{\mathcal{U}}_h^1-\tilde{\mathcal{U}}_h^2\|_{\mathbf{X}}\|\mathcal{U}_h\|_{\mathbf{X}}\|\mathcal{V}_h\|_{\mathbf{X}}\,.
\end{aligned}
$$

∎

4.2. Abstract approximation result. This section introduces an abstract approximation result concerning regular branches of solutions for problems of the form

$$F^h(\lambda,\mathcal{U}_h)=0\,. \tag{4.20}$$

We assume that $F^h(\lambda,\mathcal{U}_h)$ is a continuous, differentiable mapping defined on space $\Lambda\times\mathbf{X}^h\subset\mathbf{R}^+\times\mathbf{X}$. For this result, we let $\mathcal{U}_h(\lambda)\in\mathbf{X}^h$ and define

$$\epsilon_h(\lambda)=\|F^h(\lambda,\mathcal{U}_h(\lambda))\|_{\mathbf{X}}, \tag{4.21}$$

$$\gamma_h(\lambda)=\|D_{\mathcal{U}}F^h(\lambda,\mathcal{U}_h(\lambda))^{-1}\|_{L(\mathbf{X}^h,\mathbf{X}^h)}, \tag{4.22}$$

$$S(\mathcal{U}_h,\alpha)=\{\mathcal{V}_h\in\mathbf{X}^h\,|\,\|\mathcal{U}_h-\mathcal{V}_h\|_{\mathbf{X}}\le\alpha\}, \tag{4.23}$$

and

$$L_h(\lambda,\alpha)=\sup_{\mathcal{V}_h\in S(\mathcal{U}_h(\lambda),\alpha)}\|D_{\mathcal{U}}F^h(\lambda,\mathcal{U}_h(\lambda))-D_{\mathcal{U}}F^h(\lambda,\mathcal{V}_h)\|_{L(\mathbf{X}^h,\mathbf{X}^h)}\,. \tag{4.24}$$

The following abstract theorem is a version of the implicit function theorem presented in [8], modified to suit our needs.

THEOREM 4.6. *Let $\mathcal{U}_h(\lambda)\in\mathbf{X}^h$ be a given function such that $\lambda\mapsto\mathcal{U}_h(\lambda)$ is continuous and mapping $D_{\mathcal{U}}F(\lambda,\mathcal{U}_h(\lambda))$ is an isomorphism of $\mathbf{X}^h$ onto itself for all $\lambda\in\Lambda$. Assume also that there exists $h_0>0$ such that, for $h\le h_0$, the following conditions hold:*

A-1. $\gamma_h(\lambda)\le C$ *uniformly in* $\lambda\in\Lambda$*;*
A-2. $\lim_{h\to0}\sup_{\lambda\in\Lambda}\epsilon_h(\lambda)=0$*;*
A-3. $\lim_{\alpha\to0}\sup_{\lambda\in\Lambda}L_h(\lambda,\alpha)=0$.

Then there exist $\alpha>0$, $h_1>0$, and $\tilde{\mathcal{U}}_h(\lambda)\in C^0(\Lambda,\mathbf{X}^h)$ such that, for all $h\le h_1$, we have:

1. *$\{(\lambda,\tilde{\mathcal{U}}_h(\lambda)),\lambda\in\Lambda\}$ is a regular branch of solutions of $F^h(\lambda,\tilde{\mathcal{U}}_h(\lambda))=0$;*
2. *for each $\lambda\in\Lambda$, $\tilde{\mathcal{U}}_h(\lambda)$ is the only solution in $S(\mathcal{U}_h(\lambda),\alpha)$ and the upper bound*

$$\|\tilde{\mathcal{U}}_h(\lambda)-\mathcal{V}_h\|_{\mathbf{X}}\le C\|F^h(\lambda,\mathcal{V}_h)\|_{\mathbf{X}} \tag{4.25}$$

is valid for all $\mathcal{V}_h\in S(\mathcal{U}_h(\lambda),\alpha)$.

4.3. Existence of regular branches and error estimates. Here we apply Theorem 4.6 to discrete negative-norm least-squares problem (3.4). In this context, we identify abstract problem (4.20) with canonical form (3.14) for problem (3.4). We thus have

$$\epsilon_h(\lambda) = \|F_{-h}(\lambda, \mathcal{U}_h(\lambda))\|_{\mathbf{X}}, \tag{4.26}$$

$$\gamma_h(\lambda) = \|D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda))^{-1}\|_{L(\mathbf{X}^h,\mathbf{X}^h)}, \tag{4.27}$$

and

$$L_h(\lambda,\alpha) = \sup_{\mathcal{V}_h \in S(\mathcal{U}_h(\lambda),\alpha)} \|D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda)) - D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{V}_h)\|_{L(\mathbf{X}^h,\mathbf{X}^h)}\,. \tag{4.28}$$

We continue to assume that **H-1**-**H-3** hold, so that $\{\lambda, \mathcal{U}(\lambda)\}$ is a regular branch of solutions of the velocity-flux first-order Navier-Stokes equations, and that $\mathbf{X}^h \subset \mathbf{X}$ satisfies the appropriate approximation properties.

THEOREM 4.7. *There exist $\alpha > 0$, $h_1 > 0$, and $\tilde{\mathcal{U}}_h(\lambda) \in C^0(\Lambda, \mathbf{X}^h)$ such that, for all $h \le h_1$, $\{(\lambda, \tilde{\mathcal{U}}_h(\lambda)), \lambda \in \Lambda\}$ is a regular branch of solutions of $F_{-h}(\lambda, \mathcal{U}_h) = 0$ that is unique in the ball $S(\mathcal{U}_h(\lambda), \alpha)$. Furthermore, if $m \ge 1$ and $d \ge 1$ are the integers from* **H-1** *and* **H-2***, respectively, and $\tilde{d} = \min\{d, m\}$, then*

$$\|\tilde{\underline{\mathbf{U}}}_h(\lambda) - \underline{\mathbf{U}}(\lambda)\|_0 + \|\tilde{\mathbf{u}}_h(\lambda) - \mathbf{u}(\lambda)\|_1 + \|\tilde{p_h}(\lambda) - p(\lambda)\|_0 \le C\delta h^{\tilde{d}}\,, \tag{4.29}$$

where δ depends only on λ and

$$\|\mathcal{U}(\lambda)\|_{\mathbf{W}_{\tilde{d}}} = \|\underline{\mathbf{U}}(\lambda)\|_{\tilde{d}} + \|\mathbf{u}(\lambda)\|_{\tilde{d}+1} + \|p(\lambda)\|_{\tilde{d}}\,.$$

Proof. Let $\mathcal{U}_h(\lambda)$ be discrete function that satisfies **H-3**. The first part of the theorem would follow if we could show that $D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda))$ is an isomorphism of $\mathbf{X}^h$ onto itself, that the norm of its inverse is bounded, and that hypotheses **A-1** - **A-3** of Theorem 4.6 are satisfied.

To this end, we first show that $D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda))$ is an isomorphism by proving that, for any $\mathcal{U}_h^* \in \mathbf{X}^h$, problem

$$D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda)) \cdot \hat{\mathcal{U}}_h = \mathcal{U}_h^*$$

has a unique solution $\hat{\mathcal{U}}_h \in \mathbf{X}^h$. According to (3.18), this problem is equivalent to the variational problem

find $\hat{\mathcal{U}}_h \in \mathbf{X}^h$ such that

$$\mathcal{D}\mathcal{B}_{-h}[\mathcal{U}_h(\lambda)](\hat{\mathcal{U}}_h, \mathcal{V}_h) = \mathcal{B}^S_{-h}(\mathcal{U}_h^*, \mathcal{V}_h), \quad \forall \mathcal{V}_h \in \mathbf{X}^h\,.$$

Lemma 4.3 asserts coercivity of form $\mathcal{D}\mathcal{B}^G_{-h}[\mathcal{U}_h(\lambda)](\cdot,\cdot)$, and Lemma 4.2 implies that $\mathcal{B}^S_{-h}(\mathcal{U}^h_*, \cdot)$ is a continuous linear functional on $\mathbf{X}^h$. Hence, existence and uniqueness of $\hat{\mathcal{U}}_h$ for all $\lambda \in \Lambda$ follows by the Lax-Milgram lemma. Thus, $D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda))$ is an isomorphism of $\mathbf{X}^h$ onto itself.

Next, we show that the norm of $D_{\mathcal{U}} F_{-h}(\lambda, \mathcal{U}_h(\lambda))^{-1}$ is bounded using Lemma 4.3, Lemma 4.2, and a standard elliptic finite element argument:

$$\frac{1}{C}\|\hat{\mathcal{U}}_h\|^2_{\mathbf{X}} \le \mathcal{D}\mathcal{B}_{-h}[\mathcal{U}_h(\lambda)](\hat{\mathcal{U}}_h, \hat{\mathcal{U}}_h) = \mathcal{B}^S_{-h}(\mathcal{U}_h^*, \hat{\mathcal{U}}_h) \le C\|\mathcal{U}_h^*\|_{\mathbf{X}}\|\hat{\mathcal{U}}_h\|_{\mathbf{X}}\,,$$

so

$$
\begin{aligned}
\|D_{\mathcal{U}}F_{-h}(\lambda,\mathcal{U}_h(\lambda))^{-1}\|_{L(\mathbf{X}^h,\mathbf{X}^h)} &= \sup_{\mathcal{U}_h^*\in\mathbf{X}^h} \frac{\|D_{\mathcal{U}}F_{-h}(\lambda,\mathcal{U}_h(\lambda))^{-1}\cdot\mathcal{U}_h^*\|_{\mathbf{X}^h}}{\|\mathcal{U}_h^*\|_{\mathbf{X}^h}} \\
&= \sup_{\mathcal{U}_h^*\in\mathbf{X}^h} \frac{\|\hat{\mathcal{U}}_h\|_{\mathbf{X}}}{\|\mathcal{U}_h^*\|_{\mathbf{X}^h}} \\
&\le C\,.
\end{aligned}
$$

This also verifies **A-1**.

To verify remaining hypotheses **A-2** and **A-3**, we first estimate (4.26), that is, the norm $\|F_{-h}(\lambda,\mathcal{U}_h(\lambda))\|_{\mathbf{X}}$. To this end, note that

$$\mathcal{U}_h^* = F_{-h}(\lambda,\mathcal{U}_h(\lambda))$$

if and only if $\mathcal{U}_h^*$ solves the variational problem
find $\mathcal{U}_h^*\in\mathbf{X}^h$ *such that*

$$\mathcal{B}^S_{-h}(\mathcal{U}_h^*,\mathcal{V}_h) = \mathcal{B}_{-h}(\mathcal{U}_h(\lambda),\mathcal{V}_h), \quad \forall\mathcal{V}_h\in\mathbf{X}^h\,.$$

From Lemma 4.2, we know that $\mathcal{B}^S_{-h}(\cdot,\cdot)$ is coercive, and Lemma 4.4 asserts that $\mathcal{B}_{-h}(\mathcal{U}_h(\lambda),\cdot)$ is a continuous linear functional on $\mathbf{X}^h$. Hence, there exists a unique solution $\mathcal{U}_h^*$ in $\mathbf{X}^h$ that satisfies the bound

$$
\begin{aligned}
\|\mathcal{U}_h^*\|_{\mathbf{X}}^2 \le \mathcal{B}^S_{-h}(\mathcal{U}_h^*,\mathcal{U}_h^*) &= \mathcal{B}_{-h}(\mathcal{U}_h(\lambda),\mathcal{U}_h^*) \\
&\le \delta h^{\tilde{d}}\|\mathcal{U}_h^*\|_{\mathbf{X}}\,.
\end{aligned}
$$

Thus, we can conclude that

$$\epsilon_h(\lambda) \le \delta h^{\tilde{d}}, \quad \forall\lambda\in\Lambda\,, \tag{4.30}$$

which proves **A-2**.

To prove **A-3**, let $\mathcal{V}_h\in S(\mathcal{U}_h(\lambda),\alpha)$ and $\hat{\mathcal{U}}_h\in\mathbf{X}^h$ be arbitrary and define

$$\mathcal{U}_h^1 = D_{\mathcal{U}}F_{-h}(\lambda,\mathcal{U}_h(\lambda))\cdot\hat{\mathcal{U}}_h$$

and

$$\mathcal{U}_h^2 = D_{\mathcal{U}}F_{-h}(\lambda,\mathcal{V}_h)\cdot\hat{\mathcal{U}}_h\,.$$

According to (3.17), we have that

$$\mathcal{U}_h^1-\mathcal{U}_h^2 = D_{\mathcal{U}}F_{-h}(\lambda,\mathcal{U}_h(\lambda))\cdot\hat{\mathcal{U}}_h - D_{\mathcal{U}}F_{-h}(\lambda,\mathcal{V}_h)\cdot\hat{\mathcal{U}}_h$$

if and only if $\mathcal{U}_h^1-\mathcal{U}_h^2$ solves the variational problem
find $\mathcal{U}_h^1-\mathcal{U}_h^2\in\mathbf{X}^h$ *such that*

$$\mathcal{B}^S_{-h}(\mathcal{U}_h^1-\mathcal{U}_h^2,\mathcal{W}_h) = \mathcal{D}\mathcal{B}_{-h}[\mathcal{U}_h(\lambda)](\hat{\mathcal{U}}_h,\mathcal{W}_h) - \mathcal{D}\mathcal{B}_{-h}[\mathcal{V}_h](\hat{\mathcal{U}}_h,\mathcal{W}_h), \quad \forall\mathcal{W}_h\in\mathbf{X}^h\,.$$

Existence and uniqueness of the solution of this problem follows as before by Lemma 4.2 (coercivity of the left-hand side) and Lemma 4.5 (continuity of the right-hand side). Moreover, these lemmas also show that

$$
\begin{aligned}
C\|\mathcal{U}_h^1-\mathcal{U}_h^2\|_{\mathbf{X}}^2 &\le \mathcal{B}^S_{-h}(\mathcal{U}_h^1-\mathcal{U}_h^2,\mathcal{U}_h^1-\mathcal{U}_h^2) \\
&= \mathcal{D}\mathcal{B}_{-h}[\mathcal{U}_h(\lambda)](\hat{\mathcal{U}}_h,\mathcal{U}_h^1-\mathcal{U}_h^2) - \mathcal{D}\mathcal{B}_{-h}[\mathcal{V}_h](\hat{\mathcal{U}}_h,\mathcal{U}_h^1-\mathcal{U}_h^2) \\
&\le C\|\mathcal{U}_h^1-\mathcal{U}_h^2\|_{\mathbf{X}}\|\hat{\mathcal{U}}_h\|_{\mathbf{X}}\|\mathcal{U}_h(\lambda)-\mathcal{V}_h\|_{\mathbf{X}}\left(1+\|\mathcal{U}_h(\lambda)\|_{\mathbf{X}}+\|\mathcal{V}_h\|_{\mathbf{X}}\right)\,.
\end{aligned}
$$

Since $\mathcal{V}_h \in S(\mathcal{U}_h(\lambda), \alpha)$, we have that

$$\|\mathcal{U}_h(\lambda) - \mathcal{V}_h\|_{\mathbf{X}} \leq \alpha$$

and

$$1 + \|\mathcal{U}_h(\lambda)\|_{\mathbf{X}} + \|\mathcal{V}_h\|_{\mathbf{X}} \leq 1 + \alpha + 2\|\mathcal{U}_h(\lambda)\|_{\mathbf{X}}.$$

Moreover, from **H-1 - H-3** it follows that $\|\mathcal{U}_h(\lambda)\|_{\mathbf{X}}$ can be bounded uniformly from above by $\|\mathcal{U}(\lambda)\|_{\mathbf{W}}$, i.e., the term $\|\mathcal{U}_h(\lambda)\|_{\mathbf{X}} + \|\mathcal{V}_h\|_{\mathbf{X}}$ is bounded uniformly in λ. Hence,

$$\|\mathcal{U}_h^1 - \mathcal{U}_h^2\|_{\mathbf{X}} \leq \alpha\delta\|\hat{\mathcal{U}}_h\|_{\mathbf{X}}\,.$$

Since $\|\cdot\|_{L(\mathbf{X}^h,\mathbf{X}^h)}$ is the supremum over $\hat{\mathcal{U}}_h \in \mathbf{X}^h$ with $\|\hat{\mathcal{U}}_h\|_{\mathbf{X}} = 1$, then

$$L_h(\lambda,\alpha) = \sup_{\mathcal{V}_h \in S(\mathcal{U}_h(\lambda),\alpha)} \|D_{\mathcal{U}}F(\lambda,\mathcal{U}_h(\lambda)) - D_{\mathcal{U}}F(\lambda,\mathcal{V}_h)\|_{L(\mathbf{X}^h,\mathbf{X}^h)} \leq \alpha C, \qquad \forall \lambda \in \Lambda\,,$$

which proves **A-3**. Thus, problem (3.4) has a regular branch of solutions that is unique in the ball $S(\mathcal{U}_h(\lambda), \alpha)$.

To complete the proof of the theorem, it remains to prove error estimate (4.29). Using (4.25) with $\mathcal{U}_h(\lambda)$, we obtain from (4.30) that

$$\|\tilde{\mathcal{U}}_h(\lambda) - \mathcal{U}_h(\lambda)\|_{\mathbf{X}} \leq C\|F_{-h}(\lambda,\mathcal{U}_h(\lambda))\|_{\mathbf{X}} = \epsilon_h(\lambda) \leq C\delta h^{\tilde{d}}\,.$$

The error estimate now follows by observing that δ depends only on λ and $\|\mathcal{U}(\lambda)\|_{\mathbf{W}}$. □

5. Numerical examples. In this section, we report on some numerical results obtained with the negative norm least-squares method of §3. For a more detailed computational study of this method, as well as a discussion of implementation details, we refer to [1]. For simplicity, the negative norm method has been implemented using equal order interpolation for all variables. In particular, to construct $\mathbf{X}^h$, we used a bilinear finite element space defined with respect to a uniform partition of Ω into rectangles.

In our first experiment, we consider two smooth solutions given by

$$\begin{aligned} u_1 &= \exp(\pi y)(x\cos(\pi x) - \sin(\pi x)/\pi)/2 \\ u_2 &= \exp(\pi y)\sin(\pi x)x/2 \\ p &= \exp(\pi y)\cos(\pi x) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} u_1 &= \exp(x)\cos(y) + \sin(y) \\ u_2 &= -\exp(x)\sin(y) + 1 - x^3 \\ p &= \sin(y)\cos(x) + xy^2 - 1/6 - \sin(1)(1-\cos(1)). \end{aligned} \tag{5.2}$$

In both cases, Ω is taken to be the unit square in $\mathbf{R}^2$, $\underline{\mathbf{U}}$ is set equal to $\nabla\mathbf{u}^t$, and the data is computed by evaluating equations (1.1)-(1.3) at the given exact solution. The main objective of this experiment is to verify computationally the theoretical error

TABLE 5.1
Convergence rates of the negative norm least-squares method.

	L^2 error rates		H^1 error rates	
Variable	Example (5.1)	Example (5.2)	Example (5.1)	Example (5.2)
$\mathbf{u}$	1.79	1.70	**1.03**	**1.02**
$\underline{\mathbf{U}}$	**1.50**	**1.44**	0.59	0.60
p	**1.10**	**1.16**	0.53	0.55

estimates in Theorem 4.7. In view of the choice of the space $\mathbf{X}^h$, we expect that (4.29) will hold with $\tilde{d} = 1$, that is,

$$\|\mathcal{U}_h(\lambda) - \mathcal{U}(\lambda)\|_{\mathbf{X}} \le Ch\,.$$

To estimate convergence rates numerically, computations were carried out using grids with 16x16 and 32x32 square cells. These results are reported in Table 1. Here bold face symbols are used to denote the rates of the errors included in (4.29). Note that these observed rates are at least as good as the rate $\tilde{d} = 1$ predicted by Theorem 4.7.

For the second experiment, we consider the two-dimensional driven cavity flow. For this flow, Ω is the unit square in $\mathbf{R}^2$, $\mathbf{f} = \mathbf{0}$, and the velocity boundary condition (1.4) is given by

$$\mathbf{u} = \begin{cases} (1,0)^t & \text{for } y = 1 \\ \mathbf{0} & \text{otherwise} \end{cases}\,.$$

Computations were carried out for $Re = 100$ using grids with 20x20 and 32x32 square cells. In Fig. 1, we compare velocity profiles computed using the negative norm method with the benchmark results of [7]. Profiles of the first velocity component (denoted by u in Fig.1) are measured along the line $x = 0.5$, whereas profiles of the second component (denoted by v in Fig.1) are measured along the horizontal line $y = 0.5$. Although grids employed in our computations are relatively coarse, results obtained for the 32x32 case show good agreement with the benchmark data. One can also expect that the use of nonuniform grids which cluster the points along the top wall of the cavity will yield even better results.

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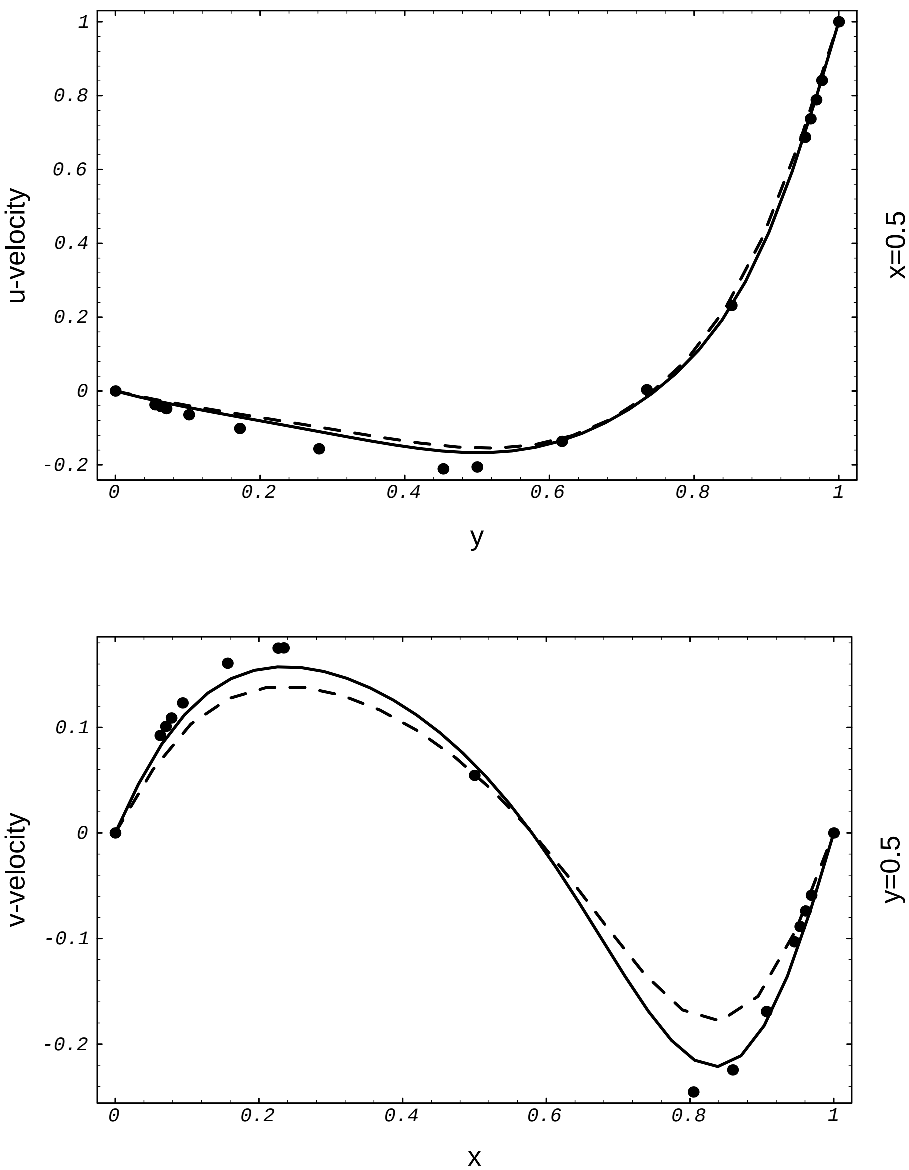

Fig. 5.1. *Velocity profiles: negative norm least-squares for 20x20 (dashed line) and 32x32 (solid line) vs. benchmark results of [7] (dots).*

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