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COMPATIBLE DISCRETIZATIONS OF SECOND–ORDER ELLIPTIC PROBLEMS

ABSTRACT. Differential forms provide a powerful abstraction tool to encode the structure of many partial differential equation problems. Discrete differential forms offer the same possibility with regard to compatible discretizations of these problems, i.e., for finite-dimensional models that exhibit similar conservation properties and invariants. We consider the application of a discrete exterior calculus to the approximation of second-order elliptic boundary-value problems. We show that there exist three possible discretization patterns. In the context of finite element methods, two of these lead to familiar classes of discrete problems, while the third offers a novel perspective about least-squares variational principles, namely how they can arise from particular choices for discrete Hodge–* operators.

**Dedicated to the memory
of Olga A. Ladyzhenskaya**

1. Introduction

Partial differential equations (PDEs) are a fundamental modeling tool in science and engineering. With the exception of few special cases, their exact analytical solution is difficult, if not impossible to obtain. As a result, the approximate numerical solution of PDEs is a task of tremendous practical importance. A key ingredient in this task is discretization. Discretization is a model reduction process wherein a system of differential equations, posed in infinite-dimensional spaces, is replaced by a finite-dimensional algebraic model that can be solved on a digital computer. Over the last few decades, discretizations evolved along three separate and seemingly distinct routes which relied on different types of reduction ideas to define the algebraic equations.

Finite difference quotient methods replace differential operators by difference operators acting upon grid functions. Finite volume methods

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reduce an equivalent integral form of the equations to an algebraic system by breaking down the computational domain into a finite number of cells (volumes) and approximating the boundary fluxes by polynomial expressions. Finally, finite element methods transform the PDE into an equivalent variational equation and then restrict this equation to a finite-dimensional function space. While in some cases all three methods lead to exactly the same reduced model, historically adherents of each discretization approach placed the main focus of their research on the distinctions rather than on the similarities between the three.

Recently, researchers have focused more attention on the fact that regardless of which particular reduction approach is used, stable and accurate discretizations of many PDE problems share more than an accidental similarity with each other. This similarity becomes even more obvious when one adopts the formalism of exterior calculus and differential forms and uses them to express the PDE equation problem. Then, one quickly realizes that successful discretizations of the problem try to mimic the abstract structure revealed by differential forms by using a parallel system of discrete differential forms.

Bossavit [6–8] was among the first to recognize the importance of these connections in the context of computational electromagnetics. His work spurred further research activity that revealed fundamental connections between, e.g., stability conditions in mixed methods [11–13] and discrete differential form structures [2, 4] and led to the further development of tools for and understanding of compatibility in discretization processes [5, 10, 15, 20, 26]. Similar developments occurred in finite difference methods [17–19] and finite volume methods [21, 22].

In this paper, we consider a general framework for compatible discretizations of second-order elliptic PDEs. Our starting point is a factorization diagram (see [6, 16, 27, 28] among others) that breaks down the second-order PDE into a set of metric-independent primal and dual *equilibrium* relations augmented by a set of metric-dependent *constitutive* laws. To solve the PDE problem, we consider a four-field constrained optimization approach in which the error in the constitutive laws is minimized subject to the equilibrium equations. The optimization problem is discretized using an exact sequence of finite-dimensional spaces. We show that, depending on the choices made for the approximation of the various fields, one recovers many familiar methods. Included among these are a mixed-type and Ritz-like finite element methods that can be associated with the use of the constitutive laws and/or one of the equilibrium equations to eliminate the primal or dual variables.

Our main focus is on another kind of formulation where the elimination of variables in the four-field problem is effected by using the equilibrium equations alone. Under certain assumptions on the equilibrium equations, this elimination method reduces the four-field constrained optimization problem to a two-field *unconstrained* minimization problem. The latter involves one primal and one dual variable and can be shown to combine the best properties of the other elimination methods. Its discretization also leads to symmetric and positive definite algebraic systems that are easier to solve than some of the systems arising with the other elimination choices. A remarkable fact about the reduced two-field problem is that it remains meaningful even if some of the assumptions necessary for the initial elimination process are not satisfied.

The idea of using constitutive error minimization was perhaps first suggested by Bossavit [6] for magnetostatics problems; see also [1] for a similar approach applied to the full Maxwell equations using vector potentials. It was further developed in the magnetostatics setting in [9] and analyzed for the two-dimensional case in [23]. These works focus on the solution of the constrained optimization problem by Lagrange multipliers rather than elimination, and consider the field-based formulation as being separate from mixed and Ritz-like methods. The approach adopted in this paper allows us to show that minimization of the constitutive error is in fact the most general variational setting that, depending on the discretization choices, defaults to three basic schemes – a mixed type, a Ritz type, and a least-squares type.

2. Quotation of results

2.1. Exterior and differential forms.

We recall [3] that, in n dimensions, an exterior form of degree k, $k \leqslant n$, or a $k-$form, is a mapping

$$\omega : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k} \mapsto \mathbb{R}$$

that is $k-$linear and antisymmetric:

$$\omega(\ldots, \mu\boldsymbol{\xi}' + \nu\boldsymbol{\xi}'', \ldots) = \mu\omega(\ldots, \boldsymbol{\xi}', \ldots) + \nu\omega(\ldots, \boldsymbol{\xi}'', \ldots)$$

and

$$\omega(\boldsymbol{\xi}_{i_1}, \ldots, \boldsymbol{\xi}_{i_k}) = (-1)^{\nu}, \quad \text{where} \quad \nu = \begin{cases} 0 \text{ if } & (i_1, i_2, \ldots, i_k) \text{ is even} \\ 1 \text{ if } & (i_1, i_2, \ldots, i_k) \text{ is odd.} \end{cases}$$

The set of all $k-$forms is a linear space with dimension $\binom{n}{k}$.

Exterior forms operate on groups of vectors in Euclidean space. Differential forms do the same thing, except that they draw their arguments from tangent spaces TM to differentiable manifolds M. Locally, a differentiable $k-$manifold embedded in $\mathbb{R}^n$ is defined by the $n-k$ equations

$$f_i(\mathbf{x}) = 0 \quad i = 1, \ldots, n-k\,,$$

where $f_1, \ldots, f_{n-k} : \mathbb{R}^n \mapsto \mathbb{R}$ are smooth functions such that ∇f_i are linearly independent. The tangent space at the point $\mathbf{x} \in M$ is defined by

$$TM_{\mathbf{x}} = (\mathrm{span}\{\nabla f_1, \ldots, \nabla f_{n-k}\})^{\perp}.$$

The union of all tangent spaces $TM = \cup_{\mathbf{x}\in M} TM_{\mathbf{x}}$ is called the *tangent bundle* of M. In what follows, we restrict attention to $M = \mathbb{R}^n$. A differential $k-$form $\omega_k|_{\mathbf{x}}$ at a point $\mathbf{x} \in M$ is an exterior k-form on $TM_{\mathbf{x}}$. If such a form ω_k is given at every point $\mathbf{x} \in M$ and if it is differentiable, it is called $k-$form on M. A differential 0–form is a function on M. The set of all k–forms on M will be denoted by $W^k(M)$.

We recall the operators $\wedge : W^k \times W^l \mapsto W^{k+l}$ for $k + l \leqslant n$ and $d : W^k \mapsto W^{k+1}$ for $k = 0, 1, \ldots, n-1$ with the properties that

$$\omega_k \wedge \omega_l = (-1)^{kl} \omega_l \wedge \omega_k\,,$$

$$dd\,\omega = 0\,,$$

and, if $k + l < n$,

$$d(\omega_k \wedge \omega_l) = (d\omega_k) \wedge \omega_l + (-1)^k \omega_k \wedge (d\omega_l)\,. \tag{1}$$

Integration of a $k-$form on a $k-$cell is another fundamental operation that illustrates the power of differential form abstraction. It allows the merging of the Newton-Leibniz theorem, the Stokes circulation theorem, and the Gauss divergence theorem into one simple, elegant formula:

$$\text{for any} \quad \omega \in W^k(M) \quad \text{and a k-cell C,} \quad \int_{\partial C} \omega = \int_C d\omega\,. \tag{2}$$

As a corollary to this theorem and (1), we have, for $k + l + 1 = n$, the integration by parts formula

$$\int_{\partial C} \omega_k \wedge \omega_l = \int_C (d\omega_k) \wedge \omega_l + (-1)^k \int_C \omega_k \wedge (d\omega_l)\,. \tag{3}$$

In what follows, we will use the symbol d^* to denote the "dual" operator

$$d^* = (-1)^{n-l} d \tag{4}$$

so that (3) takes the form

$$\int_{\partial C} \omega_k \wedge \omega_l = \int_C (d\omega_k) \wedge \omega_l - \int_C \omega_k \wedge (d^* \omega_l).$$

2.2. De Rham co-homology and the Hodge star operator.

Structures consisting of spaces and an operator $\mathcal{L}$ between them that has the property $\mathcal{L}\mathcal{L} \equiv 0$ are called homological complexes. The homological complex consisting of differential forms $W^k(M)$ and the operator d is called the *De Rham complex.* In $\mathbb{R}^3$, the De Rham complex contains the forms $W^0(M)$, $W^1(M)$, $W^2(M)$, and $W^3(M)$. If M is contractible or star-shaped, then the sequence

$$\mathbb{R} \hookrightarrow W^0(M) \stackrel{d}{\longmapsto} W^1(M) \stackrel{d}{\longmapsto} W^2(M) \stackrel{d}{\longmapsto} W^3(M) \longmapsto 0 \tag{5}$$

is *exact*, i.e., the closed forms in $W^0(M)$ are the constants and all closed forms in $W^k(M)$, $k = 1, 2, 3$, are differentials of $(k-1)$–forms. Therefore, for contractible and star-shaped regions, the Betti numbers are $b_0 = 1$ and $b_1 = b_2 = 0$; see [7].

Note that for any 3–form in $\mathbb{R}^3$, we have that $d\omega_3 = 0$. Therefore, $W^3(M)$ contains only closed forms. The last link in (5), i.e., $W^3(M) \longmapsto 0$, means that for regions without peculiarities, all forms in $W^3(M)$ are in fact differentials, i.e., they are exact forms. But this also means that d is a surjective map from $W^2(M)$ into $W^3(M)$.

For simplicity, in what follows we assume that $M \equiv \Omega \subset \mathbb{R}^3$ is contractible or star-shaped so that the De Rham complex (5) is exact. In three dimensions, every 1 and 2–form can be associated with a vector field according to

$$\omega_1^{\mathbf{u}} = \mathbf{u}_1(\mathbf{x}) dx_1 + \mathbf{u}_2(\mathbf{x}) dx_2 + \mathbf{u}_3(\mathbf{x}) dx_3$$

and

$$\omega_2^{\mathbf{u}} = \mathbf{u}_1(\mathbf{x}) dx_2 \wedge dx_3 + \mathbf{u}_2(\mathbf{x}) dx_3 \wedge dx_1 + \mathbf{u}_3(\mathbf{x}) dx_1 \wedge dx_2 \,,$$

respectively. If ϕ is a smooth function, then the 0–form ω_0^ϕ is the function ϕ itself, and the 3–form ω_3^ϕ is given by

$$\omega_3^\phi = \phi(\mathbf{x}) dx_1 \wedge dx_2 \wedge dx_3 \,.$$

Using these identifications, it is easy to see that

$$d\omega_0^{\phi} = \omega_1^{\nabla\phi}\,; \quad d\omega_1^{\mathbf{u}} = \omega_2^{\nabla\times\mathbf{u}}\,; \quad \text{and} \quad d\omega_2^{\mathbf{u}} = \omega_3^{\nabla\cdot\mathbf{u}}\,, \tag{6}$$

i.e., exterior differentiation of a $0,1,2-$form is equivalent to application of the gradient, curl, and divergence operators, respectively, to its associated scalar or vector field.

A manifold Ω can be endowed with a metric, i.e., a quadratic, positive definite, symmetric form. The properties of differential forms discussed so far do not depend on the choice of this metric. The Hodge–$*_k$ operator is a mapping $W^k(\Omega) \mapsto W^{n-k}(\Omega)$ that depends on the metric selection on Ω. For a precise definition of this operator, we refer to [25, p. 356]. For the Euclidean metric, the action of $*_k$ on the basic forms is given by (see [29, p. 30])

$$*_0 1 = dx_1 \wedge dx_2 \wedge dx_3\,;$$

$$*_1 dx_1 = dx_2 \wedge dx_3, \quad *_1 dx_2 = dx_3 \wedge dx_3, \quad *_1 dx_3 = dx_1 \wedge dx_2\,;$$

$$*_2(dx_1 \wedge dx_2) = dx_3, \quad *_2(dx_2 \wedge dx_3) = dx_1, \quad *_2(dx_3 \wedge dx_1) = dx_2\,;$$

and

$$*_3(dx_1 \wedge dx_2 \wedge dx_3) = 1\,.$$

To understand the action of this operator, it is convenient to think of k–forms as scalar or a vector function with attachments that can measure k–cells. Then, we can view the Hodge–* operator as a device that leaves the function unchanged but swaps its k-measure for an $n-k$–measure. Since the sum of the orders of a k–form and its image under the Hodge–* is always n, the Hodge operator can be used to define an inner product on $W^k(\Omega)$ according to

$$(\omega_k^1, \omega_k^2)_0 = \int_\Omega *_k \omega_k^1 \wedge \omega_k^2\,. \tag{7}$$

In this paper, we will consider forms whose associated scalar and vector fields are L^2 functions. With this in mind, we introduce the spaces

$$W^k(d,\Omega) = \{\omega \in W^k(\Omega) \ | \ \|\omega\|_0^2 + \|d\omega\|_0^2 < \infty\}\,,$$

where $\|\cdot\|_0$ is the norm induced by (7). In view of (6), it is clear that for $k = 0,1,2$, the spaces $W^k(d,\Omega)$ respectively coincide with the Sobolev spaces $H(\Omega, \mathbf{grad})$, $H(\Omega, \mathbf{curl})$, and (Ω, div) of square integrable functions whose gradient, curl, and divergence are also square integrable.

The space $W^3(d,\Omega)$ is simply $L^2_0(\Omega)$. Under the hypotheses on Ω, the sequence

$$\mathbb{R} \hookrightarrow H(\Omega, \mathbf{grad}) \stackrel{\nabla}{\longmapsto} H(\Omega, \mathbf{curl}) \stackrel{\nabla\times}{\longmapsto} H(\Omega, \mathrm{div}) \stackrel{\nabla\cdot}{\longmapsto} L^2_0(\Omega) \longmapsto 0 \quad (8)$$

is exact. We call (8) De Rham differential complex.

For PDEs, one has to account for various boundary conditions. We will consider a simple situation wherein the boundary of Ω consists of two smooth, disjoint pieces Γ and Γ^*. If we incorporate the boundary conditions in the spaces, the domains of the gradient, curl, and divergence, relative to Γ, are

$$\begin{aligned} H_0(\Omega, \mathbf{grad}) &= \{\phi \in H(\Omega, \mathbf{grad}) \mid \phi = 0 \quad \text{on} \quad \Gamma\} \\ H_0(\Omega, \mathbf{curl}) &= \{\mathbf{w} \in H(\Omega, \mathbf{curl}) \mid \mathbf{w} \times \mathbf{n} = 0 \quad \text{on} \quad \Gamma\} \\ H_0(\Omega, \mathrm{div}) &= \{\mathbf{w} \in H(\Omega, \mathrm{div}) \mid \mathbf{w} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma\}. \end{aligned}$$

These spaces define a De Rham differential complex relative to Γ:

$$\mathbb{R} \hookrightarrow H_0(\Omega, \mathbf{grad}) \stackrel{\nabla}{\longmapsto} H_0(\Omega, \mathbf{curl}) \stackrel{\nabla\times}{\longmapsto} H_0(\Omega, \mathrm{div}) \stackrel{\nabla\cdot}{\longmapsto} L^2_0(\Omega) \longmapsto 0\,. \quad (9)$$

The dual De Rham complex is defined relative to Γ^* and involves the "dual" operators $\nabla^* = -\nabla\cdot$, $(\nabla\times)^* = \nabla\times$ and $(\nabla\cdot)^* = -\nabla$. The domains of these operators are the Sobolev spaces $H^*_0(\Omega, \mathrm{div})$, $H^*_0(\Omega, \mathbf{curl})$, and $H^*_0(\Omega, \mathbf{grad})$, where, e.g. $H^*_0(\Omega, \mathbf{grad}) = \{\phi \in H(\Omega, \mathbf{grad}) \mid \phi = 0 \quad \text{on} \quad \Gamma^*\}$.

The De Rham complex (9) and its dual form exact sequences. We note for further reference that the last links in these two complexes are surjective. In what follows, we will use the simpler notation $W^k(d,\Omega)$ and $W^{k,*}(d^*,\Omega)$ to denote the spaces in the De Rham complex and its dual. Here, d^* is the operator defined in (4).

2.3. Factorization diagrams.

Consider the exact sequence $\{W^k(d,\Omega)\}$ and its dual $\{W^{k,*}(d^*,\Omega)\}$. The Hodge–$*$ operator connects the two sequences into the following structure:

$$\begin{array}{ccccccc} W^0(d,\Omega) & \stackrel{d}{\longrightarrow} & W^1(d,\Omega) & \stackrel{d}{\longrightarrow} & W^2(d,\Omega) & \stackrel{d}{\longrightarrow} & W^3(d,\Omega) \\ *_0 \downarrow & & *_1 \downarrow & & *_2 \downarrow & & *_3 \downarrow \\ W^{3,*}(d^*,\Omega) & \stackrel{d^*}{\longleftarrow} & W^{2,*}(d^*,\Omega) & \stackrel{d^*}{\longleftarrow} & W^{1,*}(d^*,\Omega) & \stackrel{d^*}{\longleftarrow} & W^{0,*}(d^*,\Omega) \end{array} \quad (10)$$

We will call (10) a *primal-dual* complex. For simplicity, we will use bold face roman letters to denote forms from the primal complex and bold face Greek letters for the forms in the dual complex.

The primal-dual complex can be traversed along its horizontal and vertical links, which correspond to the application of the exterior derivative or a Hodge operator, respectively. In this way we can obtain different diagrams relating primal and dual forms. In this paper, we focus on the diagrams

$$\begin{array}{ccc} W^{k-1}(d,\Omega) & \xrightarrow{d} & W^k(d,\Omega) \\ *_{k-1}\downarrow & & *_k\downarrow \\ W^{n-(k-1),*}(d^*,\Omega) & \xleftarrow{d^*} & W^{n-k,*}(d^*,\Omega) \end{array} \tag{11}$$

where $k = 1, 2$, or 3. Each diagram in (11) represents a second-order elliptic boundary value problem by virtue of

$$\begin{array}{ccc} \mathbf{u} & \xrightarrow{d} & a\mathbf{v}\ \boxed{f} \\ *_{k-1}\downarrow & & \downarrow *_k \\ \boxed{\xi}\ -\alpha\boldsymbol{\mu} & \xleftarrow{d^*} & \boldsymbol{\nu} \end{array} \tag{12}$$

where $(\mathbf{u},\mathbf{v}) \in W^{k-1}(d,\Omega) \times W^k(d,\Omega)$ and $(\boldsymbol{\mu},\boldsymbol{\nu}) \in W^{n-k+1,*}(d^*,\Omega) \times W^{n-k,*}(d^*,\Omega)$ are the primal and dual variables, respectively, f and ξ are additive data functions, and a and α are multiplicative, non-negative data functions with at least one of them being positive. Homogeneous boundary conditions on the primal and dual variables are already included in the definition of the spaces and need not appear explicitly in (12). The use of constructs like (11) or (12) to represent physical models was pioneered by Tonti [27, 28] and further popularized by Bossavit [6], Hiptmair [16], and many others. For further details and more examples, we refer to [7, 16, 20, 26].

Remark 1. A diagram such as (12) reflects the fact that, in many cases, a physical quantity can be described in two complementary ways. Consider, for example, the flow of a fluid in a pipe. We can either measure the velocity of the fluid along a line or measure its flux across a cross-section of the pipe. In the first case, we are using a 1–form description for the flow, while the second case describes the same phenomena using a 2-form. Then, the Hodge-* operator is the facility that enables conversion between the two descriptions. When setting up a model to describe the flow, we use the description that is consistent with the kind of balance we want to express. If the flow is irrotational, we use the 1-form of the velocity to relate it to the pressure gradient. To express mass conservation, we switch to a 2-form description.

Remark 2. Which set of variables and spaces are labeled as being primal and which are labeled as being dual is almost always a matter of arbitrary choice, i.e., one can just as well reverse their roles.

A fundamental property of (11) and (12) is that they separate topological from metric relationships. The primal and dual equations

$$d\mathbf{u} = a\mathbf{v} + f \quad \text{and} \quad d^*\boldsymbol{\nu} = -\alpha\boldsymbol{\mu} + \xi$$

express two independent topological "equilibrium" relations while the metric-dependent properties of the problem are completely isolated in the two "constitutive" equations

$$\boldsymbol{\mu} = *_{k-1}\mathbf{u} \quad \text{and} \quad \boldsymbol{\nu} = *_k\mathbf{v}$$

that involve Hodge-* operators; see [16, 24].

3. Compatible discretizations

3.1. Discrete exact sequences.

Factorization diagrams are a succinct tool for representing the PDE structure. *Compatible* discretization can be viewed as a process wherein one attempts to create a finite-dimensional analogue of a factorization diagram. Such model reduction requires two principal ingredients:

1. two sets $\{W_h^k(d_h,\Omega)\}$ and $\{W_h^{k,*}(d_h^*,\Omega)\}$ of *primal* and *dual* discrete differential forms that are exact with respect to the operations d_h and d_h^*;
2. discrete Hodge operators
$$*_k^h : W_h^k(d_h,\Omega) \mapsto W_h^{n-k,*}(d_h^*,\Omega)$$

that connect the primal and dual spaces into a discrete primal-dual complex

$$\begin{array}{ccccccc} W_h^0(d_h,\Omega) & \xrightarrow{d_h} & W_h^1(d_h,\Omega) & \xrightarrow{d_h} & W_h^2(d_h,\Omega) & \xrightarrow{d_h} & W_h^3(d_h,\Omega) \\ *_0^h\downarrow & & *_1^h\downarrow & & *_2^h\downarrow & & *_3^h\downarrow \\ W_h^{3,*}(d_h^*,\Omega) & \xleftarrow{d_h^*} & W_h^{2,*}(d_h^*,\Omega) & \xleftarrow{d_h^*} & W_h^{1,*}(d_h^*,\Omega) & \xleftarrow{d_h^*} & W_h^{0,*}(d_h^*,\Omega) \end{array} \tag{13}$$

Each one of the primal and dual sequences in (13) is a model for a "discrete vector calculus" and offers discrete analogues of, e.g., the Stokes formula (2). Let n_k and n_k^* denote the dimensions of $W_h^k(d_h,\Omega)$ and $W_h^{k,*}(d_h^*,\Omega)$, respectively. Since $*_k^h$ is a mapping between two finite-dimensional spaces, it must be expressible in the form of a matrix relation. Hiptmair [16] pointed out that a generic form of this operator is given by (see also [24])

$$\mathbb{M}\omega_k^h = \mathbb{K}\omega_{n-k}^{h,*}\,, \tag{14}$$

where $\mathbb{M}$ is $n_k \times n_k$ symmetric "mass" matrix and $\mathbb{K}$ is a rectangular $n_k \times n_{n-k}^*$ matrix.

Remark 3. We will say that $W_h^k(d_h,\Omega)$ provides an *internal approximation* of $W^k(d,\Omega)$ if it contains discrete k-forms. This notion is weaker than, e.g., the notion of a conforming finite element space which requires $W_h^k(d_h,\Omega)$ to be a subspace of the *function* space $W^k(d,\Omega)$. In contrast, $W_h^k(d_h,\Omega)$ can be an internal approximation without even having to be a function space. This situation occurs in, e.g., mimetic methods [17, 18] and co-volume schemes [20–22], where $W_h^k(d_h,\Omega)$ is represented by co-chains. The values of such objects are associated with particular mesh locations; an additional reconstruction operator must be defined to obtain their values at other locations. For comparison, for finite element methods the reconstruction operator is built into the definition of $W_h^k(d_h,\Omega)$ by virtue of a shape function choice.

3.2. Discrete factorization diagrams.

Once a discrete primal-dual complex is available, a compatible discretization of a problem represented by (12) can be defined by its restriction to the discrete complex (13):

$$\begin{array}{ccc} \mathbf{u}^h & \xrightarrow{d_h} & a\mathbf{v}^h\,\boxed{f^h} \\ *_{k-1}^h \downarrow & & \downarrow *_k^h \\ \boxed{\xi^h}\,-\alpha\boldsymbol{\mu}^h & \xleftarrow{d_h^*} & \boldsymbol{\nu}^h \end{array} \tag{15}$$

The diagram (15) is a generic discretization template that maps the PDE structure onto discrete spaces. An important feature of this template is that the discrete equilibrium equations will be satisfied exactly as long as $W_h^k(d_h,\Omega)$ and $W_h^{k,*}(d_h^*,\Omega)$ are exact with respect to d_h and d_h^*, respectively. This means that any discretization based on (15) will also imitate the conservation properties that are inherent to (12). However, this situation describes one possibility among many for the realization of (15). Practical considerations may dictate other choices for the components in (15), leading to methods with very different properties.

3.3. An optimization approach to compatibility.

One of the main strengths of finite element methods is rooted in their reliance on variational principles. This in turn opens up a possibility to draw upon the rich foundations of functional analysis and operator theory when studying finite element methods. As a result, finite element theory has been extremely strong in all matters related to metric aspects of discretizations such as error estimates, convergence, and stability. Finite element theory has been less successful in providing constructive means to determine what spaces should be used in a discretization so as to satisfy various compatibility conditions.

The situation with the use of structures like (15) is exactly the opposite. They provide excellent means to identify the appropriate discrete spaces that depend on the topological relations between the fields but are less successful in assessing the accuracy and asymptotic behavior of the discrete models.

Since variational and geometric approaches complement each other so nicely, it is only natural to try to apply them both in the discretization process. Our plan is to do exactly this by using (12) in conjunction with constrained optimization to derive the discrete diagrams (15). To this end, recall that (12) represents the following boundary value problem:

$$\begin{aligned} d\mathbf{u} &= a\mathbf{v} + f \quad \boldsymbol{\mu} = *_{k-1}\mathbf{u} \\ d^*\boldsymbol{\nu} &= -\alpha\boldsymbol{\mu} + \xi \quad \boldsymbol{\nu} = *_k\mathbf{v}\,. \end{aligned} \tag{16}$$

We cast (16) into a constrained optimization problem by minimizing the error in the two constitutive laws subject to the two equilibrium equations. Thus, we consider the quadratic functional

$$J(\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}; \widetilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{\nu}}) = \frac{1}{2}\left(\|\widetilde{\boldsymbol{\nu}} - *_k\widetilde{\mathbf{v}}\|_0^2 + \|\widetilde{\boldsymbol{\mu}} - *_{k-1}\widetilde{\mathbf{u}}\|_0^2\right)$$

and the optimization problem:

$$\begin{gathered} \textit{find } (\mathbf{u},\mathbf{v}) \in W^{k-1}(d,\Omega)\times W^k(d,\Omega) \textit{and } (\boldsymbol{\mu},\boldsymbol{\nu}) \in W^{n-k,*}(d^*,\Omega) \\ \times W^{n-k+1,*}(d^*,\Omega) \quad \textit{such that for all} \\ (\widetilde{\mathbf{u}},\widetilde{\mathbf{v}}) \in W^{k-1}(d,\Omega)\times W^k(d,\Omega) \\ \textit{and } (\widetilde{\boldsymbol{\mu}},\widetilde{\boldsymbol{\nu}}) \in W^{n-k,*}(d^*,\Omega)\times W^{n-k+1,*}(d^*,\Omega) \end{gathered}$$

$$\begin{cases} J(\mathbf{u},\mathbf{v};\boldsymbol{\mu},\boldsymbol{\nu}) \leqslant J(\widetilde{\mathbf{u}},\widetilde{\mathbf{v}};\widetilde{\boldsymbol{\mu}},\widetilde{\boldsymbol{\nu}}) \\ \textit{subject to} \quad d\widetilde{\mathbf{u}} = a\widetilde{\mathbf{v}} + f \quad \textit{and} \quad d^*\widetilde{\boldsymbol{\nu}} = -\alpha\widetilde{\boldsymbol{\mu}} + \xi\,. \end{cases} \tag{17}$$

It is easy to see that any solution of (16) solves (17) and vice versa. We will develop compatible discretizations by restricting this optimization problem to suitable finite-dimensional spaces and making various choices for the Hodge−∗ operator.

4. Realizations of compatible discretizations

To reduce (16) to an algebraic model, we select the components in (13) and then restrict (17) to this structure. There are two basic templates that exist in this process. The first leads to what we call *primal* and *dual* formulations. In this case, (13) provides an internal approximation to only one of the complexes in (10). The second template leads to a class of *primal–dual* formulations. In this case, (13) uses internal approximations to both complexes in (10). Our main focus will be on a class of primal–dual methods; however, to demonstrate the capacity of (17) to serve as a general framework for compatible discretizations, we briefly examine the primal and dual reduction patterns.

4.1. Primal and dual methods.

4.1.1. Primal methods.

We consider an exact sequence $\{W_h^k(d_h,\Omega)\}$, where

$$d_h : W_h^k(d_h,\Omega) \mapsto W_h^{k+1}(d_h,\Omega); \quad d_h d_h = 0$$

is a set of *natural* exterior derivative operators. We assume that there exists a dual set $\{(d_h)^*\}$ of discretely defined *adjoint*[1] operators

$$(d_h)^* : W_h^{k+1}(d_h,\Omega) \mapsto W_h^k(d_h,\Omega); \quad (d_h)^*(d_h)^* = 0\,.$$

To define a primal[2] method, we approximate the primal variables $(\mathbf{u},\mathbf{v})$ *internally* by $(\mathbf{u}^h,\mathbf{v}^h) \in W_h^{k-1}(d_h,\Omega) \times W_h^k(d_h,\Omega)$. The dual variables $(\boldsymbol{\mu},\boldsymbol{\nu})$ are approximated *externally* by the same pair of discrete spaces, i.e., we use $(\boldsymbol{\mu}^h,\boldsymbol{\nu}^h) \in W_h^{k-1}(d_h,\Omega) \times W_h^k(d_h,\Omega)$.

[1]By this we mean that the adjoint operator $(d_h)^*$ is defined with respect to an appropriate discrete inner product. The terminology "natural" and "adjoint" are adopted from [17, 18].

[2]Again, we note that the designation of what is a primal method and what is a dual method is somewhat arbitrary.

Remark 4. One immediate consequence from this choice is that any boundary conditions imposed on the dual pair cannot be enforced strongly, e.g., by setting the degrees of freedom residing on Γ^* to values prescribed by the boundary data.

The choice of the same spaces for the primal and the dual variables implies that (14) reduces to

$$\boldsymbol{\mu}^h = \mathbb{I}_\beta \mathbf{u}^h \quad \text{and} \quad \boldsymbol{\nu}^h = \mathbb{I}_b \mathbf{v}^h \,, \tag{18}$$

where $\mathbb{I}_\beta$ and $\mathbb{I}_b$ are scaled identity matrices, i.e., diagonal matrices, and $\beta > 0$ and $b > 0$ are "constitutive" functions, e.g., material constants. Therefore, (17) specializes to the following *primal* discrete optimization problem:

$$\textit{find} \;\; (\mathbf{u}^h, \mathbf{v}^h) \in W_h^{k-1}(d_h, \Omega) \times W_h^k(d_h, \Omega)$$

$$\textit{and} \;\; (\boldsymbol{\mu}^h, \boldsymbol{\nu}^h) \in W_h^{k-1}(d_h, \Omega) \times W_h^k(d_h, \Omega) \;\; \textit{that}$$

$$\begin{cases} \textit{minimize} \quad J(\widetilde{\mathbf{u}}^h, \widetilde{\mathbf{v}}^h; \widetilde{\boldsymbol{\mu}}^h, \widetilde{\boldsymbol{\nu}}^h) = \dfrac{1}{2}\left(\|\widetilde{\boldsymbol{\mu}}^h - \mathbb{I}_\beta \widetilde{\mathbf{u}}^h\|_0^2 + \|\widetilde{\boldsymbol{\nu}}^h - \mathbb{I}_b \widetilde{\mathbf{v}}^h\|_0^2\right) \\ \textit{subject to} \quad d_h \widetilde{\mathbf{u}}^h = a \widetilde{\mathbf{v}}^h + f^h \quad \textit{and} \quad (d_h)^* \widetilde{\boldsymbol{\nu}}^h = -\alpha \widetilde{\boldsymbol{\mu}}^h + \xi^h . \end{cases} \tag{19}$$

Let $(\mathbf{u}^h, \mathbf{v}^h)$ and $(\boldsymbol{\mu}^h, \boldsymbol{\nu}^h)$ solve (19). It is easy to see that the minimum of the cost functional will be achieved when (18) is satisfied. Substituting (18) into the second constraint (the dual equilibrium equation) yields

$$(d_h)_h^* \mathbb{I}_b \mathbf{v}^h = -\alpha \mathbb{I}_\beta \mathbf{u}^h + \xi^h . \tag{20}$$

This relation along with primal equation

$$d_h \mathbf{u}^h = a \mathbf{v}^h + f^h \tag{21}$$

serve to define the approximations of the primal variables $\mathbf{u}^h$ and $\mathbf{v}^h$.

If $a \neq 0$, then we can use the primal equation (21) to eliminate $\mathbf{v}^h$ and to obtain a single second-order equation in terms of $\mathbf{u}^h$ only:

$$(d_h)_h^* \mathbb{I}_b d_h \mathbf{u}^h + a\alpha \mathbb{I}_\beta \mathbf{u}^h = a\xi^h + (d_h)^* \mathbb{I}_b f^h . \tag{22}$$

In this case, (22) is an equivalent formulation of (19). If $a = 0$, then this reduced "Ritz-Galerkin" system cannot be obtained and one instead must solve the "mixed-Galerkin" system (20)–(21).

4.1.2. Dual methods.

We consider an exact sequence $\{W_h^{k,*}(d_h^*,\Omega)\}$, where

$$d_h^* : W_h^{k,*}(d_h^*,\Omega) \mapsto W_h^{k+1,*}(d_h^*,\Omega); \quad d_h^* d_h^* = 0$$

is a set of natural exterior derivative operators. We assume that there exists a dual set $\{(d_h^*)^*\}$ of adjoint operators

$$(d_h^*)^* : W_h^{k+1,*}(d_h^*,\Omega) \mapsto W_h^{k,*}(d_h^*,\Omega); \quad (d_h^*)^*(d_h^*)^* = 0\,.$$

Remark 5. In general, since they operate between different sets of spaces, the operators d_h and $(d_h)^*$ that help define the primal discrete problem and the operators d_h^* and $(d_h^*)^*$ that help define the dual discrete problem do not satisfy $d_h^* = (d_h)^*$ or $d_h = (d_h^*)^*$.

We define a dual method by choosing to approximate *internally* the dual variables $(\boldsymbol{\mu}^h,\boldsymbol{\nu}^h)$ by the pair $W^{n-k+1,*}(d_h^*,\Omega)\times W^{n-k}(d_h^*,\Omega)$. The primal variables $(\mathbf{u},\mathbf{v})$ are approximated *externally* by the same pair, i.e., we use $(\mathbf{u}^h,\mathbf{v}^h)\in W^{n-k+1,*}(d_h^*,\Omega)\times W^{n-k}(d_h^*,\Omega)$.

Remark 6. One immediate consequence from this choice is that any boundary conditions imposed on the primal pair cannot be enforced strongly, e.g., by setting the degrees of freedom residing on Γ to values prescribed by the boundary data.

For the dual methods, (14) specializes to

$$\mathbf{u}^h = \mathbb{I}_{\beta^{-1}}\boldsymbol{\mu}^h \quad \text{and} \quad \mathbf{v}^h = \mathbb{I}_{b^{-1}}\boldsymbol{\nu}^h, \tag{23}$$

where $\mathbb{I}_{\beta^{-1}}$ and $\mathbb{I}_{b^{-1}}$ are scaled identity matrices. The *dual* discrete optimization problem, derived from (17), is:

$$\begin{gathered} \textit{find} \quad (\boldsymbol{\mu}^h,\boldsymbol{\nu}^h)\in W_h^{n-k+1,*}(d_h^*,\Omega)\times W_h^{n-k,*}(d_h^*,\Omega) \quad \textit{and} \\ (\mathbf{u}^h,\mathbf{v}^h)\in W_h^{n-k+1,*}(d_h^*,\Omega)\times W_h^{n-k,*}(d_h^*,\Omega) \quad \textit{that} \end{gathered}$$

$$\begin{cases} \textit{minimize } J(\widetilde{\mathbf{u}}^h,\widetilde{\mathbf{v}}^h;\widetilde{\boldsymbol{\mu}}^h,\widetilde{\boldsymbol{\nu}}^h) = \dfrac{1}{2}\left(\|\mathbb{I}_{\beta^{-1}}\widetilde{\boldsymbol{\mu}}^h-\widetilde{\mathbf{u}}^h\|_0^2+\|\mathbb{I}_{b^{-1}}\widetilde{\boldsymbol{\nu}}^h-\widetilde{\mathbf{v}}^h\|_0^2\right) \\ \textit{subject to } (d_h^*)^*\widetilde{\mathbf{u}}^h = a\widetilde{\mathbf{v}}^h + f^h \textit{ and } d_h^*\widetilde{\boldsymbol{\nu}}^h = -\alpha\widetilde{\boldsymbol{\mu}}^h + \xi^h. \end{cases} \tag{24}$$

If $(\mathbf{u}^h,\mathbf{v}^h)$ and $(\boldsymbol{\mu}^h,\boldsymbol{\nu}^h)$ solve (19), it is easy to see that they must satisfy (23). We use those relations to eliminate the primal variables from the primal equation to obtain

$$(d_h^*)^*\mathbb{I}_{\beta^{-1}}\boldsymbol{\mu}^h = a\mathbb{I}_{b^{-1}}\boldsymbol{\nu}^h + f^h. \tag{25}$$

This relation along with dual equation

$$d_h^* \boldsymbol{\nu}^h = -\alpha \boldsymbol{\mu}^h + \xi^h \tag{26}$$

serve to define the approximations of the dual variables $\boldsymbol{\nu}^h$ and $\boldsymbol{\mu}^h$.

If $\alpha \neq 0$, then we can use the dual equation (26) to eliminate $\boldsymbol{\mu}^h$ and to obtain a single second-order equation in terms of $\boldsymbol{\nu}^h$ only:

$$(d_h^*)^* \mathbb{I}_{\beta^{-1}} d_h^* \boldsymbol{\nu}^h + a\alpha \mathbb{I}_{b^{-1}} \boldsymbol{\nu}^h = -\alpha f^h + (d_h^*)^* \mathbb{I}_{\beta^{-1}} \xi^h. \tag{27}$$

In this case, (27) is an equivalent formulation of (24). If $\alpha = 0$, then this reduced "Ritz-Galerkin" system cannot be obtained and one instead must solve the "mixed-Galerkin" system (25)–(26).

4.2. Primal-dual methods.

The methods defined in the last section utilize the same set of forms in the discrete complex (13). This set can provide an internal approximation either with respect to the primal or the dual variables, but not for both of them simultaneously. This is why in primal and dual methods we have to use a combination of natural and adjoint differential operators. The adjoint operators act as surrogates for the true exterior derivative for the set of variables that are not approximated internally.

In this section, we consider another class of methods defined by using internal approximations for both the primal and dual variables. In this case, the exterior derivative for both sides is represented by the natural differential operators and there is no need for using an adjoint operator. We will call such methods *primal–dual* to highlight the participation of both groups of variables.

To define primal–dual methods, we require a primal and a dual exact sequence, denoted by $\{W_h^k(d_h, \Omega)\}$ and $\{W_h^{k,*}(d_h^*, \Omega)\}$, respectively. We also have the primal

$$d_h : W_h^k(d_h, \Omega) \mapsto W_h^{k+1}(d_h, \Omega); \quad d_h d_h = 0$$

and the dual

$$d_h^* : W_h^{k,*}(d_h^*, \Omega) \mapsto W_h^{k+1,*}(d_h^*, \Omega); \quad d_h^* d_h^* = 0$$

sets of *natural* exterior derivative operators. The factor $(-1)^{n-l}$ is included in the definition of d_h^*.

Remark 7. The operators d_h and d_h^* now defined in the context of primal-dual methods do not necessarily need to be the same as the same-named operators used in the primal and dual methods, respectively.

In conjunction with a discrete Hodge operator, these two sequences lead to a discrete complex (13) that provides an internal approximation of the complex in (10). In the realization of (13), there are two distinct possibilities. The first is to use primal and dual sequences defined on topologically dual grids. For such grids there exists a one-to-one relationship between the primal $k-$cells and the dual $(n-k)-$cells and the spaces $W_h^k(d_h,\Omega)$ and $W^{(n-k),*}(d_h^*,\Omega)$ have the same dimension. As a result, the discrete Hodge operator defines an isomorphism between these spaces and (14) reduces to

$$\omega_{n-k}^{h,*} = \mathbb{M}\omega_k^h \,,$$

where $\mathbb{M}$ is a symmetric, positive definite matrix. This setting leads to some well-known finite volume and co-volume methods [14, 21, 22] and staggered finite difference schemes such as the Yee scheme [30].

Our focus, however, is on another class of primal-dual methods that exploits the use of *unrelated* sets of primal and dual spaces. In particular, one possibility afforded by this approach is to consider primal and dual spaces defined with respect to the same partition of Ω into $k-$cells. Since on a single grid it is impossible to have a one-to-one relationship between the $k-$ and the $(n-k)-$cells, the spaces $W_h^k(d_h,\Omega)$ and $W^{(n-k),*}(d_h^*,\Omega)$ defined on this grid cannot be isomorphic. In this case, the discrete Hodge operator is represented by a pair of matrices as in (14). However, the use of a single grid for both complexes offers numerous advantages and simplifications that, in our opinion, outweigh the disadvantage of not having an isomorphic discrete Hodge operator. Furthermore, instead of attempting to find explicit representations for the discrete Hodge operators, we will let the optimization process in (17) define the operators implicitly. Thus, to define this primal-dual method we consider the optimization problem:

$$\begin{gathered}
\textit{find} \quad (\mathbf{u}^h,\mathbf{v}^h)\in W_h^{k-1}(d_h,\Omega)\times W_h^k(d_h,\Omega) \quad \textit{and} \\
(\boldsymbol{\mu}^h,\boldsymbol{\nu}^h)\in W_h^{(n-k+1),*}(d_h^*,\Omega)\times W_h^{(n-k),*}(d_h^*,\Omega) \quad \textit{that}
\end{gathered}$$

$$\begin{cases}
\textit{minimize } J(\widetilde{\mathbf{u}}^h,\widetilde{\mathbf{v}}^h;\widetilde{\boldsymbol{\mu}}^h,\widetilde{\boldsymbol{\nu}}^h)=\dfrac{1}{2}\left(\|\alpha(\widetilde{\boldsymbol{\mu}}^h-\beta\widetilde{\mathbf{u}}^h)\|_0^2+\|a(b^{-1}\widetilde{\boldsymbol{\nu}}^h-\widetilde{\mathbf{v}}^h)\|_0^2\right)\\
\textit{subject to } d_h\widetilde{\mathbf{u}}^h=a\widetilde{\mathbf{v}}^h+f^h \textit{ and } d_h^*\widetilde{\boldsymbol{\nu}}^h=-\alpha\widetilde{\boldsymbol{\mu}}^h+\xi^h.
\end{cases} \tag{28}$$

The "constitutive" functions $\beta>0$ and $b>0$ insure, among other things, dimensional consistency in the functional $J(\cdot,\cdot;\cdot,\cdot)$. The appearance of the weight factors a and α is natural, given how the variables $\mathbf{v}^h$ and $\boldsymbol{\mu}^h$ appear in the constraint equations. We recall that in (28) both d_h and d_h^* are natural differentiation operators. To solve (28), one could use Lagrange multipliers to enforce the two constraints. A better and more efficient way to solve this problem is to take advantage of the fact that, thanks to the use of the natural operators, *both* constraint equations are exact.

First, consider the case for which both $a \neq 0$ and $\alpha \neq 0$. Let $(\mathbf{u}^h, \mathbf{v}^h)$ and $(\boldsymbol{\mu}^h, \boldsymbol{\nu}^h)$ solve (28). Then, we can eliminate $\mathbf{v}^h$ and $\boldsymbol{\mu}^h$ from (28) using the primal and the dual equations, respectively. The result is the *unconstrained* minimization problem in terms of one of the primal and one of the dual variables:

$$\textit{find } \mathbf{u}^h \in W_h^{k-1}(d_h, \Omega) \textit{ and } \boldsymbol{\nu}^h \in W_h^{(n-k),*}(d_h^*, \Omega) \textit{ that minimize}$$

$$J_{a\alpha}(\widetilde{\mathbf{u}}^h, \widetilde{\boldsymbol{\nu}}^h) = \frac{1}{2}\left(\|d_h^*\widetilde{\boldsymbol{\nu}}^h + \alpha\beta\widetilde{\mathbf{u}}^h - \xi^h\|_0^2 + \|d_h\widetilde{\mathbf{u}}^h - ab^{-1}\widetilde{\boldsymbol{\nu}}^h - f^h\|_0^2\right). \tag{29}$$

Next, consider the case[3] $a = 0$ and $\alpha > 0$. We can consider this case as corresponding to the limit, as $a \to 0$, of a sequence of problems with $a > 0$ and $\alpha > 0$. Since for positive values of a and α the *two-field* formulation (29) and the *four-field* formulation (28) are completely equivalent, we can derive formulations for $a = 0$ by taking the limit $a \to 0$ either in the original problem (28), or its reduced version (29). This, however, leads to two different compatible formulations that are *not equivalent* with each other. In other words, taking the limit $a \to 0$ does not commute with eliminating variables from (28). Indeed, starting from (29) and setting $a = 0$ gives the *unconstrained* optimization problem:

$$\textit{find } \mathbf{u}^h \in W_h^{k-1}(d_h, \Omega) \textit{ and } \boldsymbol{\nu}^h \in W_h^{(n-k),*}(d_h^*, \Omega) \textit{ that minimize}$$

$$J_{\alpha}(\widetilde{\mathbf{u}}^h, \widetilde{\boldsymbol{\nu}}^h) = \frac{1}{2}\left(\|d_h^*\widetilde{\boldsymbol{\nu}}^h + \alpha\beta\widetilde{\mathbf{u}}^h - \xi^h\|_0^2 + \|d_h\widetilde{\mathbf{u}}^h - f^h\|_0^2\right). \tag{30}$$

On the other hand, setting $a = 0$ in (28) leads to a problem where the primal variable $\mathbf{v}^h$ does not appear in the constraints or the functional $J(\cdot, \cdot; \cdot, \cdot)$. We can still use the second constraint equation to eliminate $\boldsymbol{\mu}^h$, however, there's nothing that can be done to rid the problem of the

[3] Recall that it has been assumed that not both a and α can vanish.

first equation which continues to act as a constraint. As a result, we are led to a simpler, but still constrained, *two-field* optimization problem

$$\textit{find } \mathbf{u}^h \in W_h^{k-1}(d_h,\Omega) \textit{ and } \boldsymbol{\nu}^h \in W_h^{(n-k),*}(d_h^*,\Omega) \textit{ that minimize}$$

$$J(\widetilde{\mathbf{u}}^h;\widetilde{\boldsymbol{\nu}}^h) = \frac{1}{2}\|d_h^*\widetilde{\boldsymbol{\nu}}^h + \alpha\beta\widetilde{\mathbf{u}}^h - \xi^h\|_0^2 \quad \textit{subject to} \quad d_h\widetilde{\mathbf{u}}^h = f^h. \tag{31}$$

Finally, consider the case $a > 0$ and $\alpha = 0$. Taking the limit $\alpha \to 0$ in (29) gives the unconstrained optimization problem

$$\textit{find } \mathbf{u}^h \in W_h^{k-1}(d_h,\Omega) \textit{ and } \boldsymbol{\nu}^h \in W_h^{(n-k),*}(d_h^*,\Omega) \textit{ that minimize}$$

$$J_a(\widetilde{\mathbf{u}}^h,\widetilde{\boldsymbol{\nu}}^h) = \frac{1}{2}\left(\|d_h^*\widetilde{\boldsymbol{\nu}}^h - \xi^h\|_0^2 + \|d_h\widetilde{\mathbf{u}}^h - ab^{-1}\widetilde{\boldsymbol{\nu}}^h - f^h\|_0^2\right). \tag{32}$$

Since (30) and (32) are the same as (29) specialized to the cases $a = 0$ and $\alpha = 0$, respectively, all three optimization problems can be cast in terms of the single functional (29).

When the limit $\alpha \to 0$ is taken in (28), the situation is exactly the opposite to what happened earlier: now the dual variable $\boldsymbol{\mu}^h$ does not appear in the constraints or the functional $J(\cdot,\cdot;\cdot,\cdot)$ and $\mathbf{v}^h$ can be eliminated from the first constraint equation. This gives another reduced, but constrained, optimization problem:

$$\textit{find } \mathbf{u}^h \in W_h^{k-1}(d_h,\Omega) \textit{ and } \boldsymbol{\nu}^h \in W_h^{(n-k),*}(d_h^*,\Omega) \textit{ that minimize}$$

$$J(\widetilde{\mathbf{u}}^h;\widetilde{\boldsymbol{\nu}}^h) = \frac{1}{2}\|d_h\widetilde{\mathbf{u}}^h - ab^{-1}\widetilde{\boldsymbol{\nu}}^h - f^h\|_0^2 \quad \textit{subject to} \quad d_h^*\widetilde{\boldsymbol{\nu}}^h = \xi^h. \tag{33}$$

There is an obvious symmetry between (31) and (33). Both problems use the same pair of variables but have different constraints and their cost functionals employ dual operators.

Regarding the solution of (29), using standard calculus of variations tools it is not hard to see that the first-order necessary condition for $(\mathbf{u}^h,\boldsymbol{\nu}^h) \in W_h^{k-1}(d_h,\Omega) \times W_h^{(n-k),*}(d_h,\Omega)$ to minimize (29) is given by the variational (Euler-Lagrange) equation:

$$\textit{find } \mathbf{u}^h \in W_h^{k-1}(d_h,\Omega) \textit{ and } \boldsymbol{\nu}^h \in W_h^{(n-k),*}(d_h^*,\Omega) \textit{ such that}$$

$$Q(\mathbf{u}^h,\boldsymbol{\nu}^h;\widetilde{\mathbf{u}}^h,\widetilde{\boldsymbol{\nu}}^h) = F(\widetilde{\mathbf{u}}^h,\widetilde{\boldsymbol{\nu}}^h) \tag{34}$$

$$\textit{for all } (\widetilde{\mathbf{u}}^h,\widetilde{\boldsymbol{\nu}}^h) \in W_h^{k-1}(d_h,\Omega) \times W_h^{(n-k),*}(d_h^*,\Omega), \textit{ where}$$

$$Q(\mathbf{u}^h,\boldsymbol{\nu}^h;\widetilde{\mathbf{u}}^h,\widetilde{\boldsymbol{\nu}}^h) = \left(d_h^*\boldsymbol{\nu}^h + \alpha\beta\mathbf{u}^h\ ,\ d_h^*\widetilde{\boldsymbol{\nu}}^h + \alpha\beta\widetilde{\mathbf{u}}^h\right)_0 + \left(d_h\mathbf{u}^h - ab^{-1}\boldsymbol{\nu}^h\ ,\ d_h\widetilde{\mathbf{u}}^h - ab^{-1}\widetilde{\boldsymbol{\nu}}^h\right)_0 \tag{35}$$

and

$$F(\widetilde{\mathbf{u}}^h,\widetilde{\boldsymbol{\nu}}^h) = \left(\xi^h\ ,\ d_h^*\widetilde{\boldsymbol{\nu}}^h + \alpha\beta\widetilde{\mathbf{u}}^h\right)_0 + \left(f^h\ ,\ d_h\widetilde{\mathbf{u}}^h - ab^{-1}\widetilde{\boldsymbol{\nu}}^h\right)_0 .$$

An analysis of (34) in the case of $k = 1$ and exact sequences of finite element spaces can be found in [4]. A formal analysis of this formulation in three dimensions and general discrete exact sequences is a subject of a forthcoming paper.

Here we proceed with examples that illustrate the various methods in the context of the eddy current subset of the Maxwell equations when the exact sequences are defined by finite element spaces. Among other things, the examples illustrate the attractive computational properties of formulations based on the primal-dual variational equation (34).

5. Examples

In [4], examples of primal, dual, and primal-dual finite element methods for a div-grad system (for which $k = 1$) are considered. Here, we consider the eddy current problem

$$\nabla\times\mathbf{H} = \mathbf{J} \quad \text{in } \Omega\times(0,T] \tag{36a}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \text{in} \quad \Omega\times(0,T] \tag{36b}$$

that is derived from the full set of the Maxwell equations by neglecting the displacement current. In (36a)–(36b), $\mathbf{H}$ denotes the magnetic field, $\mathbf{J}$ the current density, $\mathbf{E}$ the electric field, and $\mathbf{B}$ the magnetic flux density. Note that (36a) and (36b) imply

$$\nabla\cdot\mathbf{B} = 0 \qquad \text{in} \quad \Omega \tag{37a}$$

$$\nabla\cdot\mathbf{J} = 0 \qquad \text{in} \quad \Omega\,. \tag{37b}$$

Any initial values $\mathbf{B}_0$ for $\mathbf{B}$ is required to satisfy (37a). The system (36a)–(36b) is closed by adding the constitutive relations

$$\mathbf{B} = \mu\mathbf{H} \quad \text{and} \quad \mathbf{J} = \sigma\mathbf{E}\,, \tag{38}$$

where σ and μ denote the conductivity and permeability of the media, respectively, and the boundary conditions

$$\mathbf{n}\times\mathbf{H} = \mathbf{n}\times\mathbf{H}_{\mathrm{b}} \quad \text{or} \quad \mathbf{n}\cdot\mathbf{J} = \mathbf{n}\cdot\mathbf{J}_{\mathrm{b}} \quad \text{on} \quad \Gamma \tag{39}$$

$$\mathbf{n}\times\mathbf{E}=\mathbf{n}\times\mathbf{E}_{\mathrm{b}} \quad \text{or} \quad \mathbf{n}\cdot\mathbf{B}=\mathbf{n}\cdot\mathbf{B}_{\mathrm{b}} \quad \text{on} \quad \Gamma^* \tag{40}$$

posed on disjoint parts Γ and Γ^* of the boundary $\partial\Omega$. The pairs of boundary conditions in (39) and (40) are not independent and only one from each pair can be posed on Γ and Γ^*. Here we use

$$\mathbf{n}\times\mathbf{H}=\mathbf{n}\times\mathbf{H}_{\mathrm{b}} \quad \text{on} \quad \Gamma \quad \text{and} \quad \mathbf{n}\times\mathbf{E}=\mathbf{n}\times\mathbf{E}_{\mathrm{b}} \quad \text{on} \quad \Gamma^*. \tag{41}$$

We will apply the framework of the previous sections to a semidiscrete version of (36a)–(36b). For simplicity, assume that $[0,T]$ is divided into N subintervals of equal length $\Delta t = T/N$. The implicit Euler method to advance solution by one time step requires the solution of the elliptic problem

$$\nabla\times\mathbf{H}=\mathbf{J} \tag{42a}$$

$$\nabla\times\mathbf{E}=-\frac{1}{\Delta t}\mathbf{B}+\frac{1}{\Delta t}\widetilde{\mathbf{B}} \tag{42b}$$

along with (41). In (42b), $\widetilde{\mathbf{B}}$ is the magnetic flux at the beginning of the time step. Clearly, for the first time step, $\widetilde{\mathbf{B}}=\mathbf{B}_0$.

To apply the framework, we assume constant material properties, homogeneous boundary conditions, and make the following identifications. First, $k=2$, $a=1$, $f=0$, the primal variables are

$$\mathbf{u}=\mathbf{H}\in W^1(d,\Omega)=H_0(\Omega,\mathbf{curl}) \text{ and } \mathbf{v}=\mathbf{J}\in W^2(d,\Omega)=H_0(\Omega,\mathrm{div}), \tag{43}$$

and (42a) is the primal equilibrium equation. Second, $\alpha=1/\Delta t$, $\xi=\alpha\widetilde{\mathbf{B}}$, the dual variables are

$$\boldsymbol{\mu}=\mathbf{B}\in W^{2,*}(d,\Omega)=H_0^*(\Omega,\mathrm{div}) \text{ and } \boldsymbol{\nu}=\mathbf{E}\in W^{1,*}(d,\Omega)=H_0^*(\Omega,\mathbf{curl}), \tag{44}$$

and (42b) is the dual equation. Finally, (38) is the set of material laws where $\beta=\mu$ and $b=1/\sigma$.

For a primal method, we need to choose spaces[4] $W_h^1(d_h,\Omega)$ and $W_h^2(d_h,\Omega)$. For the sake of clarity, we will denote the operators d_h and $(d_h)^*$ by $\mathbb{C}$ and $\mathbb{C}^*$, respectively. Here $\mathbb{C}$ is the *natural* curl and $\mathbb{C}^*$ is its discretely defined adjoint curl. Since μ and σ are assumed constant, (22) specializes to

$$\frac{1}{\sigma}\mathbb{C}^*\mathbb{C}\mathbf{H}^h+\frac{\mu}{\Delta t}\mathbf{H}^h=\frac{1}{\Delta t}\widetilde{\mathbf{B}}^h. \tag{45}$$

[4] We recall that the notion of internal approximation, discussed in Remark 3, does not require these spaces to be proper subspaces of $W^1(d,\Omega)$ and $W^2(d,\Omega)$.

For a dual method, we need to choose spaces $W_h^{1,*}(d_h^*,\Omega)$ and $W_h^{2,*}(d_h^*,\Omega)$. For the sake of clarity, we will denote the operators d_h^* and $(d_h^*)^*$ by $\widetilde{\mathbb{C}}^*$ and $\widetilde{\mathbb{C}}$, respectively, where now $\widetilde{\mathbb{C}}^*$ is the natural curl and $\widetilde{\mathbb{C}}$ is its discrete adjoint curl. The dual problem (27) then specializes to

$$\frac{\Delta t}{\mu}\widetilde{\mathbb{C}}\widetilde{\mathbb{C}}^*\mathbf{E}^h + \sigma\mathbf{E}^h = \frac{1}{\mu}\widetilde{\mathbb{C}}\widetilde{\mathbf{B}}^h. \tag{46}$$

Note that, in general, $\widetilde{\mathbb{C}}^* \neq \mathbb{C}$ even though both are natural curl operators. For the problem considered here, $a > 0$ and $\alpha > 0$ so that in both cases we were able to eliminate all but one variable and obtain a Ritz-type method. Note that both the primal (45) and dual (46) equations are essentially (potentially different) discretizations of the same curl-curl differential operator.

5.1. Finite element realizations.

We now demonstrate the methods one obtains when the discrete exact sequences are built from finite element spaces. For the sake of brevity, we only consider the dual formulation (46) and the primal-dual method (28) or (34).

To specialize the dual method (46) to the case of finite element spaces, we need both the natural curl $\widetilde{\mathbb{C}}^*$ and its discrete adjoint operator $\widetilde{\mathbb{C}}$. Finite element spaces provide proper, finite-dimensional subspaces of the Sobolev spaces that form (10). As a result, the natural differential operators are defined by restriction of d^* to the discrete spaces, i.e., $\widetilde{\mathbb{C}}^*$ is a matrix that gives the action of the actual curl operator on finite element functions. For the sake of clarity, we will use the latter instead of the matrix form, i.e., we will write the dual equation restricted to finite element spaces as

$$\nabla\times\mathbf{E}^h = -\frac{1}{\Delta t}\mathbf{B}^h + \frac{1}{\Delta t}\widetilde{\mathbf{B}}^h. \tag{47}$$

The adjoint operator $\widetilde{\mathbb{C}}$ should act on discrete functions that form a subspace of $W^{2,*}(d^*,\Omega)$. Such functions are in the domain of the divergence but not in the domain of the curl. In the finite element case, this problem is resolved by using a weak variational form of the curl operator obtained by integration by parts: $\widetilde{\mathbb{C}} : W_h^{2,*}(d_h^*,\Omega) \mapsto W_h^{1,*}(d_h^*,\Omega)$ and $\widetilde{\mathbb{C}}\mathbf{B}^h = \mathbf{E}^h$ if and only if

$$\int_\Omega \widetilde{\mathbb{C}}\,\mathbf{B}^h\widehat{\mathbf{E}}^h\,d\Omega = \int_\Omega \mathbf{B}^h\nabla\times\widehat{\mathbf{E}}^h\,d\Omega \quad \forall\,\widehat{\mathbf{E}}^h \in W_h^{1,*}(d_h^*,\Omega)\,. \tag{48}$$

The operator $\widetilde{\mathbb{C}}$ defined by (48) is not local. Finding its explicit matrix representation entails the solution of a linear system of equations and is not practical. Fortunately, it is also not needed, as the primal variables $\mathbf{H}$ and $\mathbf{J}$ can be eliminated from (36a) by directly using (48). The ensuing weak form of this equation is

$$\int_\Omega \frac{1}{\mu}\mathbf{B}^h \cdot \nabla \times \widehat{\mathbf{E}}^h \, d\Omega = \int_\Omega \sigma \mathbf{E}^h \cdot \widehat{\mathbf{E}}^h \, d\Omega \quad \forall\, \widehat{\mathbf{E}}^h \in W_h^{1,*}(d_h^*, \Omega). \tag{49}$$

Using (47) to eliminate $\mathbf{B}^h$ from (49), we obtain the dual finite element method: *seek* $\mathbf{E}^h \in W^{1,*}(d_h^*, \Omega)$ *such that*

$$\int_\Omega \frac{\Delta t}{\mu}\left(\nabla \times \mathbf{E}^h\right) \cdot \left(\nabla \times \widehat{\mathbf{E}}^h\right) d\Omega + \int_\Omega \sigma \mathbf{E}^h \cdot \widehat{\mathbf{E}}^h \, d\Omega = \int_\Omega \frac{1}{\mu}\widetilde{\mathbf{B}}^h \cdot \left(\nabla \times \widehat{\mathbf{E}}^h\right) d\Omega \tag{50}$$

for all $\widehat{\mathbf{E}}^h \in W^{1,*}(d_h^*, \Omega)$.

Remark 8. If inhomogeneous boundary conditions are specified in (41), they will contribute an additional trace term

$$\int_b (\mathbf{n} \times \mathbf{H}_{\mathrm{b}}) \cdot \widehat{\mathbf{E}}_h \, d\Gamma$$

to (48), (49), and (50); this term will modify the source function. The boundary condition on $\mathbf{E}$ has to be strongly imposed, in some appropriate approximate manner, on the discrete approximation $\mathbf{E}^h$. Thus, for the dual problem, the boundary condition on $\mathbf{H}$ is a natural one while the one of $\mathbf{E}$ is an essential one. For further details about the dual formulation with such boundary conditions, see [5].

The primal finite element counterpart of (50) will lead to a similar curl-curl equation but in terms of the *primal* variable $\mathbf{H}$; the roles of the essential and natural boundary conditions are reversed.

Let us now specialize the primal-dual method (28)–(29) to the semidiscrete eddy current problem (42a)-(42b) and finite element spaces. First, the constrained optimization problem (28) now reads: *find* $(\mathbf{H}^h, \mathbf{J}^h) \in W_h^1(d_h, \Omega) \times W_h^2(d_h, \Omega)$ *and* $(\mathbf{B}^h, \mathbf{E}^h) \in W_h^{2,*}(d_h^*, \Omega) \times W_h^{1,*}(d_h^*, \Omega)$ *that*

$$\begin{cases} \textit{minimize} \;\; J(\widetilde{\mathbf{H}}^h, \widetilde{\mathbf{J}}^h; \widetilde{\mathbf{B}}^h, \widetilde{\mathbf{E}}^h) = \dfrac{1}{2}\left(\|\widetilde{\mathbf{B}}^h - \mu\widetilde{\mathbf{H}}^h\|_0^2 + \|\widetilde{\mathbf{E}}^h - \sigma^{-1}\widetilde{\mathbf{J}}^h\|_0^2\right) \\ \textit{subject to} \;\; \nabla \times \widetilde{\mathbf{H}}^h = \widetilde{\mathbf{J}}^h \text{ and } \nabla \times \widetilde{\mathbf{E}}^h = -\dfrac{1}{\Delta t}\widetilde{\mathbf{B}}^h + \dfrac{1}{\Delta t}\widetilde{\mathbf{B}}^h. \end{cases}$$

In the present case, $a > 0$ and $\alpha > 0$ and so we can proceed to eliminate $\mathbf{J}^h$ from the primal variables and $\mathbf{B}^h$ from the dual set to obtain a minimization problem in terms of $\mathbf{H}^h$ and $\mathbf{E}^h$ only. The analogue of (29) is therefore: *find* $\mathbf{H}^h \in W_h^1(d_h, \Omega)$ and $\mathbf{E}^h \in W_h^{1,*}(d_h^*, \Omega)$ *that minimize*

$$J(\widehat{\mathbf{H}}^h, \widehat{\mathbf{E}}^h) = \frac{1}{2}\left(\|\Delta t \nabla \times \widehat{\mathbf{E}}^h + \mu \widehat{\mathbf{H}}^h - \widetilde{\mathbf{B}}^h\|_0^2 + \|\widehat{\mathbf{E}}^h - \sigma^{-1}\nabla \times \widehat{\mathbf{H}}^h\|_0^2\right). \tag{51}$$

The form in (35) is given by

$$\begin{aligned} Q(\mathbf{H}^h, \mathbf{E}^h; \widehat{\mathbf{H}}^h, \widehat{\mathbf{E}}^h) &= \left(\Delta t \nabla \times \mathbf{E}^h + \mu \mathbf{H}^h, \Delta t \nabla \times \widehat{\mathbf{E}}^h + \mu \widehat{\mathbf{H}}^h\right) \\ &\quad + \left(\mathbf{E}^h - \sigma^{-1}\nabla \times \mathbf{H}^h, \widehat{\mathbf{E}}^h - \sigma^{-1}\nabla \times \widehat{\mathbf{H}}^h\right) \end{aligned}$$

and the right hand-side functional is

$$F^h(\widehat{\mathbf{H}}^h, \widehat{\mathbf{E}}^h) = \left(\widetilde{\mathbf{B}}^h, \Delta t \nabla \times \widehat{\mathbf{E}}^h + \mu \widehat{\mathbf{H}}^h\right). \tag{52}$$

Another way to derive the reduced problem (51) is by application of least-squares principles directly to the semidiscrete equations (42a)–(42b), in which $\mathbf{J}$ and $\mathbf{B}$ were eliminated by using the constitutive laws in (38). However, this way of deriving (51) obscures the origins of each variable and makes the choice of the correct discrete spaces less transparent. Coupled with the extreme robustness of least-squares principles, this has led to the widespread misconception that the choice of elements in (51) is irrelevant and that standard C^0 nodal spaces can be used for $\mathbf{E}$ and $\mathbf{H}$ instead of the spaces $W_h^{1,*}(d_h^*, \Omega)$ and $W_h^1(d_h, \Omega)$.

Indeed, a least-squares principle remains stable on any proper subspace of the relevant function spaces by virtue of the fact that it computes an orthogonal projection of the true solution onto that subspace. Since nodal C^0 spaces do form proper subspaces of $W^{1,*}(d^*, \Omega)$ and $W^1(d, \Omega)$, such an implementation of (51) will reliably compute the best C^0 approximations of $\mathbf{E}$ and $\mathbf{H}$. The trouble with this approach is that the best C^0 approximation is not necessarily the most accurate possible approximation of $\mathbf{E}$ and $\mathbf{H}$ since the latter pair are required to have only tangential continuity (and not continuity of all their components) along element interfaces. Consequently, while always stable, nodal implementations of (51) may lack optimal accuracy in many physically relevant settings.

6. Conclusions

The approach developed in this paper combines variational and geometric ideas into a single framework for compatible discretizations. Depending on the choice of components, the framework allows the recovery of familiar examples of discretization schemes and to derive new types of formulations. This capability has been demonstrated by using as an example problem the eddy current equations.

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