A Multiscale Discontinuous Galerkin Method

Pavel Bochev[1], Thomas J.R. Hughes[2], and Guglielmo Scovazzi[3]

[1] Computational Mathematics and Algorithms Department,
Sandia National Laboratories*, Albuquerque, NM 87185, USA
pbboche@sandia.gov
[2] Institute for Computational Engineering and Science,
The University of Texas at Austin, Austin, TX 78712, USA
hughes@ices.utexas.edu
[3] Computational Physics R&D Department,
Sandia National Laboratories*, Albuquerque, NM 87185
gscovaz@sandia.gov

Abstract. We propose a new class of Discontinuous Galerkin (DG) methods based on variational multiscale ideas. Our approach begins with an additive decomposition of the discontinuous finite element space into continuous (coarse) and discontinuous (fine) components. Variational multiscale analysis is used to define an interscale transfer operator that associates coarse and fine scale functions. Composition of this operator with a donor DG method yields a new formulation that combines the advantages of DG methods with the attractive and more efficient computational structure of a continuous Galerkin method. The new class of DG methods is illustrated for a scalar advection-diffusion problem.

1 Introduction

Discontinuous Galerkin (DG) methods offer several important computational advantages over their continuous Galerkin counterparts. For instance, DG methods are particularly well-suited for application of h and p-adaptivity strategies. DG methods are also felt to have advantages of robustness over conventional Galerkin methods for problems of hyperbolic type [13, 11, 12]. There has also been recent interest in applying DG to elliptic problems so that advective-diffusive phenomena can be modeled; see Brezzi *et al.* [3], Dawson [7], and Hughes, Masud and Wan [10]. For a summary of the current state-of-the-art and introduction to the literature we refer to [1] and [6].

Despite the increased interest in DG methods, there are shortcomings that limit their practical utility. Foremost among these is the size of the DG linear system. Storage and solution cost are, obviously, adversely affected, which seems the main reason for the small industrial impact the DG method has had so far.

In [8] we proposed a new multiscale DG method that has the computational structure of a standard continuous Galerkin method. In this paper we extend

* Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under contract DE-AC04-94-AL85000.

I. Lirkov, S. Margenov, and J. Waśniewski (Eds.): LSSC 2005, LNCS 3743, pp. 84–93, 2006.
© Springer-Verlag Berlin Heidelberg 2006

this idea to a general multiscale framework for DG methods. Our approach starts with an additive decomposition of a given discontinuous finite element space into continuous (coarse) and discontinuous (fine) components. Then, variational multiscale analysis is used to define an interscale transfer operator that associates coarse and fine scale functions. Composition of this operator with a donor DG method yields a new formulation that combines the advantages of DG methods with the attractive and more efficient computational structure of a continuous Galerkin method. Variational multiscale analysis leads to a natural definition of local, elementwise problems that allow for an efficient computation of the interscale operator.

2 Notation

Throughout this paper Ω will denote an open bounded region in $\mathbb{R}^n$, $n = 2, 3$ with a polyhedral boundary $\partial\Omega$. We recall the standard Sobolev spaces $L^2(\Omega)$ and $H^1(\Omega)$. Let $\mathcal{T}_h$ be a regular partition of Ω into finite elements K that contains only regular nodes [4]. For simplicity, we limit our discussion to two space dimensions. Extension to three dimensions is straightforward.

Every element $K \in \mathcal{T}_h$ is an image of a reference element $\hat{K}$ that can be a triangle $\hat{T}$ or a square $\hat{Q}$. The vertices $\mathbf{v}$ and the edges $\mathbf{e}$ of K form the sets $V(K)$ and $E(K)$, respectively; $V(\mathcal{T}_h) = \cup_{K\in\mathcal{T}_h} V(K)$, $E(\mathcal{T}_h) = \cup_{K\in\mathcal{T}_h} E(K)$, Γ_h^0 is the set of all internal edges and Γ_h is the set of all edges on $\partial\Omega$.

The local space. The reference space $S^{p(\hat{K})}(\hat{K})$ on $\hat{K}$ is defined as follows:

$$S^{p(\hat{K})}(\hat{K}) = \begin{cases} \varphi = \sum_{i,j} a_{ij}\xi_1^i\xi_2^j\,, \quad 0 \le i,j \le p(\hat{K});\ i+j \le p(\hat{K}) & \text{if } \hat{K} = \hat{T} \\ \varphi = \sum_{i,j} a_{ij}\xi_1^i\xi_2^j\,, \quad 0 \le i,j \le p(\hat{K}) & \text{if } \hat{K} = \hat{Q} \end{cases} \tag{1}$$

The local element spaces $S^{p(K)}(K)$ are defined by a mapping of the reference space (1) to the physical space.

The discontinuous finite element space. Given two integers $0 \le p_{\min} < p_{\max}$ we consider the following finite element subspace of $L^2(\Omega)$

$$\Phi_h(\Omega) = \left\{ \varphi_h \in L^2(\Omega) \,|\, \varphi_h{}_{|_K} \in S^{p(K)}(K), \quad p_{\min} \le p(K) \le p_{\max};\ \forall K \in \mathcal{T}_h \right\}. \tag{2}$$

We will assume that $p_{\min} \ge 1$. Note that $\Phi_h(\Omega)$ is a formal union of the local spaces $S^{p(K)}(K)$.

The continuous finite element space. The additive decomposition of $\Phi_h(\Omega)$ is induced by a finite element subspace $\overline{\Phi}_h(\Omega)$ of $H^1(\Omega)$, defined with respect to the same partition $\mathcal{T}_h$ of Ω into finite elements. The space $\overline{\Phi}_h(\Omega)$ can be defined in many possible ways. However, to ensure H^1 conformity, functions in this space are constrained to be continuous across element interfaces; see [5]. Here, for simplicity we consider a minimal choice of $\overline{\Phi}_h(\Omega)$ given by (see Fig. 2)

$$\overline{\Phi}_h(\Omega) = \left\{ \overline{\varphi}_h \in H^1(\Omega) \,|\, \overline{\varphi}_h|_K \in S^1(K) \right\}. \tag{3}$$

In $\overline{\Phi}_h(\Omega)$ we consider a nodal basis $\{\overline{V}_{\overline{\mathbf{v}}}\}$; $\overline{\mathbf{v}} \in V(\mathcal{T}_h)$ such that $\overline{V}_{\overline{\mathbf{v}}_i}(\overline{\mathbf{v}}_j) = \delta_{ij}$. The basis functions have local supports given by $\mathrm{supp}(\overline{V}_{\overline{\mathbf{v}}}) = \cup_{\overline{\mathbf{v}} \in V(K)} K$. For $K \in \mathrm{supp}(\overline{V}_{\overline{\mathbf{v}}})$, $\overline{V}_{\overline{\mathbf{v}}}|_K = V_{\mathbf{v}}$ where $\mathbf{v} \in V(K)$ is the local vertex that corresponds to the global vertex $\overline{\mathbf{v}} \in V(\mathcal{T}_h)$. Owing to the assumption $p_{\min} \geq 1$ the space $\overline{\Phi}_h(\Omega)$ is contained in $\Phi_h(\Omega)$. While the actual choice of $\overline{\Phi}_h(\Omega)$ and the resulting decomposition will have an impact on the accuracy of the multiscale DG, it will not affect formulation of the overall framework.

Orientations, jumps and averages. We briefly review the relevant notation following the Brezzi conventions. We assume that all edges in $E(\mathcal{T}_h)$ are endowed by orientation. A convenient way to orient an edge is to pick a normal direction to that edge; see Fig. 2. An element can be oriented by selecting one of the two possible normal directions to its boundary ∂K. Without loss of generality, all elements are oriented by using the outward normal.

An internal edge $\mathbf{e} \in \Gamma_h^0$ is shared by exactly two elements. The outward normal on one of these elements will coincide with the normal used to orient $\mathbf{e}$; we call this element K^-. The outward normal on the other element will have the opposite direction to the normal on $\mathbf{e}$; we call this element K^+; see Fig. 2. Edge orientation also induces partition of the boundary of an internal element into $\partial^+ K$, consisting of all edges whose normal direction coincides with the outer

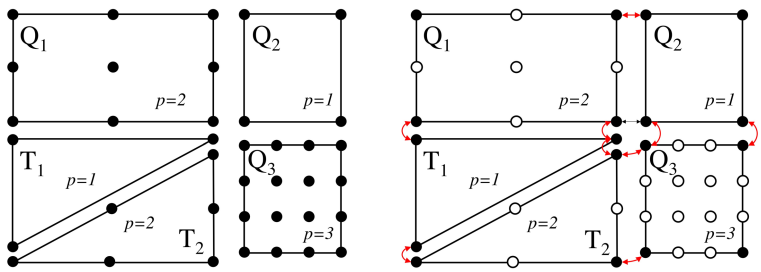

Fig. 1. The space $\Phi_h(\Omega)$ (left) and the corresponding minimal C^0 space $\overline{\Phi}_h(\Omega)$ (right)

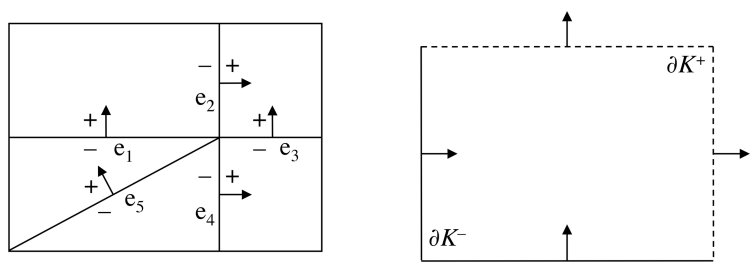

Fig. 2. Orientation of internal edges in $\mathcal{T}_h$ and $+/-$ elements with respect to an edge (left). Partition of element boundary into $\partial^+ K$ and $\partial^- K$ (right).

normal on ∂K and $\partial^- K$, consisting of all edges $\mathbf{e}$ whose normal direction is opposite to the outer normal on ∂K.

Let φ be a scalar field, and $\varphi^\pm := \varphi|_{K^\pm}$. For $\mathbf{e} \in \Gamma_h^0$ we define the *average* and the *jump* as $\langle\varphi\rangle := \frac{1}{2}(\varphi^+ + \varphi^-)$ and $[\varphi] := \varphi^+\boldsymbol{n}^+ + \varphi^-\boldsymbol{n}^-$, respectively. Analogously, if $\mathbf{u}$ is a vector field, $\langle\mathbf{u}\rangle := \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-)$ and $[\mathbf{u}] := \mathbf{u}^+\cdot\boldsymbol{n}^+ + \mathbf{u}^-\cdot\boldsymbol{n}^-$. Note that, by definition of "$[\,\cdot\,]$", the jump of a scalar quantity is a vector and the jump of a vector quantity is a scalar. For edges belonging to Γ_h, $[\varphi] = \varphi\,\boldsymbol{n}$ and $\langle\mathbf{u}\rangle = \mathbf{u}$. It will not be necessary to define $\langle\varphi\rangle$ and $[\mathbf{u}]$ on the boundary Γ, because they are never utilized.

3 Multiscale Discontinuous Galerkin Method

We consider an abstract linear boundary value problem

$$\mathcal{L}(\mathbf{x}, D)\varphi = f \text{ in } \Omega \quad \text{and} \quad \mathcal{R}(\mathbf{x}, D)\varphi = g \text{ on } \Gamma\,. \tag{4}$$

The multiscale DG framework for problem (4) has two basic components. The first is a *donor* DG formulation for (4): *find* $\varphi_h \in \Phi_h(\Omega)$ *such that*

$$B_{DG}(\varphi_h; \psi_h) = F_{DG}(\psi_h) \quad \forall \psi_h \in \Phi_h(\Omega)\,. \tag{5}$$

In (5), $B_{DG}(\cdot;\cdot)$ is a continuous bilinear form $\Phi_h(\Omega) \times \Phi_h(\Omega) \mapsto \mathbb{R}$ and $F_{DG}(\cdot)$ is a bounded linear functional $\Phi_h(\Omega) \mapsto \mathbb{R}$. We assume that (5) has a unique solution φ_h that depends continuously on the data and converges (in a suitable norm) to all sufficiently smooth solutions φ of (4). The second component is an interscale transfer (or expansion) operator

$$T : \overline{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega)\,. \tag{6}$$

We assume that T is a bounded linear operator, however, it is not required to be surjective, or invertible. Thus, in general $T(\overline{\Phi}_h(\Omega))$ will be a proper subspace of the discontinuous space $\Phi_h(\Omega)$.

We define the Multiscale DG (MDG) method by a composition of the donor DG scheme with the interscale transfer operator T: *find* $\overline{\varphi}_h \in \overline{\Phi}_h(\Omega)$ *such that*

$$B_{DG}(T\overline{\varphi}_h; T\overline{\psi}_h) = F_{DG}(T\overline{\psi}_h) \quad \forall \overline{\psi}_h \in \overline{\Phi}_h(\Omega)\,. \tag{7}$$

Substitution of discontinuous test and trial functions in the donor DG method by $T\overline{\psi}_h$ and $T\overline{\varphi}_h$ reduces the number of degrees-of-freedom in the MDG formulation to that of a standard Galerkin method posed on $\overline{\Phi}_h(\Omega)$. Since $T(\overline{\Phi}_h(\Omega)) \subset \Phi_h(\Omega)$, (7) occupies a middle ground between a DG and a CG method for (4).

3.1 Definition of the Interscale Operator

The definition of the interscale operator T is key to a robust, efficient and accurate MDG method. For instance, it is desirable to compute T locally on each element. To discuss definition of this operator assume that

$$B_{DG}(\varphi_h; \psi_h) = \sum_{K\in\mathcal{T}_h} B_K(\varphi_h; \psi_h) + \sum_{\mathbf{e}\in\Gamma_h} B_\Gamma(\varphi_h; \psi_h) + \sum_{\mathbf{e}\in\Gamma_h^0} B_{\mathbf{e}}\left(\{\varphi_h^-, \varphi_h^+\}; \{\psi_h^-, \psi_h^+\}\right) \tag{8}$$

where $B_K(\cdot;\cdot)$ is a bilinear *local* element form defined for every $K \in \mathcal{T}_h$, $B_\Gamma(\cdot;\cdot)$ is a bilinear form defined on $\mathbf{e} \in \Gamma_h$, and $B_{\mathbf{e}}(\{\cdot\};\{\cdot\})$ is an *edge* bilinear form defined for $\mathbf{e} \in \Gamma_h^0$.

To define T we proceed to formally split functions $\varphi_h \in \Phi_h(\Omega)$ into a continuous ("coarse" scale) part $\overline{\varphi}_h \in \overline{\Phi}_h(\Omega)$ and a discontinuous ("fine" scale) component $\varphi'_h \in \Phi_h(\Omega)$, viz. $\varphi_h = \overline{\varphi}_h + \varphi'_h$. Then, (5) takes the following form:

$$\begin{aligned} B_{DG}(\overline{\varphi}_h;\overline{\psi}_h) + B_{DG}(\varphi'_h;\overline{\psi}_h) &= F_{DG}(\overline{\psi}_h) \quad \forall \overline{\varphi}_h \in \overline{\Phi}_h(\Omega) \\ B_{DG}(\varphi'_h;\psi'_h) + B_{DG}(\overline{\varphi}_h;\psi'_h) &= F_{DG}(\psi'_h) \quad \forall \psi'_h \in \Phi_h(\Omega) \end{aligned} \tag{9}$$

The first line in (9) is the coarse scale equation. The second line is the fine scale equation that will be used to define T. Treating the coarse scale function as data we write this equation as: *find* $\varphi'_h \in \Phi_h(\Omega)$ *such that*

$$B_{DG}(\varphi'_h;\psi'_h) = F_{DG}(\psi'_h) - B_{DG}(\overline{\varphi}_h;\psi'_h) \quad \forall \psi'_h \in \Phi_h(\Omega)\,. \tag{10}$$

We restrict (10) to an element K by choosing test functions $\psi'_h \in S^{p(K)}(K)$ that vanish outside of this element. With the above selection of a test function, $(\psi'_h)^+ = \chi(\partial^- K)\psi'_h$ and $(\psi'_h)^- = \chi(\partial^+ K)\psi'_h$ where $\chi(\cdot)$ is the characteristic function. Using these identities and that $(\overline{\varphi}_h)^+ = (\overline{\varphi}_h)^- = \overline{\varphi}_h$, for a C^0 function, the restricted fine scale problem can be expressed as follows: *find* $\varphi'_h \in S^{p(K)}(K)$ *such that*

$$\begin{aligned} &B_K(\varphi'_h;\psi'_h) + B_\Gamma(\varphi'_h;\psi'_h) + \sum_{\mathbf{e}\in E(K)} B_{\mathbf{e}}\left(\{(\varphi'_h)^-,(\varphi'_h)^+\};\{\chi(\partial^+K)\psi'_h,\chi(\partial^-K)\psi'_h\}\right) \\ &= F_{DG}(\psi'_h) - B_K(\overline{\varphi}_h;\psi'_h) - B_\Gamma(\overline{\varphi}_h;\psi'_h) \\ &\quad - \sum_{\mathbf{e}\in E(K)} B_{\mathbf{e}}\left(\{\overline{\varphi}_h,\overline{\varphi}_h\};\{\chi(\partial^+K)\psi'_h,\chi(\partial^-K)\psi'_h\}\right) \ \forall \psi'_h \in S^{p(K)}(K)\,. \end{aligned} \tag{11}$$

Problem (11) relates fine scales to the coarse scales, but remains coupled to the contiguous elements through the numerical flux terms in (11). Therefore, it does not meet our criteria for localized computation of the interscale transfer operator T. However, we make the important observation that our goal is not to solve the DG problem (9) but rather use it to define a local computation procedure for T that maps $\overline{\varphi}_h$ into the local space $S^{p(K)}(K)$. We note that this objective is reminiscent of other applications of variational multiscale analysis in which the fine scale problem is used for *estimation* rather than *approximation* of the unresolved solution component. This process can be accomplished by a modification of the numerical flux inherited from the donor DG formulation, or by using a new flux defined only in terms of the local function $\varphi'_h \in S^{p(K)}(K)$. Let $B'_{\mathbf{e}}(\{\cdot\};\{\cdot\})$ be the new numerical flux. The local fine scale problem obtained from (11) is: *find* $\varphi'_h \in S^{p(K)}(K)$ *such that*

$$\begin{aligned} &B_K(\varphi'_h;\psi'_h) + B_\Gamma(\varphi'_h;\psi'_h) + \sum_{\mathbf{e}\in E(K)} B'_{\mathbf{e}}\left(\{\varphi'_h\};\{\psi'_h\}\right) \\ &\quad = F_{DG}(\psi'_h) - B_K(\overline{\varphi}_h;\psi'_h) - B_\Gamma(\overline{\varphi}_h;\psi'_h) \\ &\quad - \sum_{\mathbf{e}\in E(K)} B_{\mathbf{e}}\left(\{\overline{\varphi}_h,\overline{\varphi}_h\};\{\chi(\partial^+K)\psi'_h,\chi(\partial^-K)\psi'_h\}\right) \ \forall \psi'_h \in S^{p(K)}(K)\,. \end{aligned} \tag{12}$$

Problem (12) is a local equation that can be solved on an element by element basis. This problem defines an operator $T_K : \overline{\Phi}_h(\Omega) \mapsto S^{p(K)}(K)$ that maps any given C^0 finite element function $\overline{\varphi}_h$ to a function in the local element space $S^{p(K)}(K)$. Therefore,

$$T : \overline{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega); \quad T|_K = T_K \quad \forall K \in \mathcal{T}_h \tag{13}$$

defines an interscale transfer operator T for the MDG method. The abstract variational equation (7) and the local problem (12) complete the definition of the MDG framework.

4 Multiscale DG for a Scalar Advection-Diffusion Problem

We consider a model advection diffusion problem written in conservative form:

$$\begin{cases} \nabla \cdot (F_a + F_d) = f \text{ in } \Omega; & -(F_a + F_d) \cdot \boldsymbol{n} = h^- \text{ on } \Gamma_n^- \\ \varphi = g \text{ on } \Gamma_g; & -(F_d) \cdot \boldsymbol{n} = h^+ \text{ on } \Gamma_n^+ \end{cases} \tag{14}$$

where $F_d = -\kappa \nabla \varphi$ and $F_a = \mathbf{a}\varphi$ denote diffusive and advective flux, respectively. The total flux is $F = F_a + F_d$. The Neumann boundary condition can be written compactly as $-(\chi(\Gamma_n^-)F_a + F_d) \cdot \boldsymbol{n} = h$; where $h = \chi(\Gamma_n^-)h^- + \chi(\Gamma_n^+)h^+$.

4.1 A Donor DG Method for the Model Problem

When dealing with advection-diffusion problems it is profitable to coordinate edge orientations with the advective direction. Given an edge $\mathbf{e}$ we choose the normal $\boldsymbol{n}_{\mathbf{e}}$ for which $\boldsymbol{n}_{\mathbf{e}} \cdot \mathbf{a} \geq 0$. A general weighted residual form of a Discontinuous Galerkin method for (14) is given by: *find* $\varphi \in \Phi_h(\Omega)$ *such that*

$$\begin{aligned} 0 = \sum_{i=1}^{n_{el}} -\int_{K_i} (F_i \cdot \nabla \psi + f\psi)\, d\Omega + \int_{\Gamma_n} (\chi(\Gamma_n^+)F_a \cdot \boldsymbol{n} - h)\psi d\mathbf{l} + \int_{\Gamma_g} (F \cdot \boldsymbol{n})\psi d\mathbf{l} + \\ \int_{\Gamma_g} \epsilon(\varphi - g)W(\psi) d\mathbf{l} + \sum_{\mathbf{e} \in \Gamma_h^0} \int_{\mathbf{e}} \Big(F_b^h(\varphi^+;\varphi^-) \cdot [\psi] + F_c^h(\psi^+;\psi^-) \cdot [\varphi] + \alpha[\varphi][\psi] \Big) d\mathbf{l} \end{aligned} \tag{15}$$

for all $\psi \in \Phi_h(\Omega)$. Above, $W(\psi)$ is a weight function that enforces the Dirichlet boundary condition weakly,

$$F_b^h \stackrel{\text{def}}{=} s_{11}F_a^h + s_{12}F_d^h \quad \text{and} \quad F_c^h \stackrel{\text{def}}{=} s_{21}F_a^h + s_{22}F_d^h \tag{16}$$

are numerical models of the total flux across $\mathbf{e} \in \Gamma_h^0$ and

$$F_a^h \stackrel{\text{def}}{=} F_a^h(\varphi^+, \varphi^-) \quad \text{and} \quad F_d^h \stackrel{\text{def}}{=} F_d^h(\varphi^+, \varphi^-) \tag{17}$$

are constitutive relations for the advective and the diffusive fluxes across $\mathbf{e}$ in terms of the solution states φ^+ and φ^- from the two elements that share $\mathbf{e}$. The component bilinear forms in (8) can be easily identified from (15):

$$B_K(\varphi;\psi) = \int_K -F_K \cdot \nabla\psi \, d\Omega \tag{18}$$

$$B_\Gamma(\varphi;\psi) = \int_{\Gamma_n} (\chi(\Gamma_n^+)F_a \cdot \boldsymbol{n})\psi \, d\mathbf{l} + \int_{\Gamma_g} (F \cdot \boldsymbol{n})\psi \, d\mathbf{l} + \epsilon \int_{\Gamma_g} \varphi W(\psi) \, d\mathbf{l} \tag{19}$$

$$B_{\mathbf{e}}\left(\{\varphi^+;\varphi^-\};\{\psi^+;\psi^-\}\right) = \int_{\mathbf{e}} \Big(F_b^h(\varphi^+;\varphi^-)\cdot[\psi] + F_c^h(\psi^+;\psi^-)\cdot[\varphi] + \alpha[\varphi][\psi] \Big) \, d\mathbf{l} \,. \tag{20}$$

particular donor DG method is obtained from (15) by specification of ϵ, α, the numerical fluxes in (16)–(17) for the internal edges Γ_h^0, and the weight function $W(\psi)$.

We set $\epsilon = \alpha = \delta\kappa/h_\perp$, where $\delta > 0$ is non-dimensional parameter and $h_\perp = (\text{meas}(K^+) + \text{meas}(K^-))/(2\ \text{meas}(\mathbf{e}))$. Roughly speaking, $h_\perp$ is a length scale in the direction perpendicular to the edge e, close to the length of the segment joining the barycenters of K^- and K^+.

A standard choice for F_a^h is the upwinded advective flux $F_a^h(\psi^+;\psi^-) = F_a(\psi^-) = \mathbf{a}\varphi^-$. Possible choices for the numerical diffusive flux are the averaged flux $F_d^h(\psi^+;\psi^-) = \langle F_d(\psi)\rangle = -\frac{1}{2}\left(\kappa\nabla\psi^+ + \kappa\nabla\psi^-\right)$ or the upwinded flux $F_d^h(\psi^+;\psi^-) = F_d(\psi^-) = -\kappa\nabla\psi^-$. To define F_b^h and F_c^h we set $s_{11} = s_{12} = 1$, $s_{21} = 0$ and $s_{22} = s \in \{-1, 0, +1\}$ in (16). This leads to two different donor DG methods: DG-A which uses averaged diffusive flux, and DG-B which uses the upwinded version of that flux; see [8]. Flux and weight function definitions for the two methods are summarized in Table 1.

The effect of the parameter s has been extensively studied in the discontinuous Galerkin literature (see Arnold *et al.* [1], Baumann and Oden [2], and Hughes *et al.* [9]). The symmetric formulation ($s = -1$) is adjoint-consistent, guaranteeing optimal L_2-convergence rates in the diffusive limit. Ostensibly, the skew formulation ($s = +1$) has superior stability properties but the ϵ and α-terms can be used to improve the stability behavior of the neutral (i.e., $s = 0$) and symmetric formulations. For more details about the implementation of the donor DG and numerical results we refer to [8].

For DG-B the numerical flux F_b^h is simply the upwinded total flux $F(\varphi^-)$. DG-A and DG-B have the same element form $B_K(\cdot;\cdot)$ (given by (18)) and the same boundary form:

$$B_\Gamma(\varphi;\psi) = \int_{\Gamma_n} (\chi(\Gamma_n^+)F_a \cdot \boldsymbol{n})\psi \, d\mathbf{l} + \int_{\Gamma_g} (F \cdot \boldsymbol{n})\psi \, d\mathbf{l} + \epsilon \int_{\Gamma_g} \varphi \underbrace{(\psi - s\kappa\nabla\psi \cdot \boldsymbol{n})}_{W(\psi)} \, d\mathbf{l} \tag{21}$$

The internal edge form for DG-A is

$$\begin{aligned} B_{\mathbf{e}}\left(\{\varphi^+;\varphi^-\};\{\psi^+;\psi^-\}\right) &= \int_{\mathbf{e}} \alpha[\varphi][\psi] \, d\mathbf{l} \\ &+ \int_{\mathbf{e}} \Big(\underbrace{(\mathbf{a}\varphi^- - (\kappa\nabla\varphi^+ + \kappa\nabla\varphi^-)/2)}_{F_b^h} \cdot [\psi] - \underbrace{s(\kappa\nabla\psi^+ + \kappa\nabla\psi^-)/2}_{F_c^h} \cdot [\varphi] \Big) \, d\mathbf{l} \end{aligned} \tag{22}$$

while for DG-B this form is given by

$$B_{\mathbf{e}}\left(\{\varphi^+;\varphi^-\};\{\psi^+;\psi^-\}\right) = \int_{\mathbf{e}} \alpha[\varphi][\psi] \, d\mathbf{l} + \int_{\mathbf{e}} \Big(\underbrace{(\mathbf{a}\varphi^- - \kappa\nabla\varphi^-)}_{F_b^h} \cdot [\psi] - \underbrace{s\kappa\nabla\psi^-}_{F_c^h} \cdot [\varphi] \Big) \, d\mathbf{l} \,. \tag{23}$$

Table 1. Specialization of fluxes and weight function for the donor DG methods

Function	DG-A	DG-B
$F_b^h(\varphi^+;\varphi^-)$	$F_a(\varphi^-)+\langle F_d(\varphi)\rangle$	$F_a(\varphi^-)+F_d(\varphi^-)$
$F_c^h(\psi^+;\psi^-)$	$s\langle F_d(\psi)\rangle$	$sF_d(\psi^-)$
$W(\psi)$	$\psi+sF_d(\psi)\cdot\boldsymbol{n}$	

4.2 The Interscale Operator

We develop a consistent approach that reduces the edge form $B_{\mathbf{e}}(\{\cdot\};\{\cdot\})$ in the donor DG method to a form defined in terms of the local (fine scale) variable φ' and test function ψ'. In doing so we aim to preserve as much as possible from the structure of the donor DG method in the local problem.

For this purpose we redefine the calculation of the jump, the average and the states $\varphi^\pm$, $\psi^\pm$ as follows: given $\psi\in S^{p(K)}(K)$ its states are defined by

$$\psi^+=\chi(\partial^-K)\psi \quad \text{and} \quad \psi^-=\chi(\partial^+K)\psi \tag{24}$$

its jump is the vector

$$[\psi]=\boldsymbol{n}_K\psi\,, \tag{25}$$

and its average is the function itself:

$$\langle\psi\rangle=\psi\,. \tag{26}$$

The rules in (24)-(26) have the following interpretation. To compute the states and the jump of ψ, extend by zero to a function $\psi_0\in L^2(\Omega)$. Then $[\psi_0]=\boldsymbol{n}^+\chi(\partial^-K)\psi_0+\boldsymbol{n}^-\chi(\partial^+K)\psi_0=\boldsymbol{n}_K\psi_0$. Definition (26) can be motivated by noting that for affine elements ψ can be trivially extended to a function$\psi_\infty\in C^\infty(\Omega)$ for which $\langle\psi_\infty\rangle=\frac{1}{2}(\psi_\infty+\psi_\infty)=\psi_\infty$ giving (26). The local definitions of the numerical fluxes obtained through (24)-(26) are summarized in Table 2.

Local Problem for DG-A. The localized edge form for DG-A method is

$$B'_{\mathbf{e}}(\{\varphi\};\{\psi\})=\int_{\mathbf{e}}\Big(\underbrace{(\mathbf{a}\chi(\partial^+K)\varphi-\kappa\nabla\varphi)}_{F_b^h}\cdot\boldsymbol{n}_K\psi-\underbrace{s\kappa\nabla\psi}_{F_c^h}\cdot\boldsymbol{n}_K\varphi+\alpha\varphi\psi\Big)\,d\mathbf{l}\,. \tag{27}$$

The last two terms can be combined into a single weight function $W_\alpha(\psi)=\alpha\psi-s\kappa\nabla\psi\cdot\boldsymbol{n}_K$. Thus, the local problem obtained from DG-A is: given a $\overline{\varphi}\in\overline{\Phi}_h(\Omega)$ *find* $\varphi'\in S^{p(K)}(K)$ *such that*

Table 2. Specialization of fluxes for the local problem

Function	DG-A	DG-B
$F_b^h(\varphi)$	$F_a(\chi(\partial^+K)\varphi)+F_d(\varphi)$	$F_a(\chi(\partial^+K)\varphi)+F_d(\chi(\partial^+K)\varphi)$
$F_c^h(\psi)$	$sF_d(\psi)$	$sF_d(\chi(\partial^+K)\psi)$

$$B_K(\varphi';\psi') + B_\Gamma(\varphi';\psi') + \sum_{\mathbf{e}\in\partial K}\int_{\mathbf{e}}\Big((\mathbf{a}\chi(\partial^+K)\varphi' - \kappa\nabla\varphi')\cdot\boldsymbol{n}_K\psi' + \varphi' W_\alpha(\psi')\Big)\,d\mathbf{l}$$
$$= F_{DG}(\psi') - B_K(\overline{\varphi};\psi') - B_\Gamma(\overline{\varphi};\psi')$$
$$- \sum_{\mathbf{e}\in\partial K} B_{\mathbf{e}}\left(\{\overline{\varphi},\overline{\varphi}\};\{\chi(\partial^+K)\psi',\chi(\partial^-K)\psi'\}\right) \quad \forall\psi'\in S^{p(K)}(K)\,. \tag{28}$$

Remark 1. This local problem is identical to the one used in [8].

Local Problem for DG-B. For DG-B we have the localized edge form:

$$B'_{\mathbf{e}}\left(\{\varphi\};\{\psi\}\right) = \int_{\mathbf{e}}\Big(\underbrace{\chi(\partial^+K)(\mathbf{a}\varphi-\kappa\nabla\varphi)}_{F_b^h}\cdot\boldsymbol{n}_k\psi - \underbrace{s\chi(\partial^+K)\kappa\nabla\psi}_{F_c^h}\cdot\boldsymbol{n}_K\varphi + \alpha\varphi\psi\Big)\,d\mathbf{l}\,. \tag{29}$$

The last two terms can be combined into the weight function $W_\alpha^-(\psi) = \alpha\psi - s\chi(\partial^+K)\nabla\psi\cdot\boldsymbol{n}_K$, which is an "upwinded" version of $W_\alpha(\psi)$. The local problem is: given a $\overline{\varphi}\in\overline{\Phi}_h(\Omega)$ *find* $\varphi'\in S^{p(K)}(K)$ *such that*

$$B_K(\varphi';\psi') + B_\Gamma(\varphi';\psi') + \sum_{\mathbf{e}\in\partial K}\int_{\mathbf{e}}\Big(\chi(\partial^+K)(\mathbf{a}\varphi' - \kappa\nabla\varphi')\cdot\boldsymbol{n}_K\psi' + \varphi' W_\alpha(\psi')\Big)\,d\mathbf{l}$$
$$= F_{DG}(\psi') - B_K(\overline{\varphi};\psi') - B_\Gamma(\overline{\varphi};\psi')$$
$$- \sum_{\mathbf{e}\in\partial K} B_{\mathbf{e}}\left(\{\overline{\varphi},\overline{\varphi}\};\{\chi(\partial^+K)\psi',\chi(\partial^-K)\psi'\}\right) \quad \forall\psi'\in S^{p(K)}(K)\,. \tag{30}$$

5 Conclusions

In this work we extended the DG method developed in [8] to a general framework for multiscale DG methods that have the computational structure of continuous Galerkin methods. This represents a solution to a fundamental and long-standing problem in discontinuous-Galerkin technology, namely, restraining the proliferation of degrees-of-freedom. Numerical results reported in [8] indicate that for a scalar advection-diffusion equation the new method at least attains, and even somewhat improves upon, the performance of the associated continuous Galerkin method. Within the framework of the multiscale discontinuous Galerkin method, the local problem provides a vehicle for incorporating the necessary stabilization features such as discontinuity capturing and upwinding. There seems to be a potential connection here with ideas from wave propagation methods based on solutions of the Riemann problem, which is worth exploring in more detail.

The MDG formulation can be also viewed as an approach that enables uncoupling of *storage* locations of the data from the *computational* locations where this data is used. For example, one can envision a situation where information is

stored at the nodes and then mapped to flux and circulation degrees-of-freedom by the operator T. Such an extension of MDG appears to be a fruitful direction for further research.

Acknowledgements

The authors would like to thank Giancarlo Sangalli and the anonymous referees for helpful remarks. The work of T. J. R. Hughes was supported by Sandia Contract No. A0340.0 with the University of Texas at Austin. This support is gratefully acknowledged.

References

1. D. N. Arnold, F. Brezzi, B. Cockburn and L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM Journal on Numerical Analysis, **39** (5), (2002), 1749–1779
2. C.E. Baumann and J.T. Oden, A discontinuous *hp* finite element method for convection-diffusion problems, Comp. Meth. Appl. Mech. Engrg., **175**, (1999) 311–341
3. F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A. Russo, Discontinuous Galerkin approximations for elliptic problems, Numerical methods in PDEs, **16** (4), (2000), 365–378
4. C. Schwab, $p-$ and $hp-$ finite element methods. Theory and applications in solid and fluid mechanics. Clarendon Press, Oxford, 1998.
5. P. Ciarlet, The finite element method for elliptic problems. Classics in applied mathematics, SIAM, Philadelphia, 2002.
6. B. Cockburn, G. E. Karniadakis, and C.-W. Shu (eds.), Discontinuous Galerkin Methods: Theory, Computation and Applications, Springer-Verlag, Lecture Notes in Computational Science and Engineering, **11**, (2000)
7. C. Dawson, The P^{k+1}-S^k local discontinuous Galerkin method for elliptic equations, SIAM J. Num. Anal., **40** (6), (2003), 2151–2170
8. T.J.R. Hughes, G. Scovazzi, P. Bochev and A. Buffa, A multiscale discontinuous Galerkin method with the computational structure of a continuous Galerkin method. Comp. Meth. Appl. Mech. Engrg., submitted
9. T.J.R. Hughes, G. Engel, L. Mazzei and M.G. Larson, A comparison of discontinuous and continuous Galerkin methods based on error estimates, conservation, robustness and efficiency, In: Discontinuous Galerkin Methods: Theory, Computation and Applications, Edited by Cockburn, B. and Karniadakis, G.E. and Shu, C.-W., Lecture Notes in Computational Science and Engineering, **11**, Springer-Verlag, (2000).
10. T. J. R. Hughes, A. Masud, and J. Wan, A stabilized mixed discontinuous Galerkin method for Darcy flow, Comp. Meth. Appl. Mech. Enrgr. To appear.
11. C. Johnson, U. Nävert and J. Pitkäranta, Finite element methods for linear hyperbolic problems, Comp. Meth. Appl. Mech. Engrg., **45**, (1984), 285–312
12. C. Johnson, and J. Pitkäranta, An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation, Mathematics of Computation, **46** (173), (1986), 1–26
13. W. H. Reed and T. R. Hill, Triangular mesh methods for the neutron transport equation, LA-UR-73-479, Los Alamos Scientific Laboratory, (1973)