Analysis and Computation of Least-squares Methods for a Compressible Stokes Problem

Pavel Bochev,[1,*] **Sang Dong Kim,**[2] **Byeong-Chun Shin**[3]

[1] *Computational Mathematics and Algorithms, Sandia National Laboratories, Albuquerque, NM 87185-1110*

[2] *Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea*

[3] *Department of Mathematics, Chonnam National University, Gwangju 500-757, Korea*

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We develop and analyze a negative norm least-squares method for the compressible Stokes equations with an inflow boundary condition. Least-squares principles are derived for a first-order form of the equations obtained by using $\omega = \nabla \times \mathbf{u}$ and $\phi = \nabla \cdot \mathbf{u}$ as new dependent variables. The resulting problem is incompletely elliptic, i.e., it combines features of elliptic and hyperbolic equations. As a result, well-posedness of least-squares functionals cannot be established using the ADN elliptic theory and so we use direct approaches to prove their norm-equivalence. The article concludes with numerical examples that illustrate the theoretical convergence rates. Numer Methods Partial Differential Eq 22: 000–000, 2006

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I. INTRODUCTION

This article deals with least-squares finite element methods for the approximate numerical solution of the boundary value problem

$$\begin{cases} -\mu\Delta\,\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u} + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p = g & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \\ p = 0 & \text{on } \Gamma_{\text{in}} \end{cases} \tag{1.1}$$

Correspondence to: Byeong-Chun Shin, Department of Mathematics, Chonnam National University, Gwangju 500-757, Korea (email: bcshin@chonnam.ac.kr)
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In (1.1) Ω is a bounded open region in $\Re^2$ with a Lipschitz continuous boundary $\partial\Omega$, $\boldsymbol{\beta} = (\beta_1, \beta_2)^t$ with $\beta_1 \geq C_0 > 0$ is a given C^1 vector function on $\bar{\Omega}$, $\mathbf{f}$ and g are given C^1 functions, and μ and ν are two constants such that $\mu > 0$ and $\mu > -\nu$.

Problem (1.1) can be derived by linearization of the barotropic Navier-Stokes equations around a given ambient flow; see [1]. In what follows we shall refer to (1.1) as the *compressible Stokes* equations. Accordingly, we will refer to $\mathbf{u} = (u_1, u_2)^t$, p and μ and ν as the *velocity*, *pressure*, and the *viscous constants*, respectively.

We draw attention to the fact that the second equation in (1.1) contains a streamline derivative of the pressure and so the pressure must be specified on the inflow boundary $\Gamma_{\text{in}} = \{(x, y) \in \partial\Omega \mid \boldsymbol{\beta} \cdot \mathbf{n} < 0\}$, where $\mathbf{n}$ is the outward unit vector normal to the boundary $\partial\Omega$. This means that with respect to the pressure, (1.1) has a hyperbolic character, while with respect to the velocity it is elliptic. Systems of equations that exhibit such mixed behavior are called *incompletely elliptic*.

Since the structure of (1.1) resembles that of the usual Stokes problem, finite element methods for (1.1) can be obtained by an extension of the mixed Galerkin formulation to the compressible case. This approach has been adopted in [2–4]. Because the continuity equation in the compressible case already includes a pressure term, it is relatively easy to obtain mixed formulations that do not require the inf-sup stability condition; see [5]. Such formulations resemble the so-called *stabilized* finite element methods where the continuity equation is relaxed by adding a suitable pressure term.

An alternative path to the finite element solution of (1.1) is to use least-squares minimization principles. While for the incompressible Stokes, or Navier-Stokes equations the main reason to consider least-squares would be the desire to circumvent the inf-sup stability condition; see [6–9], for the compressible case this reason would be the possibility to obtain a symmetric and positive definite algebraic system. Such least-squares methods for (1.1) have been introduced only recently in [8] and [9]. The method of [8] is based on the second-order problem (1.1), while [9] uses a *velocity-flux* first-order system that was proposed in [8]. In the velocity flux formulation all first derivatives of the velocity $\mathbf{U} = \nabla\mathbf{u}$ are used as new dependent variables leading to a significant increase in the problem size. This drawback can be partially offset for problems with smooth solutions because then one can consider an augmented velocity flux formulation that is H^1-elliptic. In this case algebraic equations can be solved efficiently by multigrid.

In this article we develop a new least-squares method for (1.1) based on a formulation that falls between the two aforementioned compressible Stokes formulations. We do replace (1.1) by a first-order system, but unlike the one in [9], it is based on the *physically* important variables $\omega = \nabla \times \mathbf{u}$ and $\phi = \nabla \cdot \mathbf{u}$. This allows us to reap the benefits of a direct approximation of physically meaningful fields without unnecessary increase in complexity. This is especially important in cases when solutions are not smooth enough for the augmented flux formulation to make sense and so the advantages of fast multigrid solvers cannot be exploited. In addition, unlike [8], we allow advective fields $\boldsymbol{\beta}$ that do not vanish on the outflow boundary. This more general case considerably complicates the analysis because now the pressure does not have a vanishing mean and Necas's inequality is not applicable.

The article is organized as follows. Notation and technical results are summarized at the end of this section. In Section II we prove an auxiliary *a priori* estimate for the second-order system (1.1) that is later used to establish norm-equivalence of first-order system least-squares functional. This system and the associated *continuous* least-squares principle are formulated in Section III. Here we also show the well-posedness of the least-squares minimization problem. Our results are established under additional assumptions on $\|\nabla \cdot \boldsymbol{\beta}\|_\infty$ that are not uncommon for the compressible case; see [8] or [3,5]. The continuous least-squares principle in §III uses negative norms and is not a practical foundation for a finite element method. In section IV we use it as a template to define a proper discrete least-squares functional that can be implemented in a practical manner. We show that the discrete least-squares principle leads to finite element approximations that converge at optimal rates to all sufficiently smooth solutions of (1.1). These theoretical error estimates are further confirmed by a series of numerical experiments collected in Section V.

A. Notations

We use standard notations and definitions for the Sobolev spaces $H^s(\Omega)$, with their associated inner products $\langle\cdot,\cdot\rangle_s$, and norms $\|\cdot\|_s$, $s \geq 0$. For $s = 0$, $H^0(\Omega)$ is the usual $L^2(\Omega)$ with norm and inner product denoted by $\|\cdot\|_0 = \|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. As usual, $L^2_0(\Omega)$ is the subspace of all square integrable functions with zero mean. The closure of $D(\Omega)$ in $H^s(\Omega)$ will be denoted by $H^s_0(\Omega)$, where $D(\Omega)$ is the linear space of infinitely differentiable functions with compact supports on Ω with respect to the norm $\|\cdot\|_s$. We identify it with the subspace of all $H^s(\Omega)$ functions that vanish on $\partial\Omega$. $H^{-1}(\Omega)$ denotes the dual of H^1_0 equipped with the norm

$$\|\phi\|_{-1} = \sup_{\psi \in H^1_0(\Omega)} \frac{(\phi, \psi)}{\|\psi\|_1}. \tag{1.1}$$

The product spaces $H^s_0(\Omega)^2$ and $L^2(\Omega)^2$ are defined in the usual manner. We recall that

$$H(\mathrm{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \nabla\cdot\mathbf{v} \in L^2(\Omega)\} \tag{1.2}$$

equipped with a norm

$$\|\mathbf{v}\|_{H(\mathrm{div};\Omega)} := (\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2)^{1/2}$$

is a Hilbert space. We will make a frequent use of the space

$$Q(\beta,\Omega) = \left\{q \in L^2(\Omega) : \boldsymbol{\beta}\cdot\nabla q \in L^2(\Omega), q = 0 \text{ on } \Gamma_{\mathrm{in}}\right\} \tag{1.3}$$

equipped with the norm

$$\|q\|_Q = (\|q\|^2 + \|\boldsymbol{\beta}\cdot\nabla q\|^2)^{1/2}. \tag{1.4}$$

In two dimensions the *curl* of $\mathbf{v} = (v_1, v_2)^t$ is the scalar function

$$\nabla\times\mathbf{v} = \partial_{x_1} v_2 - \partial_{x_2} v_1,$$

and the curl of a scalar function p is the planar vector

$$\nabla\times p = (\partial_{x_2} p,\ -\partial_{x_1} p)^t.$$

II. AN *A PRIORI* ESTIMATE FOR THE COMPRESSIBLE STOKES EQUATIONS

Below we establish an *a priori* estimate for the second-order problem (1.1). This auxiliary result will be used later on to show norm-equivalence of least-squares functionals based on first-order forms of (1.1). Let λ be the Poincaré constant, that is,

$$\|\mathbf{u}\|^2 \leq \lambda\,\|\nabla\mathbf{u}\|^2, \qquad \forall\,\mathbf{u} \in H^1_0(\Omega)^2.$$

Furthermore, let $\mu_0 = \min\{\mu, \mu + 2\nu\}$,

$$\gamma_0(\boldsymbol{\beta}) := \frac{1}{2}\|\nabla \cdot \boldsymbol{\beta}\|_\infty \qquad \text{and} \qquad \gamma_1(\boldsymbol{\beta}) := \frac{\mu_0 - \lambda\gamma_0(\boldsymbol{\beta})}{1+\lambda}.$$

The following Lemma extends a result of [3, Lemma 2] to the present case.

Lemma 2.1. *Assume that $\gamma_1(\beta) > 0$. For any $\mathbf{u} \in H_0^1(\Omega)^2$,*

$$\gamma_1(\boldsymbol{\beta})\|\mathbf{u}\|_1^2 \le \langle -\mu\Delta\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u} + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u}, \mathbf{u}\rangle. \tag{2.1}$$

For any $p \in Q(\boldsymbol{\beta};\Omega)$,

$$-\gamma_0(\boldsymbol{\beta})\|p\|^2 \le \langle \boldsymbol{\beta}\cdot\nabla p, p\rangle. \tag{2.2}$$

Proof. From the Green's Theorem and the fact that $\|\nabla\cdot\mathbf{u}\| \le \sqrt{2}\|\nabla\mathbf{u}\|$, it follows that

$$\begin{aligned}
&\langle -\mu\Delta\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u} + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u}, \mathbf{u}\rangle \\
&\quad = \mu\langle\nabla\mathbf{u}, \nabla\mathbf{u}\rangle + \nu\langle\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{u}\rangle + \frac{1}{2}\big\langle\boldsymbol{\beta}, \nabla|\mathbf{u}|^2\big\rangle \\
&\quad = \mu\|\nabla\mathbf{u}\|^2 + \nu\|\nabla\cdot\mathbf{u}\|^2 - \frac{1}{2}\big\langle\nabla\cdot\boldsymbol{\beta}, |\mathbf{u}|^2\big\rangle \\
&\quad \ge \mu_0\|\nabla\mathbf{u}\|^2 - \gamma_0(\boldsymbol{\beta})\|\mathbf{u}\|^2 \ge (\mu_0 - \lambda\gamma_0(\boldsymbol{\beta}))\|\nabla\mathbf{u}\|^2.
\end{aligned}$$

Since $\gamma_1(\boldsymbol{\beta}) > 0$, Poincaré's inequality yields (2.1). Because $p = 0$ on Γ_{in} for any $p \in Q(\boldsymbol{\beta};\Omega)$ and $\boldsymbol{\beta}\cdot\mathbf{n} \ge 0$ on Γ_{out}, it follows that

$$\begin{aligned}
\langle\boldsymbol{\beta}\cdot\nabla p, p\rangle &= \frac{1}{2}\langle\boldsymbol{\beta}, \nabla p^2\rangle = -\frac{1}{2}\int_\Omega (\nabla\cdot\boldsymbol{\beta})p^2\,d\Omega + \frac{1}{2}\int_{\partial\Omega}(\boldsymbol{\beta}\cdot\mathbf{n})p^2\,ds \\
&\ge -\gamma_0(\boldsymbol{\beta})\|p\|^2 + \frac{1}{2}\int_{\Gamma_{\text{out}}}(\boldsymbol{\beta}\cdot\mathbf{n})p^2\,ds \ge -\gamma_0(\boldsymbol{\beta})\|p\|^2.
\end{aligned}$$

This completes the proof of the Lemma. ∎

To state the main result of this section we recall that (see [5])

$$\|p\| \le C_1\|\boldsymbol{\beta}\cdot\nabla p\|, \qquad \forall\, p \in Q(\boldsymbol{\beta};\Omega). \tag{2.3}$$

Theorem 2.1. *Assume that $\gamma_0(\boldsymbol{\beta})$ is such that*

$$\gamma_0(\boldsymbol{\beta}) < \frac{\mu_0}{\lambda + 4C_1^2(1+\lambda)}, \tag{A_0}$$

where C_1 is given in (2.3). Then, there exist constants c and C such that for any $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times Q(\boldsymbol{\beta};\Omega)$

$$c\big(\|\mathbf{u}\|_1^2 + \|p\|_Q^2\big) \le |||\mathbf{u}, p|||^2 \le C\big(\|\mathbf{u}\|_1^2 + \|p\|_Q^2\big), \tag{2.4}$$

where

$$|||\mathbf{u}, p|||^2 = \|-\mu\Delta\mathbf{u} - \nu\nabla(\nabla\cdot\mathbf{u}) + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}^2 + \|\boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\|^2.$$

Proof. The upper bound easily follows by repeated application of the triangle inequality and the definition of $H^{-1}(\Omega)$-norm. To prove the lower bound, note that (2.3) and the triangle inequality imply

$$\|p\| \leq C_1(\|\boldsymbol{\beta} \cdot \nabla p + \nabla \cdot \mathbf{u}\| + \|\nabla \cdot \mathbf{u}\|). \tag{2.5}$$

If $\gamma_0(\boldsymbol{\beta})$ satisfies (A_0), then $\gamma_1(\boldsymbol{\beta}) > 0$. Hence, from (2.1), integrating by parts and using the Cauchy-Schwarz inequality yields that

$$\begin{aligned}
\gamma_1(\boldsymbol{\beta})\|\mathbf{u}\|_1^2 &\leq \langle -\mu\Delta\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u} + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p, \mathbf{u}\rangle + \langle p, \nabla\cdot\mathbf{u}\rangle \\
&\leq \|-\mu\Delta\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u} + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}\|\mathbf{u}\|_1 + \langle p, \nabla\cdot\mathbf{u}\rangle \\
&\leq C|||\mathbf{u}, p|||^2 + \frac{\gamma_1(\boldsymbol{\beta})}{4}\|\mathbf{u}\|_1^2 + \langle p, \nabla\cdot\mathbf{u}\rangle.
\end{aligned} \tag{2.6}$$

On the other hand, from (2.2), (2.5), and the Cauchy-Schwarz's inequality follows that

$$\begin{aligned}
\langle p, \nabla\cdot\mathbf{u}\rangle &= \langle p, \boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\rangle - \langle p, \boldsymbol{\beta}\cdot\nabla p\rangle \\
&\leq \|\boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\|\,\|p\| + \gamma_0(\boldsymbol{\beta})\|p\|^2 \\
&\leq C|||\mathbf{u}, p|||^2 + \left(\frac{\gamma_1(\boldsymbol{\beta})}{4} + 2C_1^2\gamma_0(\boldsymbol{\beta})\right)\|\mathbf{u}\|_1^2.
\end{aligned} \tag{2.7}$$

Combining (2.6), (2.7), and (A_0) gives that

$$\|\mathbf{u}\|_1^2 \leq C\|-\mu\Delta\mathbf{u} - \nu\nabla(\nabla\cdot\mathbf{u}) + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}^2 + \|\boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\|^2. \tag{2.8}$$

Now, using (2.5), (2.8), and triangle inequality

$$\|p\|_Q^2 \leq C\|-\mu\Delta\mathbf{u} - \nu\nabla(\nabla\cdot\mathbf{u}) + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}^2 + \|\boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\|^2.$$

This completes the proof of the theorem. ∎

We consider an alternative restriction for $\boldsymbol{\beta}$ in the next corollary.

Corollary 2.1. *Assume that*

$$\nabla\cdot\boldsymbol{\beta} \leq 0 \quad \text{in } \Omega \qquad \text{and} \qquad \gamma_0(\boldsymbol{\beta}) < \frac{\mu_0}{\lambda}. \tag{A_1}$$

Then, (2.4) holds for any $(\mathbf{u}, p) \in H_0^1(\Omega)^2 \times Q(\boldsymbol{\beta}; \Omega)$.

Proof. If $\nabla\cdot\boldsymbol{\beta} \leq 0$ in Ω, then instead of (2.2) we have that

$$(\boldsymbol{\beta}\cdot\nabla p, p) \geq 0 \qquad \forall\, p \in Q(\boldsymbol{\beta}; \Omega). \tag{2.9}$$

Thus, instead of (2.7) we obtain the alternative bound

$$\begin{aligned}
(p, \nabla\cdot\mathbf{u}) &= (p, \boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}) - (p, \boldsymbol{\beta}\cdot\nabla p) \\
&\leq \|\boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\|\,\|p\|.
\end{aligned} \tag{2.10}$$

If $\gamma_0(\boldsymbol{\beta}) < \mu_0/\lambda$, then $\gamma_1(\boldsymbol{\beta}) > 0$, and so from (2.5), (2.6), and (2.10) easily follows that

$$\gamma_1(\boldsymbol{\beta})\|\mathbf{u}\|_1^2 \le C\|-\mu\Delta\mathbf{u} - \nu\nabla(\nabla\cdot\mathbf{u}) + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}^2 + \|\boldsymbol{\beta}\cdot\nabla p + \nabla\cdot\mathbf{u}\|^2.$$

Validity of (2.9) is an immediate consequence of this inequality. ■

Theorem 2.1 and Corollary 2.1 can be used to set up a well-posed least-squares principle for the second-order system (1.1). We will not pursue this direction here because our main focus is on more practical least-squares finite element methods based on first-order systems. Thus, Theorem 2.1 will serve an auxiliary role in proving the norm equivalence of the first-order system least-squares functionals. For examples of second-order formulations the interested reader can consult [8]. There, *a priori* estimates similar to those in Theorem 2.1 are established under the assumption that $\boldsymbol{\beta}$ vanishes on the boundary $\partial\Omega$. In contrast, our analysis in Theorem 2.1 is carried under more general assumption that $\boldsymbol{\beta}$ vanishes only on Γ_{in}. The main complication in this case stems from the fact that p is not in L_0^2 and so the inequality $\|p\| \le C\|\nabla p\|_{-1}$ does not hold.

III. CONTINUOUS LEAST-SQUARES PRINCIPLE

In this section we formulate an equivalent first-order form of the compressible Stokes problem (1.1) and then develop a least-squares variational principle for this system. The main focus of the section is on demonstrating the norm-equivalence of the least-squares functional.

To obtain the first-order system, we introduce the vorticity $\omega = \nabla\times\mathbf{u}$ and the divergence $\phi = \nabla\cdot\mathbf{u}$ as new dependent variables. Using the vector identity

$$\nabla\times(\nabla\times\mathbf{u}) = -\Delta\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}), \tag{3.1}$$

to rewrite the momentum equation in (1.1) in terms of $\omega, \mathbf{u}$, and p, and adding the definitions of the new variables to (1.1) yields the equivalent system

$$\begin{cases} \mu\nabla\times\omega - (\mu+\nu)\nabla\phi + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega,\\ \phi + \boldsymbol{\beta}\cdot\nabla p = g, & \text{in } \Omega,\\ \omega - \nabla\times\mathbf{u} = 0, & \text{in } \Omega,\\ \phi - \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega,\\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega\\ p = 0, & \text{on } \Gamma_{\text{in}}, \end{cases} \tag{3.2}$$

where ϕ satisfies a zero mean constraint,

$$\int_\Omega \phi\, dx = 0.$$

To formulate a least-squares principle for (3.2) let

$$\mathcal{V} = L_0^2(\Omega)\times L_0^2(\Omega)\times H_0^1(\Omega)^2\times Q(\boldsymbol{\beta};\Omega),$$

with norm

$$\|(\omega,\phi,\mathbf{u},p)\|_{\mathcal{V}}^2 = \|\omega\|^2 + \|\phi\|^2 + \|\mathbf{u}\|_1^2 + \|p\|_Q^2.$$

We consider the quadratic least-squares functional

$$\mathcal{J}(\omega,\phi,\mathbf{u},p;\,\mathbf{f},g) = \|\mu\nabla\times\omega - (\mu+\nu)\nabla\phi + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p - \mathbf{f}\|_{-1}^2 + \|\phi + \boldsymbol{\beta}\cdot\nabla p - g\|^2 + \|\omega - \nabla\times\mathbf{u}\|^2 + \|\phi - \nabla\cdot\mathbf{u}\|^2, \quad (3.3)$$

and the least-squares principle: seek $(\omega,\phi,\mathbf{u},p)\in\mathcal{V}$ such that

$$\mathcal{J}(\omega,\phi,\mathbf{u},p;\,\mathbf{f},g) \le \mathcal{J}(\chi,\psi,\mathbf{v},q;\,\mathbf{f},g), \quad (3.4)$$

for all $(\chi,\psi,\mathbf{v},q)\in\mathcal{V}$. Next Theorem shows that (3.4) is a well-posed minimization problem.

Theorem 3.1. *Assume that (A_0) or (A_1) hold. Then, there exist two positive constants c and C, which depend on $\mu,\nu,\boldsymbol{\beta}$, and Ω, and such that, for all $(\omega,\phi,\mathbf{u},p)\in\mathcal{V}$,*

$$c\|(\omega,\phi,\mathbf{u},p)\|_{\mathcal{V}}^2 \le \mathcal{J}(\omega,\phi,\mathbf{u},p;\mathbf{0},0) \le C\|(\omega,\phi,\mathbf{u},p)\|_{\mathcal{V}}^2. \quad (3.5)$$

Proof. The upper bound in (3.5) follows easily from the triangle inequality. To prove the lower bound note that if either (A_0) or (A_1) holds, then the *a priori* estimate (2.4) holds too. Let $(\omega,\phi,\mathbf{u},p)$ denote an arbitrary element of $\mathcal{V}$. Using (2.4), the triangle inequality and the fact that

$$\|\nabla\times q\|_{-1}\le\|q\| \qquad \text{and} \qquad \|\nabla q\|_{-1}\le\|q\|, \qquad \forall q\in L^2(\Omega),$$

we obtain the bound

$$\begin{aligned}
\|\mathbf{u}\|_1^2 + \|p\|_Q^2 &\le C|||\mathbf{u},p|||^2\\
&= C\big(\|-\mu\Delta\mathbf{u} - \nu\nabla\nabla\cdot\mathbf{u} + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}^2 + \|\nabla\cdot\mathbf{u} + \boldsymbol{\beta}\cdot\nabla p\|^2\big)\\
&\le C\big(\|\mu\nabla\times\omega - (\mu+\nu)\nabla\phi + (\boldsymbol{\beta}\cdot\nabla)\mathbf{u} + \nabla p\|_{-1}^2 + \|\nabla\times(\omega - \nabla\times\mathbf{u})\|_{-1}^2\\
&\quad + \|\nabla(\phi - \nabla\cdot\mathbf{u})\|_{-1}^2 + \|\phi + \boldsymbol{\beta}\cdot\nabla p\|^2 + \|\phi - \nabla\cdot\mathbf{u}\|^2\big)\\
&\le C\,\mathcal{J}(\omega,\phi,\mathbf{u},p;\mathbf{0},0), \qquad (3.6)
\end{aligned}$$

where C is a constant depending on $\mu,\nu,\boldsymbol{\beta}$, and Ω. Furthermore,

$$\|\omega\|\le\|\omega - \nabla\times\mathbf{u}\| + \|\nabla\times\mathbf{u}\| \qquad \text{and} \qquad \|\phi\|\le\|\phi - \nabla\cdot\mathbf{u}\| + \|\nabla\cdot\mathbf{u}\|. \quad (3.7)$$

The lower bound of (3.5) and the theorem follow from combining (3.6) and (3.7). ∎

IV. DISCRETE LEAST-SQUARES PRINCIPLES

Formally, restriction of (3.4) to a finite dimensional subspace $\mathcal{V}_h$ of $\mathcal{V}$ will yield a well-posed discrete problem. However, the presence of negative norms in (3.3) means that computation of the discrete minimizer out of $\mathcal{V}_h$ will not be possible in a practical manner.

In this section we use (3.3) as a template to define a discrete least-squares functional that is practical and at the same time retains the attractive properties of its continuous counterpart. To discuss the discrete functional and the associated finite element methods, we will assume that $\mathcal{T}_h$ is a uniformly regular triangulation of Ω into finite elements, parametrized by the mesh parameter

h. Furthermore, we assume that $\mathcal{V}_h := V_{h,1} \times V_{h,2} \times V_{h,3} \times V_{h,4}$ is a finite dimensional subspace of $\mathcal{V}$ such that for $(\omega, \phi, \mathbf{u}, p) \in \mathcal{V} \cap [H^r(\Omega) \times H^r(\Omega) \times H^{r+1}(\Omega)^2 \times H^{r+1}(\Omega)](r \geq 1)$,

$$\inf_{\chi_h \in V_{h,1}} \{\|\omega - \chi_h\| + h\|\omega - \chi_h\|_1\} \leq Ch^r \|\omega\|_r, \tag{4.1}$$

$$\inf_{\psi_h \in V_{h,2}} \{\|\phi - \psi_h\| + h\|\phi - \psi_h\|_1\} \leq Ch^r \|\phi\|_r, \tag{4.2}$$

$$\inf_{\mathbf{v}_h \in V_{h,3}} \{\|\mathbf{u} - \mathbf{v}_h\| + h\|\mathbf{u} - \mathbf{v}_h\|_1\} \leq Ch^{r+1} \|\mathbf{u}\|_{r+1}, \tag{4.3}$$

$$\inf_{q_h \in V_{h,4}} \{\|p - q_h\| + h\|p - q_h\|_1\} \leq Ch^{r+1} \|p\|_{r+1}, \tag{4.4}$$

for some positive C independent of h. The spaces $V_{h,i}$ can be constructed from the standard Lagrangian finite element spaces S_h^d of continuous, piecewise polynomial functions whose degree on each element $K \in \mathcal{T}_h$ equals d. We recall that for such spaces on regular triangulations there exists a positive constant C such that

$$\|\phi_h\|_1 \leq h^{-1} C \|\phi_h\|_0 \qquad \forall \phi_h \in S_h^d. \tag{4.5}$$

We recall (see Lemma 2.1 in [10]) that for all $\mathbf{g} \in H^{-1}(\Omega)^2$

$$\|\mathbf{g}\|_{-1}^2 = (T\mathbf{g}, \mathbf{g}), \tag{4.6}$$

where $T : H^{-1}(\Omega)^2 \to H_0^1(\Omega)^2$ is the solution operator of the boundary value problem

$$-\Delta\, \mathbf{w} + \mathbf{w} = \mathbf{g} \quad \text{in } \Omega \qquad \text{and} \qquad \mathbf{w} = \mathbf{0} \quad \text{on } \partial\Omega.$$

Using (4.6) we can rewrite the continuous least-squares functional (3.3) as

$$\begin{aligned} \mathcal{J}(\omega, \phi, \mathbf{u}, p\,; \mathbf{f}, g) = (T(&\mu\nabla \times \omega - (\mu + \nu)\nabla\phi + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{f}), \\ &\mu\nabla \times \omega - (\mu + \nu)\nabla\phi + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{f}) \\ &+ \|\phi + \boldsymbol{\beta} \cdot \nabla p - g\|^2 + \|\omega - \nabla \times \mathbf{u}\|^2 + \|\phi - \nabla \cdot \mathbf{u}\|^2. \end{aligned} \tag{4.7}$$

Following [10] and [11], we will define a practical least-squares functional by replacing the continuous operator T by a discrete operator T_h. To this end, let $\tilde{B}_h : H^{-1}(\Omega)^2 \to V_{h,3}$ denote a solution operator of the discrete weak equation: seek $\mathbf{w}_h \in V_{h,3}$ such that

$$(\nabla\mathbf{w}_h, \nabla\mathbf{v}_h) + (\mathbf{w}_h, \mathbf{v}_h) = (\mathbf{g}, \mathbf{v}_h), \qquad \forall\, \mathbf{v}_h \in V_{h,3}. \tag{4.8}$$

In other words, given $\mathbf{g} \in H^{-1}(\Omega)$, $\tilde{B}_h\, \mathbf{g} = \mathbf{w}_h$ if and only if $\mathbf{w}_h$ solves (4.8).

Computation of $\tilde{B}_h$ requires solution of a Poisson equation on the same mesh where we pose our primary problem. To define T_h, we assume that there is a preconditioner $B_h : H^{-1}(\Omega)^2 \to V_{h,3}$ for $\tilde{B}_h$, which is symmetric positive definite operator with respect to the $L^2(\Omega)$-inner product and spectrally equivalent to $\tilde{B}_h$, i.e.,

$$C_1(\tilde{B}_h\mathbf{v}_h, \mathbf{v}_h) \leq (B_h\mathbf{v}_h, \mathbf{v}_h) \leq C_2(\tilde{B}_h\mathbf{v}_h, \mathbf{v}_h) \qquad \text{for all } \mathbf{v}_h \in V_{h,3}, \tag{4.9}$$

but which is cheaper to evaluate. Let $Q_h : L^2(\Omega)^2 \to V_{h,3}$ be the L^2-orthogonal projection onto $V_{h,3}$. We assume that Q_h is bounded in H^1, i.e.,

$$\|Q_h \mathbf{v}_h\|_1 \le C\|\mathbf{v}_h\|_1 \qquad \forall \mathbf{v}_h \in V_{h,3}.$$

Then, one can show; see [10], that

$$\|(I - Q_h)\mathbf{v}\|_{-1} \le Ch\|\mathbf{v}\|, \qquad \text{for all } \mathbf{v} \in L^2(\Omega)^2 \tag{4.10}$$

and

$$\|Q_h \mathbf{v}\|_{-1}^2 \le C(\tilde{B}_h \mathbf{v}, \mathbf{v}) \le C\|\mathbf{v}\|_{-1}^2 \qquad \text{for all } \mathbf{v} \in L^2(\Omega)^2. \tag{4.11}$$

Finally, from the symmetry of B_h and the identities $B_h = B_h Q_h$ and $\tilde{B}_h = \tilde{B}_h Q_h$, it follows that (4.9) holds for any $\mathbf{v} \in L^2(\Omega)^2$.

We now proceed to define the operator

$$T_h := h^2 I + \gamma B_h, \tag{4.12}$$

where I denotes identity and $\gamma > 0$ is a real parameter. Using (4.12) in place of T gives the discrete counterpart of (3.3):

$$\begin{aligned} \mathcal{J}_h(\omega, \phi, \mathbf{u}, p; \mathbf{f}, g) = (&T_h(\mu\nabla \times \omega - (\mu + \nu)\nabla\phi + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{f}), \\ &\mu\nabla \times \omega - (\mu + \nu)\nabla\phi + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u} + \nabla p - \mathbf{f}) \\ &+ \|\phi + \boldsymbol{\beta} \cdot \nabla p - g\|^2 + \|\omega - \nabla \times \mathbf{u}\|^2 + \|\phi - \nabla \cdot \mathbf{u}\|^2. \end{aligned} \tag{4.13}$$

The discrete least-squares principle associated with (4.13) is

$$\min_{(\chi_h, \psi_h, \mathbf{v}_h, q_h) \in \mathcal{V}_h} \mathcal{J}_h(\chi_h, \psi_h, \mathbf{v}_h, q_h; \mathbf{f}, g). \tag{4.14}$$

The optimality condition for (4.14) is to seek $U_h = (\omega_h, \phi_h, \mathbf{u}_h, p_h) \in \mathcal{V}_h$ such that

$$\mathcal{B}_h(U_h, V_h) = \mathcal{F}_h(V_h) \qquad \text{for all } V_h = (\chi_h, \psi_h, \mathbf{v}_h, q_h) \in \mathcal{V}_h, \tag{4.15}$$

and where

$$\begin{aligned} \mathcal{B}_h(U_h, V_h) = (&\mu\nabla \times \omega_h - (\mu + \nu)\nabla\phi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h + \nabla p_h, \\ &T_h\,(\mu\nabla \times \chi_h - (\mu + \nu)\nabla\psi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{v}_h + \nabla q_h)) \\ &+ (\phi_h + \boldsymbol{\beta} \cdot \nabla p_h, \psi_h + \boldsymbol{\beta} \cdot \nabla q_h) + (\omega_h - \nabla \times \mathbf{u}_h, \chi_h - \nabla \times \mathbf{v}_h) \\ &+ (\phi_h - \nabla \cdot \mathbf{u}_h, \psi_h - \nabla \cdot \mathbf{v}_h), \end{aligned} \tag{4.16}$$

$$\mathcal{F}_h(V_h) = (\mathbf{f}, T_h(\mu\nabla \times \chi_h - (\mu + \nu)\nabla\psi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{v}_h + \nabla q_h)) + (g, \psi_h + \boldsymbol{\beta} \cdot \nabla q_h). \tag{4.17}$$

Our next result reveals the norm-equivalence properties of (4.13).

Theorem 4.1. *Assume that (A_0) or (A_1) hold. Then, there exist two positive constants c and C, which depend on $\mu, \nu, \boldsymbol{\beta}$, and Ω, and such that for all $(\omega_h, \phi_h, \mathbf{u}_h, p_h) \in \mathcal{V}_h$,*

$$c\|(\omega_h, \phi_h, \mathbf{u}_h, p_h)\|_{\mathcal{V}}^2 \le \mathcal{J}_h(\omega_h, \phi_h, \mathbf{u}_h, p_h) \le C\|(\omega_h, \phi_h, \mathbf{u}_h, p_h)\|_{\mathcal{V}}^2. \tag{4.18}$$

Proof. We follow the arguments of ([10]). It is clear that for any finite element function $(\omega_h, \phi_h, \mathbf{u}_h, p_h) \in \mathcal{V}_h$ the function

$$\mathbf{v} \equiv \mu\nabla \times \omega_h - (\mu + \nu)\nabla\phi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h + \nabla p_h \tag{4.19}$$

is in $L^2(\Omega)^2$. From (4.11) it is easy to see that

$$(T_h\mathbf{v}, \mathbf{v}) \le C\big(h^2\|\mathbf{v}\|^2 + \|\mathbf{v}\|_{-1}^2\big) \qquad \forall \mathbf{v} \in \mathbf{L}^2(\Omega).$$

This, together with the triangle inequality and the inverse inequality (4.5), implies the upper bound in (4.18). To prove the lower bound in (4.18), it is enough, according to Theorem 3.1, to show that

$$\begin{aligned}
&\|\mu\nabla \times \omega_h - (\mu + \nu)\nabla\phi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h + \nabla p_h\|_{-1}^2 \\
&\quad \le C(T_h(\mu\nabla \times \omega_h - (\mu + \nu)\nabla\phi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u} + \nabla p), \\
&\qquad \mu\nabla \times \omega_h - (\mu + \nu)\nabla\phi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h + \nabla p_h).
\end{aligned}$$

From (4.10) and (4.11), for any $\mathbf{v} \in L^2(\Omega)^2$, we have

$$\|\mathbf{v}\|_{-1}^2 \le 2\big(\|(I - Q_h)\mathbf{v}\|_{-1}^2 + \|Q_h\mathbf{v}\|_{-1}^2\big) \le C\big(h^2\|\mathbf{v}\|^2 + (\tilde{B}_h\mathbf{v}, \mathbf{v})\big) \le C(T_h\mathbf{v}, \mathbf{v}).$$

Setting $\mathbf{v}$ equal to the function in (4.19) completes the proof. ■

Corollary 4.1. *Problem (4.14) has a unique minimizer $U_h \in \mathcal{V}_h$.*

Proof. From (4.18) it follows that form $\mathcal{B}_h$ is coercive and continuous on $\mathcal{V}_h \times \mathcal{V}_h$. It is not difficult to see that $\mathcal{F}_h$ is continuous linear functional $\mathcal{V}_h \mapsto \Re$ and so Lax-Milgram lemma implies that (4.15) has a unique solution. ■

Theorem 4.1 asserts that (4.13) has the same norm-equivalence properties as its continuous prototype (3.3) on any C^0 conforming finite element subspace of $\mathcal{V}$. In the next theorem we use this fact to establish optimal convergence rates for the finite element solutions of (4.13).

Theorem 4.2. *Let the assumption (2.4) hold and assume further that*

$$U = (\omega, \phi, \mathbf{u}, p) \in \mathcal{V} \cap \big[H^r(\Omega) \times H^r(\Omega) \times H^{r+1}(\Omega)^2 \times H^{r+1}(\Omega)\big], \qquad r \ge 1$$

is the solution of (3.2). Let $U_h = (\omega_h, \phi_h, \mathbf{u}_h, p_h) \in \mathcal{V}_h$ be the unique minimizer of (4.14). Then, there exists a positive constant C depending on $\mu, \nu, \boldsymbol{\beta}$, and Ω, such that

$$\begin{aligned}
\|U - U_h\|_{\mathcal{V}} &\equiv \|(\omega - \omega_h, \phi - \phi_h, \mathbf{u} - \mathbf{u}_h, p - p_h)\|_{\mathcal{V}} \\
&\le C\, h^r(\|\omega\|_r + \|\phi\|_r + \|\mathbf{u}\|_{r+1} + \|p\|_{r+1})\,.
\end{aligned} \tag{4.20}$$

Proof. Let U_h^I be the element of $\mathcal{V}_h$ that verifies (4.1)–(4.4). Then,

$$\begin{aligned}\|U - U_h\|_{\mathcal{V}} &\leq \|U - U_h^I\|_{\mathcal{V}} + \|U_h^I - U_h\|_{\mathcal{V}} \\ &\leq C\, h^r(\|\omega\|_r + \|\phi\|_r + \|\mathbf{u}\|_{r+1} + \|p\|_{r+1}) + \|U_h^I - U_h\|_{\mathcal{V}}.\end{aligned}$$

Using (4.18) in Theorem 4.1 for the discrete error gives

$$\begin{aligned}c\|U_h^I - U_h\|_{\mathcal{V}}^2 &\leq \mathcal{B}_h\left(U_h^I - U_h, U_h^I - U_h\right) = \mathcal{B}_h\left(U_h^I - U, U_h^I - U_h\right) \\ &\leq C\|U_h^I - U\|_{\mathcal{V}}\|U_h^I - U_h\|_{\mathcal{V}},\end{aligned}$$

where the last inequality follows from the error orthogonality. As a result,

$$\|U_h^I - U_h\|_{\mathcal{V}} \leq \frac{C}{c}\|U_h^I - U\|_{\mathcal{V}} \leq \tilde{C}\, h^r(\|\omega\|_r + \|\phi\|_r + \|\mathbf{u}\|_{r+1} + \|p\|_{r+1})\,.$$

This completes the proof. ∎

The norm-equivalence (4.18) holds for the discrete functional (4.13) whenever B_h is spectrally equivalent to $\tilde{B}_h$. The operator $B_h \equiv h^2 I$ does not verify (4.9) but is very simple to work with. Using this operator we have the alternative definition

$$T_h = \alpha h^2 I$$

that leads to the weighted L^2 norm least-squares functional

$$\begin{aligned}\mathcal{J}_h^w(\omega_h, \phi_h, \mathbf{u}_h, p_h; \mathbf{f}, g) = \alpha\, h^2\|\mu\nabla \times \omega_h - (\mu + \nu)\nabla\phi_h + (\boldsymbol{\beta} \cdot \nabla)\mathbf{u}_h + \nabla p_h - \mathbf{f}\|^2 \\ + \|\phi_h + \boldsymbol{\beta} \cdot \nabla p_h - g\|^2 + \|\omega_h - \nabla \times \mathbf{u}_h\|^2 + \|\phi_h - \nabla \cdot \mathbf{u}_h\|^2. \quad (4.21)\end{aligned}$$

Similar weighted functionals were studied in [6] and [12] for the incompressible Stokes problem. However, for (4.21) one can only show that

$$c\|(\omega_h, \phi_h, \mathbf{u}_h, p_h)\|_{\mathcal{V}}^2 \leq \mathcal{J}_h^w(\omega_h, \phi_h, \mathbf{u}_h, p_h) \leq C\alpha h^{-2}\|(\omega_h, \phi_h, \mathbf{u}_h, p_h)\|_{\mathcal{V}}^2, \quad (4.22)$$

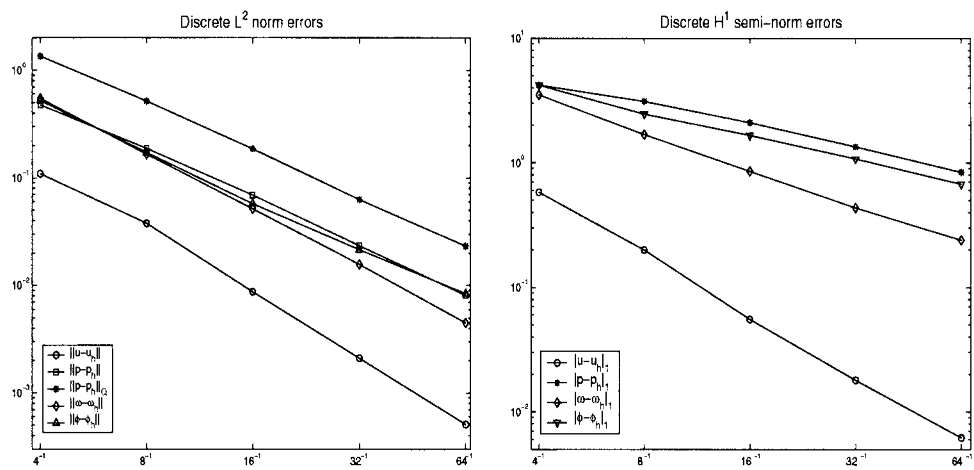

FIG. 1. Finite element solution errors as function of the mesh size: $(\mathcal{J}_h)$.

TABLE I. Numerical L^2 norm convergence rates: $(\mathcal{J}_h)$.

Level	$\|\mathbf{u}-\mathbf{u}_h\|$	$\|p-p_h\|$	$\|p-p_h\|_Q$	$\|\omega-\omega_h\|$	$\|\phi-\phi_h\|$
3	1.5278	1.3404	1.3883	1.7171	1.6006
4	2.1172	1.4399	1.4731	1.6858	1.5806
5	2.0546	1.5665	1.5695	1.7196	1.4215
6	2.0466	1.5301	1.4364	1.8084	1.3579

i.e., (4.21) is only *quasi norm-equivalent*. Because (1.1) is an incompletely elliptic system, analysis of [6] and [12] cannot be extended to the present case. Nevertheless, simplicity of the weighted functional makes it very attractive computationally. In the next section we compare this functional (4.21) with its norm-equivalent counterpart (4.13).

V. NUMERICAL EXAMPLES

Below we present numerical results obtained with the two discrete least-squares functionals introduced in the last section. One may refer to the implementation given in [13]. The goal of our experiments is twofold. First, we want to confirm the asymptotic error estimates (4.20) for the discrete functional (4.13). The second objective is to compare computationally performance of the norm-equivalent, but more expensive, functional (4.13) with that of the simpler, but only quasi-norm equivalent, weighted functional (4.21). The goal is to determine whether or not (4.21) can serve as a sensible alternative for a truly norm-equivalent least-squares functional.

All experiments were carried on with $\mu = 1, \nu = 0$, and $\boldsymbol{\beta} = (1, 0)^t$. The right-hand sides were obtained by evaluating (3.2) for a known exact solution. The operator B_h used to define T_h in (4.12) was selected to be one sweep of the multigrid $V(1,1)$-cycle algorithm with pointwise Gauss-Seidel smoothing applied to (4.8).

In all cases $\mathcal{V}_h$ was defined using equal order interpolation finite element spaces. We report numerical results with piecewise linear finite element spaces. For such spaces (4.1)–(4.4) hold with $r = 1$.

A. Numerical Results on the Unit Square

Let $\Omega = [0, 1]^2$. We consider a uniform triangulation of Ω obtained by subdividing the region into 2^{k+1} squares and then drawing the diagonal in each square. The level k used in this study range from $k = 2$ to $k = 7$. The mesh size for each level k equals 2^{-k}. The convergence rates for level $k + 1$ in $L^2(\Omega)$ and $H^1(\Omega)$ norm discretization errors are measured by

$$\log_2 \frac{\|\chi - \chi_k\|}{\|\chi - \chi_{k+1}\|} \qquad \text{and} \qquad \log_2 \frac{\|\chi - \chi_k\|_1}{\|\chi - \chi_{k+1}\|_1}$$

TABLE II. Numerical H^1 norm convergence rates: $(\mathcal{J}_h)$.

Level	$\|\mathbf{u}-\mathbf{u}_h\|_1$	$\|p-p_h\|_1$	$\|\omega-\omega_h\|_1$	$\|\phi-\phi_h\|_1$	$\|U-U_h\|_{\mathcal{V}}$
3	1.5419	0.4370	1.0543	0.7709	1.4552
4	1.8475	0.5640	0.9798	0.5645	1.5289
5	1.6239	0.6437	0.9780	0.6294	1.5690
6	1.5331	0.6765	0.8562	0.6746	1.4490

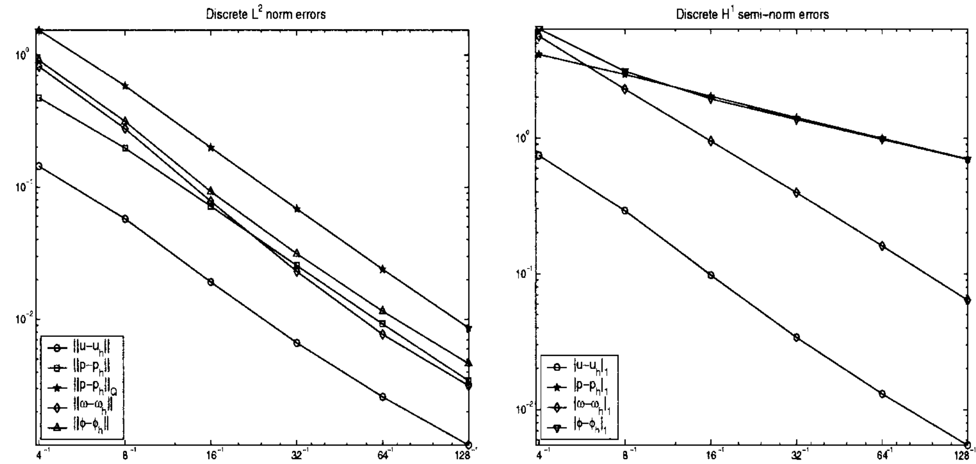

FIG. 2. Finite element solution errors as function of the mesh size: ($\mathcal{J}_h^w$).

for each variable $\chi = \mathbf{u}, p, \omega, \phi$, where χ and χ_k denote by the exact solution and approximate solution of level k, respectively. The exact solutions for the unit square domain are

$$u_1 = u_2 = \sin(\pi x)\sin(\pi y) \qquad \text{and} \qquad p = x\exp(\pi y). \tag{5.1}$$

Our results for the discrete least-squares functional $\mathcal{J}_h(\cdot;\cdot)$ in (4.13) are summarized in Fig. 1 and Tables I and II. Fig. 1 shows plots of various error norms that were used to compute numerical convergence rates reported in Tables I and II. The rates summarized in these tables show very good agreement with the theoretical error rates established in Theorem 4.2. We note that some of the numerical rates are actually higher than the theoretical rates. We also note that the L^2 error rate for the velocity appears to be optimal, even though this error is not covered by the analysis in Theorem 4.2.

Next we consider the simpler, weighted L^2 norm least-squares functional $\mathcal{J}_h^w(\cdot;\cdot)$ in (4.21). Recall that (4.21) can be viewed as a version of (4.13), where T_h was replaced by the scaled identity $h^2 I$, leading to a discrete negative norm operator $T_h = \alpha h^2 I$. In all experiments $\alpha = h$, a value that was determined to yield the best performance of the weighted least-squares functional.

Fig. 2 shows plots of finite element errors for (4.21). The convergence rates corresponding to the error plots are summarized in Tables III and IV.

From the results in Tables III and IV we see that convergence rates of the weighted least-squares method are smaller than the rates obtained with the norm-equivalent least-squares functional (4.13). This is especially true for the L^2 error in the velocity that drops to $O(h^{1.2})$ for the weighted

TABLE III. Numerical L^2 norm convergence rates: ($\mathcal{J}_h^w$).

Level	$\lVert\mathbf{u} - \mathbf{u}_h\rVert$	$\lVert p - p_h\rVert$	$\lVert p - p_h\rVert_Q$	$\lVert\omega - \omega_h\rVert$	$\lVert\phi - \phi_h\rVert$
3	1.3214	1.3971	1.2674	1.5638	1.5387
4	1.5881	1.5547	1.4621	1.8239	1.7610
5	1.5270	1.5404	1.4894	1.7662	1.5665
6	1.3570	1.5173	1.4668	1.5792	1.4419
7	1.2094	1.4692	1.4273	1.2865	1.3117

TABLE IV. Numerical H^1 norm convergence rates: ($\mathcal{J}_h^w$).

Level	$\|\mathbf{u} - \mathbf{u}_h\|_1$	$\|p - p_h\|_1$	$\|\omega - \omega_h\|_1$	$\|\phi - \phi_h\|_1$	$\|U - U_h\|_{\mathcal{V}}$
3	1.3518	0.4908	1.2989	1.0354	1.4391
4	1.5769	0.5351	1.2737	0.6771	1.6185
5	1.5222	0.5190	1.2608	0.5050	1.5599
6	1.3827	0.5086	1.3042	0.4873	1.4883
7	1.2510	0.5040	1.3259	0.4912	1.3892

functional. However, the error in the composite norm $\|U - U_h\|_{\mathcal{V}}$ is still of order $O(h^{1.4})$, i.e., above the theoretically optimal value predicted by Theorem 4.2.

B. Numerical Results on the Unit Disk

We will now test the weighted least-squares functional $\mathcal{J}_h^w(\cdot;\cdot)$ in (4.21) on the unit disk using piecewise linear finite element spaces. The triangulation of Ω is defined by using affine isoparametric families of finite elements on triangles, i.e., the computational domain $\widehat{\Omega}$ is a polygonal approximation to the unit disk domain Ω. This leads to an error of order $O(h^2)$ in the approximation of $\partial\Omega$ by $\partial\widehat{\Omega}$. Fig. 3 shows the initial triangulation of the unit disk. The refined triangulations $\mathcal{T}_2, \ldots, \mathcal{T}_j$ are generated by subdividing each triangle in $\mathcal{T}_k$ into four congruent triangles of $\mathcal{T}_{k+1}$. However, when two vertices of the triangle are located on $\partial\Omega$, the midpoint of two vertices is projected onto the curved boundary of the side with outward normal direction. A refined triangulation $\mathcal{T}_2$ is shown on the right-hand side of Fig. 3. To define the least-squares method we set $\alpha = h$ in (4.21). Then we consider an exact solution

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 - x^2 - y^2 \\ \frac{1}{2}(1+y)[(1-y^2)^{\frac{3}{2}} - |x|^3] \end{pmatrix},$$

$$p = \frac{3}{2}(1+y)(x-1)(x+\sqrt{1-y^2}).$$

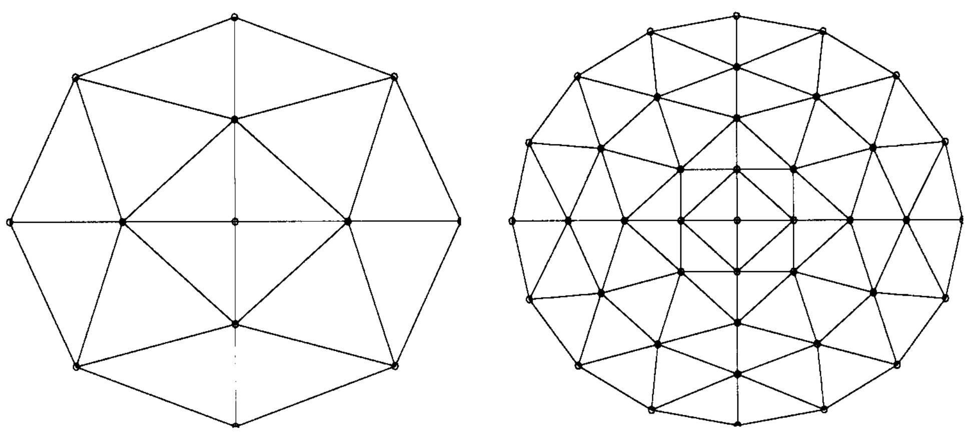

FIG. 3. Initial triangulation and refined triangulation.

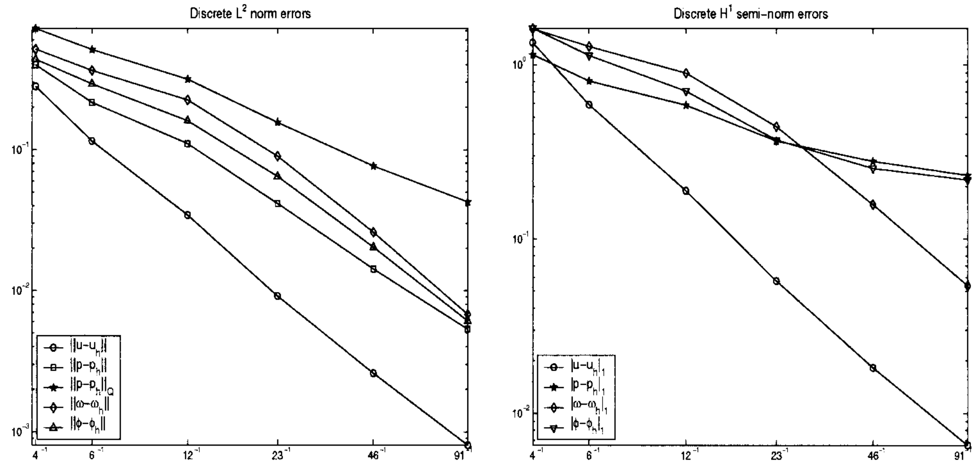

FIG. 4. L^2 norm errors and H^1 semi-norm errors: $(\mathcal{J}_h^w)$.

that was used in [5]. The derivatives $\partial_y p$ and $\partial_{yy} u_2$ of this solution have mild singularities near $(0, 1)$. In particular, near this point $\partial_y p$ behaves like $\partial_{yy} u_2$; see [5]. After substituting the exact solution in (1.1) we find that

$$
\begin{aligned}
f_1 &= -\mu\Delta u_1 + \partial_x u_1 + \partial_x p = 4\mu + x(3y+1) - \frac{3}{2}(y+1)(1-\sqrt{1-y^2}),\\
f_2 &= -\mu\Delta u_2 + \partial_x u_2 + \partial_y p = 3\mu|x|(1+y) - \frac{3}{2}x|x|(y+1) + \frac{3}{2}x(x-1)\\
&\quad + \frac{3}{2}[\mu(1+3y) + x - 1]\sqrt{1-y^2} + \frac{3}{2}(1-x-\mu y)(y+y^2)\frac{1}{\sqrt{1-y^2}},\\
g &= \partial_x u_1 + \partial_y u_2 + \partial_x p = 2(1-y^2)^{\frac{3}{2}} - \frac{1}{2}|x|^3 + x + 3y\left(x - \frac{1}{2}\right) - \frac{3}{2}.
\end{aligned}
$$

Fig. 4 shows plots of the finite element errors as a function of the number of grid points. Numerical convergence rates are reported in Tables V and VI. As expected we see a slight deterioration in the convergence rates. Nevertheless, with the exception of the H^1 error rate of the pressure variable all other rates are reasonable. We also note that the error in the convective derivative of the pressure behaves much better.

TABLE V. Discrete L^2 norm convergence rates: $(\mathcal{J}_h^w)$.

Level	$\|\mathbf{u}-\mathbf{u}_h\|$	$\|p-p_h\|$	$\|p-p_h\|_Q$	$\|\omega-\omega_h\|$	$\|\phi-\phi_h\|$
2	1.2866	0.8855	0.4991	0.5021	0.5778
3	1.7441	0.9688	0.6968	0.6938	0.8669
4	1.9063	1.4121	1.0184	1.3252	1.3118
5	1.8187	1.5424	1.0272	1.7949	1.6685
6	1.6902	1.4170	0.8453	1.9335	1.7361

TABLE VI. Discrete H^1 norm convergence rates: $(\mathcal{J}_h^w)$.

Level	$\|\mathbf{u} - \mathbf{u}_h\|_1$	$\|p - p_h\|_1$	$\|\omega - \omega_h\|_1$	$\|\phi - \phi_h\|_1$	$\|U - U_h\|_{\mathcal{V}}$
2	1.1839	0.5019	0.3400	0.5220	0.8724
3	1.6425	0.4603	0.5076	0.6811	0.9833
4	1.7193	0.6839	1.0188	0.9314	1.2052
5	1.6538	0.3873	1.4884	0.5412	1.2273
6	1.4755	0.2668	1.5466	0.2183	0.9530

VI. CONCLUSIONS

We have formulated two different least-squares functionals for the compressible Stokes problem that are based on an equivalent first-order system. The discretely norm-equivalent functional requires operator B_h that is spectrally equivalent to the discrete Poisson solution operator $\tilde{B}_h$. This method is mathematically guaranteed to be well-posed and we were able to give a rigorous quantification of its convergence rates. We also defined a simpler, weighted L^2 norm least-squares functional wherein spectral equivalence is traded for simplicity and B_h is taken to be a scaled identity operator. Rigorous error analysis for this method remains an open problem. However, our numerical results indicate that its performance is comparable to that of the norm-equivalent method, at least for smooth solutions. Thus, in such cases, the weighted method appears to be a reasonable alternative to the more complex norm-equivalent formulation.

References

1. J. R. Kweon and R. B. Kellog, Compressible Navier-Stokes equations in a bounded domain with Inflow Boundary Condition, SIAM J Math Anal 28 (1997), 94–108.
2. R. B. Kellogg and J. E. Osborn, A regularity result for the Stokes problem in a convex polygon, J Funct Anal 21 (1976), 397–431.
3. R. B. Kellogg and B. Liu, A finite element method for the compressible Stokes equations, SIAM J Numer Anal 33 (1996), 780–788.
4. B. Liu and R. B. Kellog, Discontinous solutions of linearized, steady state, viscous, compressible flows, J Math Anal Appl 180 (1993), 469–497.
5. J. R. Kweon, An optimal order convergence for a weak formulation of the compressible Stokes system with inflow boundary condition, Numer Math 86 (2000), 305–318.
6. P. B. Bochev and M. D. Gunzburger, Analysis of least-squares finite element methods for the Stokes equations, Math Comp 63 (1994), 479–506.
7. Z. Cai, T. Manteuffel, and S. McCormick, First-order system least squares for the Stokes equations, with application to linear elasticity, SIAM J Numer Anal 34 (1997), 1727–1741.
8. Z. Cai and X. Ye, A least-squares finite element approximation for the compressible Stokes equations, Numer Methods Partial Differential Equations 16 (2000), 62–70.
9. S. D. Kim and E. Lee, An analysis for the compressible Stokes equations by H^{-1} fosls finite element method, Numer Meth P.D.E. 17 (2001), 689–699.
10. J. H. Bramble, R. D. Lazarov, and J. E. Pasciak, A least-squares approach based on a discrete minus one inner product for first order system, Math Comp 66 (1997), 935–955.

11. P. B. Bochev, T. A. Manteuffel, and S. F. McCormick, Analysis of velocity-flux least-squares principles for the Navier-Stokes equations. Part II., SIAM J Numer Anal 36 (1999), 1125–1144.

12. P. B. Bochev and M. D. Gunzburger, Least-squares methods for the velocity-pressure-stress formulation of the Stokes equations, Comput Methods Appl Mech Engrg 126 (1995), 267–287.

13. S. D. Kim and B. C. Shin, H^{-1} least-squares method for the velocity-pressure-stress formulation of Stokes equations, Appl Numer Math 40 (2002), 451–465.