STABILIZATION OF LOW-ORDER MIXED FINITE ELEMENTS FOR THE STOKES EQUATIONS *

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Abstract. We present a new family of stabilized methods for the Stokes problem. The focus of the paper is on the lowest order velocity-pressure pairs. While not LBB compliant, the simplicity and the attractive computational properties make these pairs a popular choice in engineering practice. Our stabilization approach is motivated by terms that characterize the LBB "deficiency" of the unstable spaces. The stabilized methods are defined by using these terms to modify the saddle-point Lagrangian associated with the Stokes equations. The new stabilized methods offer a number of attractive computational properties. In contrast to other stabilization procedures, they are parameter free, do not require calculation of higher order derivatives or edge-based data structures, and always lead to symmetric linear systems. Furthermore, the new methods are unconditionally stable, achieve optimal accuracy with respect to solution regularity, and have simple and straightforward implementations. We present numerical results in two and three dimensions that showcase the excellent stability and accuracy of the new methods.

Key words. Stokes equations, stabilized mixed methods, equal-order interpolation, inf-sup condition.

AMS subject classifications. 76D05, 76D07, 65F10, 65F30

1. Introduction. Despite the fact that they violate the LBB stability condition, low order velocity-pressure pairs remain a popular practical choice in mixed finite element approximation of incompressible materials; see e.g. [29] and the references cited therein. This popularity results from factors such as local mass conservation for the lowest order conforming pair (piecewise linear, bilinear or trilinear C^0 velocities and piecewise constant pressures), simple and uniform data structures for the lowest equal order pair (piecewise linear, bilinear or trilinear C^0 velocities *and* pressures), and algebraic problems with manageable sizes and small bandwidths in three dimensions for both pairs. The latter is of paramount importance in engineering applications where geometry resolution requires very fine meshes and higher order elements can quickly lead to intractable algebraic problems in three space dimensions; see [27] for an example setting.

To counteract the lack of LBB stability, low-order pairs are usually supplemented by stabilization or postprocessing procedures that remove spurious pressure modes. Unlike penalty methods (see [16, 22, 24, 25]) for which the goal is to uncouple the pressure and velocity, stabilized methods aim to relax the continuity equation so as to allow application of LBB incompatible spaces. *Consistently stabilized* methods (see e.g., [1, 3, 5, 2, 15, 20, 21]) accomplish this by using the residual of the momentum equation in the added stabilization terms. However, for low-order pairs, pressure and velocity derivatives in this residual term either vanish or are poorly approximated, causing difficulties in the application of consistent stabilization. One possible remedy

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is to reformulate the Stokes problem as a first-order system so that the momentum residual contains only first-order terms [5]. This, of course, leads to more unknowns and larger problems to solve. A second approach is to reconstruct the higher order derivatives [23], or to replace the Laplace operator by a discrete operator [8]. In either case, computation of a global L^2 projection may be required.

It is possible to stabilize unstable velocity pressure pairs without using residuals. One example, motivated by fractional step algorithms for time-dependent problems, is the pressure gradient projection (PGP) method; see [6, 7, 13] and the related local pressure gradient stabilization (LPS) method [4]. In both methods incompressibility constraint is relaxed by using the difference between the discontinuous pressure gradient and its projection onto a piecewise polynomial space. The difference is that PGP projects the pressure gradient onto the continuous velocity space and gives rise to a globally coupled problem, while LPS assumes nested spaces and projects the gradient onto an element patch space, which leads to local problems. However, it is clear that both methods are not appropriate for pairs with constant pressure elements.

Another example of non-residual stabilization are the local and global pressure jump formulations for the bilinear-constant pair [28, 29]. In these methods, the constraint is relaxed by using the jumps of the discontinuous pressure across element interfaces. Application of pressure jump stabilization requires edge-based data structures, and in the case of the local formulation, subdivision of the mesh into patches. Stabilization of the bilinear-constant pair is also considered in [26] where, instead of pressure jumps, local projections onto 2×2 macroelements are employed to relax the continuity equation.

In this paper, we analyze a new, non-residual based, approach to the stabilization of low-order mixed finite element discretizations of the Stokes equations, further developing the idea of polynomial-pressure-projection-based stabilization that was presented and studied computationally in [14]. The starting point for the analysis of the method is a lower bound for a discrete negative semi-norm of the pressure gradient which quantifies the LBB "deficiency" of an unstable pair. We show that the LBB "deficiency" admits a representation in terms of operators with suitable range spaces. This very general characterization opens up a possibility for stabilizing the mixed Stokes equations in a manner that is independent of the space dimension and the shape of the finite elements and also does not require choosing any mesh-dependent parameters.

Our approach differs from existing residual and non-residual stabilization techniques in several important aspects. Most notably, the new methods do not require approximation of derivatives, specification of mesh-dependent parameters, or non-standard data structures. Furthermore, our methods are unconditionally stable, optimally accurate, and always lead to symmetric problems. Their implementation relies on operators whose action can be evaluated locally at the element level using standard finite element techniques. As a result, an existing code can be easily modified to handle the new stabilization procedures.

The paper is organized as follows. The remainder of this section introduces the notation used throughout the paper. Section 2 reviews the mixed variational formulation of the Stokes problem and a weaker form of the LBB stability condition that holds for the spaces of interest to us. The new method is formulated in Section 3. Sections 4 and 5 deal with the stability and the error analysis, respectively, of the new method while Section 6 is a succinct summary of some implementation details. The paper concludes with Section 7 in which the results of a series of numerical experiments are

collected.

1.1. Nomenclature. In what follows, Ω denotes a simply connected bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with a Lipschitz continuous boundary Γ. Throughout the paper, we employ the standard notation $H^l(\Omega)$, $\|\cdot\|_l$, $(\cdot,\cdot)_l$, $l \geq 0$, for the Sobolev spaces of all functions having square integrable derivatives up to order l on Ω, and the standard Sobolev norm and inner product, respectively. When $l = 0$ we will write $L^2(\Omega)$ instead of $H^0(\Omega)$ and drop the index from the inner product designation. $H_0^l(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_l$ and $L_0^2(\Omega)$ will denote the space of all square integrable functions with vanishing mean. Spaces consisting of vector-valued functions will be denoted in bold face. Throughout the paper we use C to denote a generic positive constant whose value may change from place to place but that remains independent of the mesh parameter h.

In this paper, we formulate methods for the Stokes equations that use pressure and velocity finite element spaces defined with respect to the same partition $\mathcal{T}_h$ of Ω into finite elements Ω_e. For instance, Ω_e can be a hexahedron or a tetrahedron in three dimensions, or a triangle or a quadrilateral in two dimensions. The boundary $\partial\Omega_e$ of an element consists of faces γ_f. In two-dimensions, each γ_f is an edge; in three dimensions, γ_f can be triangles or quadrilaterals. We assume that each face is oriented by selecting a normal direction $\mathbf{n}_f$. The set of all interior faces will be denoted by Γ_h. The norm

$$\|u\|_{\Gamma_h} = \Big(\sum_{\gamma_f \in \Gamma_h} \int_{\gamma_f} u^2 \, dS \Big)^{1/2} \tag{1.1}$$

will prove useful in the sequel.

Our main focus is on low-order velocity and pressure pairs. For simplicial elements, we consider the affine finite element families

$$P_1 = \left\{ u^h \in C^0(\Omega) \;\mid\; u^h|_{\Omega_e} \in \mathcal{P}_1(\Omega_e);\ \forall\, \Omega_e \in \mathcal{T}_h \right\}, \tag{1.2}$$

where $\mathcal{P}_1(\Omega_e)$ is the space of linear polynomials on Ω_e. For quadrilateral and hexahedral elements we consider the space

$$Q_1 = \left\{ u^h \in C^0(\Omega) \;\mid\; u^h|_{\Omega_e} = \widehat{u}^h \circ F^{-1};\ \widehat{u}^h \in \mathcal{Q}_1(\widehat{\Omega}_e) \right\}, \tag{1.3}$$

where $\widehat{\Omega}_e$ is a reference element, $F : \widehat{\Omega}_e \mapsto \Omega_e$ is a bilinear or a trilinear mapping, and $\mathcal{Q}_1(\Omega_e)$ is the space of all polynomials on $\widehat{\Omega}_e$ whose degree does not exceed 1 in each coordinate direction. Note that unless Ω_e is a parallelogram or a parallelepiped, u^h is not a piecewise polynomial function. For convenience, in what follows we will use the symbol R_1 to represent both kinds of finite element spaces. In accordance with our earlier convention, vector-valued finite element spaces will be denoted in bold face, e.g., $\mathbf{R}_1$. A well-known approximation result (see [18, p.217]) is that for every $u \in H^2(\Omega)$, there exists a function $u^h \in R_1$ such that

$$\|u - u^h\|_0 + h^{1/2}\|u - u^h\|_{\Gamma_h} + h\|u - u^h\|_1 \leq Ch^2\|u\|_2 \,. \tag{1.4}$$

In addition to the C^0 spaces R_1, we will also need the piecewise constant space

$$R_0 = \left\{ q^h \in L^2(\Omega) \;\mid\; q^h|_{\Omega_e} \in \mathcal{P}_0(\Omega_e);\ \forall\, \Omega_e \in \mathcal{T}_h \right\}, \tag{1.5}$$

where $\mathcal{P}_0(\Omega_e)$ is a constant polynomial space on Ω_e. In (1.5), $\mathcal{T}_h$ can be a simplicial or a non-simplicial partition of Ω into finite elements. The space R_0 has the following

approximation property (see [18, p.102]): for every $q \in H^1(\Omega)$, there exists $q^h \in R_0$ such that

$$\|q - q^h\|_0 \le Ch\|q\|_1 \,. \tag{1.6}$$

Finite element functions satisfy a number of useful inverse inequalities. In particular, we will use the standard inverse inequality

$$\|\nabla q^h\|_0 \le C_I h^{-1}\|q^h\|_0 \tag{1.7}$$

that holds under some mild assumptions on $\mathcal{T}_h$ for all functions in R_1, and the inverse inequality for R_0 functions

$$\|[q^h]\|_{\Gamma_h} \le C_I h^{-1/2}\|q^h\|_0, \quad \forall\, q^h \in R_0 \,, \tag{1.8}$$

where $[q^h]$ denotes the jump of $q^h \in R_0$.

2. Mixed finite element methods for the Stokes problem. We consider the incompressible Stokes problem[1]

$$-\lambda\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{2.2}$$

along with the homogeneous velocity boundary condition

$$\mathbf{u} = 0 \quad \text{on } \Gamma \,. \tag{2.3}$$

The mixed variational form of (2.1)–(2.3) is to seek $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$Q(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \quad \forall\, (\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \,, \tag{2.4}$$

where

$$F(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\Omega \,,$$

$$Q(\mathbf{u}, p; \mathbf{v}, q) = A(\mathbf{u}, \mathbf{v}) + B(\mathbf{v}, p) + B(\mathbf{u}, q) \,, \tag{2.5}$$

$$A(\mathbf{u}, \mathbf{v}) = \lambda \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} d\Omega \,, \qquad \text{and} \qquad B(\mathbf{v}, p) = -\int_\Omega p\nabla\cdot\mathbf{v} d\Omega \,. \tag{2.6}$$

[1] We work with a nondimensional form of the Stokes problem. The dimensional form of the Stokes equation has the form

$$-\mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f} \,,$$

where μ is the given (dynamic) viscosity, ρ the given fluid density, and $\mathbf{f}$ is a given body force per unit mass. We choose a reference speed u_{ref}, length ℓ_{ref}, and density ρ_{ref} which we use to respectively nondimensionalize the velocity $\mathbf{u}$, the position vector $\mathbf{x}$, and the density ρ. We then arrive at (2.1) by nondimensionalizing the pressure p using $\rho_{ref}u_{ref}^2$ and $\rho\mathbf{f}$ using $\rho_{ref}u_{ref}^2/\ell_{ref}$. Then, in (2.1), the nondimensional parameter $\lambda = \mu/(\rho_{ref}\ell_{ref}u_{ref})$ is the inverse of the "Reynolds number."

Solely for the sake of keeping the presentation simple, we make the assumption that μ is constant. For non-constant μ, the viscous term in the Stokes equations (2.1) is given by $\nabla\cdot(\lambda(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$, where λ is no longer constant, and, in (2.6), the first bilinear form is given by $A(\mathbf{u}, \mathbf{v}) = 2\int_\Omega \lambda D(\mathbf{u}) : D(\mathbf{v}) d\Omega$, where $D(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$. Only minor modifications in the analyses are needed.

The mixed variational equation (2.4) is the first-order optimality condition for the saddle-point $(\mathbf{u}, p)$ of the Lagrangian functional

$$L(\mathbf{v}, q) = \frac{\lambda}{2} \int_\Omega |\nabla \mathbf{v}|^2 \, dx - \int_\Omega q \nabla \cdot \mathbf{v} \, dx - \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx \,. \tag{2.7}$$

To define a mixed finite element method for the Stokes problem (2.1)–(2.2), we restrict (2.4) to a pair of finite elements subspaces $\mathbf{V}^h \subset \mathbf{H}_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$. Stable and accurate solution of (2.4), or equivalently, a stable and accurate approximation of the saddle-point of (2.7), requires that $\mathbf{V}^h$ and S^h satisfy the discrete inf-sup condition

$$\sup_{\mathbf{v}^h \in \mathbf{V}^h, \, \mathbf{v}^h \neq \mathbf{0}} \frac{B(p^h, \mathbf{v}^h)}{\|\mathbf{v}^h\|_1} \geq \gamma \|p^h\|_0 \,, \quad \forall \, p^h \in S^h \tag{2.8}$$

with $\gamma > 0$ independent of h; see [18, 19].

In this paper, we will formulate stabilized mixed methods for the lowest equal order C^0 pair

$$\mathbf{V}^h = \mathbf{R}_1 \cap \mathbf{H}_0^1(\Omega) \quad \text{and} \quad S^h = R_1 \cap L_0^2(\Omega) \,, \tag{2.9}$$

and for the lowest order conforming pair

$$\mathbf{V}^h = \mathbf{R}_1 \cap \mathbf{H}_0^1(\Omega) \quad \text{and} \quad S^h = R_0 \cap L_0^2(\Omega) \,. \tag{2.10}$$

For simplicial elements, (2.10) is the unstable linear-constant pair that provides a textbook example for an over-constrained velocity space; see [19, p.23]. For quadrilateral elements, it is the bilinear-constant pair that exhibits the notorious checkerboard pressure mode. A common misconception is that once this mode is taken care of, the bilinear-constant pair can be safely used. However, in [9] it is shown that this is not the case and that, in fact, for this pair the constant γ in (2.8) is of order h. The pair (2.9) are additional classical examples of unstable velocity-pressure pairs; see [19, pp.21-25].

2.1. Weak inf-sup bounds. In this section, we show that the unstable velocity-pressure pairs (2.9) and (2.10) satisfy a weaker form of the inf-sup condition (2.8). This condition identifies terms that can be used to stabilize the mixed method. To state the relevant form of the weaker inf-sup condition, we first review some results of [17, 30, 31] specialized to (2.9) and (2.10).

LEMMA 2.1. *Let $\mathbf{V}^h$ and S^h be the spaces defined in* (2.9). *Then, there exist positive constants C_1 and C_2 such that*

$$\sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{\displaystyle\int_\Omega p^h \nabla \cdot \mathbf{v}^h \, d\Omega}{\|\mathbf{v}^h\|_1} \geq C_1 \|p^h\|_0 - C_2 h \|\nabla p^h\|_0 \qquad \forall \; p^h \in S^h \,. \tag{2.11}$$

Proof. By the definition of S^h, every $p^h \in S^h$ also belongs to $L_0^2(\Omega)$. As a result, there exists $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$ such that

$$\int_\Omega p^h \nabla \cdot \mathbf{w} \, d\Omega \geq \widetilde{C}_1 \|p^h\|_0 \|\mathbf{w}\|_1 \,. \tag{2.12}$$

Let $\mathbf{w}^h$ denote the interpolant of $\mathbf{w}$ out of $\mathbf{V}^h$. Then, from (1.4)

$$\|\mathbf{w}-\mathbf{w}^h\|_0 + h^{1/2}\|\mathbf{w}-\mathbf{w}^h\|_{\Gamma_h} \le Ch\|\mathbf{w}\|_1 \quad \text{and} \quad \|\mathbf{w}^h\|_1 \le C\|\mathbf{w}\|_1 \,. \tag{2.13}$$

Using (2.12), (2.13), and the fact that all elements of S^h are C^0 functions,

$$\begin{aligned}
\frac{\left|\int_\Omega p^h \nabla\cdot\mathbf{w}^h\, d\Omega\right|}{\|\mathbf{w}^h\|_1} &\ge \frac{\left|\int_\Omega p^h \nabla\cdot\mathbf{w}^h\, d\Omega\right|}{C\|\mathbf{w}\|_1} \\
&= \frac{\left|\int_\Omega p^h \nabla\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, d\Omega + \int_\Omega p^h\nabla\cdot\mathbf{w}\, d\Omega\right|}{C\|\mathbf{w}\|_1} \\
&\ge \frac{\int_\Omega p^h\nabla\cdot\mathbf{w}\, d\Omega}{C\|\mathbf{w}\|_1} - \frac{\left|\int_\Omega \nabla p^h\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, d\Omega\right|}{C\|\mathbf{w}\|_1} \\
&\ge \frac{\widetilde{C}_1}{C}\|p^h\|_0 - \frac{\|\nabla p^h\|_0\|\mathbf{w}-\mathbf{w}^h\|_0}{C\|\mathbf{w}\|_0} \ge C_1\|p^h\|_0 - C_2 h\|\nabla p^h\|_0 \,.
\end{aligned} \tag{2.14}$$

Then, since

$$\sup_{\mathbf{v}^h\in\mathbf{V}^h,\,\mathbf{v}^h\neq\mathbf{0}} \frac{\int_\Omega p^h\nabla\cdot\mathbf{v}^h\, d\Omega}{\|\mathbf{v}^h\|_1} \ge \frac{\left|\int_\Omega p^h\nabla\cdot\mathbf{w}^h\, d\Omega\right|}{\|\mathbf{w}^h\|_1}\,, \tag{2.15}$$

the lemma is proved. □

For the velocity-pressure pair (2.10), the discontinuity of the pressure space necessitates some minor modifications in the statement and proof of the weak inf-sup condition.

LEMMA 2.2. *Let $\mathbf{V}^h$ and S^h be the spaces defined in* (2.10). *Then, there exist positive constants C_1 and C_2 such that*

$$\sup_{\mathbf{v}^h\in\mathbf{V}^h} \frac{\displaystyle\int_\Omega p^h\nabla\cdot\mathbf{v}^h\, d\Omega}{\|\mathbf{v}^h\|_1} \ge C_1\|p^h\|_0 - C_2 h^{1/2}\|[p^h]\|_{\Gamma_h} \qquad \forall\, p^h\in S^h\,. \tag{2.16}$$

Proof. The pressure space S^h defined in (2.10) is also a subspace of $L^2_0(\Omega)$. Thus, there exists a $\mathbf{w}\in\mathbf{H}^1_0(\Omega)$ and a $\mathbf{w}^h\in\mathbf{V}^h$ that satisfy (2.12) and (2.13). Proceeding as in Lemma 2.1, we find that

$$\begin{aligned}
\frac{\left|\int_\Omega p^h\nabla\cdot\mathbf{w}^h\, d\Omega\right|}{\|\mathbf{w}^h\|_1} &\ge \frac{\left|\int_\Omega p^h\nabla\cdot\mathbf{w}^h\, d\Omega\right|}{C\|\mathbf{w}\|_1} \\
&\ge \frac{\int_\Omega p^h\nabla\cdot\mathbf{w}\, d\Omega}{C\|\mathbf{w}\|_1} - \frac{\left|\int_\Omega p^h\nabla\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, d\Omega\right|}{C\|\mathbf{w}\|_1} \\
&\ge \frac{\widetilde{C}_1}{C}\|p^h\|_0 - \frac{\left|\int_\Omega p^h\nabla\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, d\Omega\right|}{C\|\mathbf{w}\|_1}\,.
\end{aligned}$$

Using the fact that p^h is constant on each element Ω_e and integrating by parts gives

$$\int_\Omega p^h\nabla\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, d\Omega = \sum_{\Omega_e}\int_{\Omega_e} p^h\nabla\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, d\Omega = \sum_{\Omega_e}\int_{\partial\Omega_e} p^h\mathbf{n}\cdot\left(\mathbf{w}^h-\mathbf{w}\right)\, dS\,.$$

Each interior face γ_f participates twice in this sum. Collecting the two integrals over the same face and using (2.13) we obtain

$$\sum_{\Omega_e} \int_{\partial\Omega_e} p^h \mathbf{n} \cdot \left(\mathbf{w}^h - \mathbf{w}\right)\, dS = \sum_{\gamma_f} \int_{\gamma_f} [p^h] \mathbf{n}_f \cdot \left(\mathbf{w}^h - \mathbf{w}\right)\, dS$$
$$\leq \left(\sum_{\gamma_f} \int_{\gamma_f} [p^h]^2\, dS \right)^{1/2} \left(\sum_{\gamma_f} \int_{\gamma_f} \left|\mathbf{w}^h - \mathbf{w}\right|^2\, dS \right)^{1/2} \leq C h^{1/2} \|[p^h]\|_{\Gamma_h} \|\mathbf{w}\|_1$$

which proves that

$$\frac{\left|\int_\Omega p^h \nabla \cdot \mathbf{w}^h\, d\Omega\right|}{\|\mathbf{w}^h\|_1} \geq C_1 \|p^h\|_0 - C_2 h^{1/2} \|[p^h]\|_{\Gamma_h} \qquad \forall\, p^h \in S^h\,. \tag{2.17}$$

Then, using (2.15), the lemma is proved. □

The terms

$$-h\|\nabla p^h\|_0 \quad \text{and} \quad -h^{1/2}\|[p^h]\|_0 \tag{2.18}$$

appearing in (2.11) and (2.16) quantify the inf-sup "deficiency" of the unstable pairs (2.9) and (2.10), respectively. This observation has been used implicitly in the design of stabilized methods; additional terms are introduced to counterbalance (2.18). For instance, consistently stabilized methods are based on the observation that adding a properly weighted residual of (2.1) to the continuity equation (2.2) will contribute a term that can offset $-h\|\nabla p^h\|_0$. The rest of the added terms are introduced to fulfill the consistency requirement and may actually be destabilizing. As a result, residual based stabilization must rely on carefully selected values of parameters to keep such terms under control. Non-residual stabilization follows the same idea but introduces balancing terms that do not involve residuals. For example, in [28], pressure jumps are added directly to the continuity equation to help offset the destabilizing effect of the $-h^{1/2}\|[p^h]\|_0$ term, while Brezzi and Pitkaranta [11] use the first term in (2.18) to obtain stabilized formulation for piecewise linear velocity-pressure pairs.

As a template for the design of stabilizing terms, (2.18) is insufficiently general. One is always led to consider either the gradient or the jumps of the pressure. The latter case also has the drawback of requiring face-based assembly and data structures. Below, we will derive an alternative characterization of the inf-sup "deficiency" for low-order spaces that is formulated in terms of abstract operators. This characterization does not involve gradients or jumps, does not depend on the space dimension or the type of the element shapes, and has no explicit dependence on mesh parameters. As a result, it leads to new classes of stabilized mixed methods with attractive computational properties. At this point it will suffice to specify only the ranges of the abstract operators needed to characterize the inf-sup deficiency. As we proceed to establish stability and prove convergence results, more assumptions will be added as needed. The first operator

$$\Pi_0 : L^2(\Omega) \mapsto R_0 \tag{2.19}$$

has a piecewise constant range; it will be used to stabilize (2.9). The second operator

$$\Pi_1 : L^2(\Omega) \mapsto R_1 \tag{2.20}$$

has a continuous range; it will be used to stabilize (2.10).

LEMMA 2.3. *There exists a positive constant C such that*

$$Ch\|\nabla p^h\|_0 \le \|p^h - \Pi_0 p^h\|_0 \qquad \forall\, p^h \in R_1\,. \tag{2.21}$$

There exists another positive constant C such that

$$Ch^{1/2}\|[p^h]\|_0 \le \|p^h - \Pi_1 p^h\|_0 \qquad \forall\, p^h \in R_0\,. \tag{2.22}$$

Proof. To prove (2.21), note that $\Pi_0 p^h$ is constant on each element Ω_e, and so $\nabla(\Pi_0 p^h)|_{\Omega_e} = 0$. As a result, using the inverse inequality (1.7),

$$\begin{aligned} h^2\|\nabla p^h\|_0^2 &= \sum_{\Omega_e} h^2\|\nabla p^h\|_{0,\Omega_e}^2 = \sum_{\Omega_e} h^2\|\nabla(p^h - \Pi_0 p^h)\|_{0,\Omega_e}^2 \\ &\le \sum_{\Omega_e} C_I\|p^h - \Pi_0 p^h\|_{0,\Omega_e}^2 = C_I\|p^h - \Pi_0 p^h\|_0^2\,. \end{aligned}$$

To prove (2.22), note that $\Pi_1 p^h \in R_1 \subset C^0(\Omega)$. Thus, $[(\Pi_1 p^h)|_{\gamma_f}] = 0$ on every interior element face γ_f. Using the inverse inequality (1.8),

$$\begin{aligned} h\|[p^h]\|_{\Gamma_h}^2 &= \sum_{\gamma_f} h\|[p^h]\|_{0,\gamma_f}^2 = \sum_{\gamma_f} h\|[p^h - \Pi_1 p^h]\|_{0,\gamma_f}^2 \\ &= h\|[p^h - \Pi_1 p^h]\|_{\Gamma_h}^2 \le C_I\|p^h - \Pi_1 p^h\|_0\,. \end{aligned}$$

□

Using (2.21) and (2.22), results of Lemmas 2.1 and 2.2 can be combined into one statement. Let

$$\Pi = \begin{cases} \Pi_0 & \text{if } S^h \text{ is defined by (2.9)} \\ \Pi_1 & \text{if } S^h \text{ is defined by (2.10)}\,. \end{cases} \tag{2.23}$$

COROLLARY 2.4. *Let $\mathbf{V}^h$ and S^h be the spaces defined in* (2.9) *or* (2.10). *Then, there exist positive constants C_1 and C_2 whose value is independent of h and such that*

$$\sup_{\mathbf{v}^h \in \mathbf{V}^h} \frac{\displaystyle\int_\Omega p^h \nabla\cdot\mathbf{v}^h\, d\Omega}{\|\mathbf{v}^h\|_1} \ge C_1\|p^h\|_0 - C_2\|(I-\Pi)p^h\|_0 \qquad \forall\, p^h \in S^h\,. \tag{2.24}$$

We note that Π_0 and Π_1 are complementary in the sense that Π_0 acts on C^0 pressures and has a discontinuous range and Π_1 acts on discontinuous pressures and has a C^0 range. Note that besides the range assumption, Corollary 2.4 does not require any additional hypotheses about Π.

3. The new stabilized mixed methods. We will stabilize (2.4) by using

$$\frac{1}{2}\|(I-\Pi)p\|_0^2 \tag{3.1}$$

to compensate for the inf-sup deficiency of the low-order finite element pairs in (2.9) and (2.10). We add this term to (2.7) to obtain the following modified Lagrangian

functional:[2]

$$\widetilde{L}_m(\mathbf{v},q)=\frac{\lambda}{2}\int_\Omega|\nabla\mathbf{v}|^2\,d\Omega-\int_\Omega q\nabla\cdot\mathbf{v}\,d\Omega-\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\Omega-\frac{1}{2}\|(I-\Pi)q\|_0^2\,. \tag{3.2}$$

The saddle-point $(\widetilde{\mathbf{u}},\widetilde{p})$ of (3.2) satisfies the variational problem

$$A(\widetilde{\mathbf{u}},\mathbf{v})+B(\widetilde{p},\mathbf{v})=F(\mathbf{v})\qquad\forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega) \tag{3.3}$$

$$B(q,\widetilde{\mathbf{u}})-G(\widetilde{p},q)=0\qquad\forall\,q\in L_0^2(\Omega)\,, \tag{3.4}$$

where

$$G(\widetilde{p},q)=\int_\Omega(\widetilde{p}-\Pi\widetilde{p})(q-\Pi q)\,d\Omega\,. \tag{3.5}$$

Equivalently, we can write (3.3)–(3.4) in the following form: seek $(\widetilde{\mathbf{u}},\widetilde{p})\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ such that

$$\widetilde{Q}(\widetilde{\mathbf{u}},\widetilde{p};\mathbf{v},q)=F(\mathbf{v},q)\quad\forall\,(\mathbf{v},q)\in\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)\,, \tag{3.6}$$

where

$$\widetilde{Q}(\mathbf{u},p;\mathbf{v},q)=A(\mathbf{u},\mathbf{v})+B(p,\mathbf{v})+B(q,\mathbf{u})-G(p,q)\,. \tag{3.7}$$

The stabilized method is obtained by a restriction of (3.6) or, equivalently, of (3.3)–(3.4) to the finite element spaces (2.9) or (2.10). Thus, we seek $(\mathbf{u}^h,p^h)$ in $\mathbf{V}^h\times S^h$, such that

$$\widetilde{Q}(\mathbf{u}^h,p^h;\mathbf{v}^h,q^h)=F(\mathbf{v}^h,q^h)\quad\forall\,(\mathbf{v}^h,q^h)\in\mathbf{V}^h\times S^h\,. \tag{3.8}$$

The stabilization term (3.1) is not a residual of the Stokes equations. As a result, (3.8) is not a consistent finite element formulation of the Stokes equations. However, as noted earlier, for low-order elements, formally consistent stabilized methods [15, 20, 21] also loose their consistency, and so lack of consistency in our method should not be viewed as a serious flaw.

3.1. Comparison with the penalty method. The last term in (3.2) resembles the term that appears in the *penalized* Lagrangian

$$L_\epsilon(\mathbf{v},q)=\frac{\lambda}{2}\int_\Omega|\nabla\mathbf{v}|^2\,d\Omega-\int_\Omega q\nabla\cdot\mathbf{v}\,d\Omega-\int_\Omega\mathbf{f}\cdot\mathbf{v}\,d\Omega-\frac{\epsilon}{2}\|q\|_0^2\,. \tag{3.9}$$

However, the method (3.8) resulting from (3.2) is fundamentally different from a classical penalty approach based on (3.9). Taking first variations of (3.9) with respect to $\mathbf{v}$ and q gives the variational equation: seek $(\mathbf{u}_\epsilon,p_\epsilon)$ in $\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ such that

$$A(\mathbf{u}_\epsilon,\mathbf{v})+B(p_\epsilon,\mathbf{v})=F(\mathbf{v})\qquad\forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega) \tag{3.10}$$

$$B(q,\mathbf{u}_\epsilon)-\epsilon D(p_\epsilon,q)=0\qquad\forall\,q\in L_0^2(\Omega)\,, \tag{3.11}$$

[2]The modified Lagrangian (3.2) is in nondimensional form; its dimensional counterpart has the form

$$\widetilde{L}_m(\mathbf{v},q)=\frac{\mu}{2}\int_\Omega|\nabla\mathbf{v}|^2\,d\Omega-\int_\Omega q\nabla\cdot\mathbf{v}\,d\Omega-\int_\Omega\rho\mathbf{f}\cdot\mathbf{v}\,d\Omega-\frac{\alpha}{2}\|(I-\Pi)q\|_0^2\,,$$

where $\alpha=(\rho_{ref}u_{ref}\ell_{ref})^{-1}=\lambda/\mu$. It is important to note that α *is not a stabilization parameter*, but is merely a parameter introduced to make the dimensional form of the modified Lagrangian dimensionally correct; this observation is made obvious by examining the nondimensional form (3.2) in which the stabilization term $-\frac{1}{2}\|(I-\Pi)p\|_0^2$ is parameter free.

where $D(\cdot,\cdot)$ is the L^2 inner product. The second equation can be used to eliminate the pressure and to obtain an equation in terms of $\mathbf{u}_\epsilon$ only:

$$A(\mathbf{u}_\epsilon, \mathbf{v}) + \frac{1}{\epsilon}\int_\Omega (\nabla\cdot\mathbf{u}_\epsilon)(\nabla\cdot\mathbf{v})\,d\Omega = F(\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\Omega)\,. \tag{3.12}$$

Restriction of (3.12) to a discrete velocity space $\mathbf{V}^h$ leads to the classical penalty method. Alternatively, one can discretize (3.10)–(3.11), eliminate the discrete pressure from the linear system, and obtain another problem in terms of the discrete velocity only. Regardless of which version of the penalty method is used, i.e., *eliminate* and then *discretize*, or *discretize* and then *eliminate*, the ensuing penalty problem continues to require a discrete inf-sup compatibility condition. For instance, well-posedness of the *eliminate* and *discretize* method is subject to an inf-sup condition between $\mathbf{V}^h$ and an implicit pressure space defined by $\epsilon p_\epsilon = -\nabla\cdot\mathbf{u}_\epsilon$; see [24, 25]. A classical example of a failure in this method is the locking phenomena that occurs for linear velocities. In this case the implicit pair $(\mathbf{V}^h, S^h)$ is equivalent to the unstable P_1-P_0 element.

Because the penalty method still requires compatible finite element spaces, it is not a stabilization procedure. Rather, it is a solution method that allows one to solve the mixed problem more easily by uncoupling the velocity and pressure. In contrast to (3.10)–(3.11), in (3.8) we seek $(\mathbf{u}^h, p^h)\in\mathbf{V}^h\times S^h$ such that

$$A(\mathbf{u}^h,\mathbf{v}^h) + B(p^h,\mathbf{v}^h) = F(\mathbf{v}^h) \quad \forall\,\mathbf{v}^h\in\mathbf{V}^h \tag{3.13}$$

$$B(q^h,\mathbf{u}^h) - G(p^h,q^h) = 0 \qquad \forall\,q\in S^h\,. \tag{3.14}$$

In addition to the absence of a penalty parameter, another difference between (3.13)–(3.14) and the penalized problem (3.10)–(3.11) is that $G(\cdot,\cdot)$ vanishes for all pressures in the range of Π. As a result, this variable cannot be eliminated from (3.14). Of course, the main difference is that, as we shall see in the next section, (3.13)–(3.14) is stable for the low-order pairs in (2.9) and (2.10), while (3.10)–(3.11) may fail as $\epsilon\to 0$.

The penalty method can be extended to a stabilization procedure by using the stronger H^1-seminorm penalty $\epsilon/2\|\nabla q\|_0^2$ instead of the classical L^2 penalty $\epsilon/2\|q\|_0^2$. This leads to a stabilized finite element method proposed by Brezzi and Pitkaranta [11]. The bound (2.21) in Lemma 2.3 implies that for R^1 pressures their method and (3.8) have similar stability properties. However, (3.8) can be extended to constant pressures, while the method of [11] cannot.

4. Stability. To show that (3.8) is a stable variational problem, we have to additionally assume that Π is continuous as an operator $L^2(\Omega)\mapsto L^2(\Omega)$:

$$\|\Pi p\|_0 \le C\|p\|_0 \quad \forall\,p\in L^2(\Omega)\,. \tag{4.1}$$

Using (4.1), it is easy to show that $\widetilde{Q}$ is continuous, i.e.,

$$\widetilde{Q}(\mathbf{u}^h,p^h;\mathbf{v}^h,q^h) \le C\left(\|\mathbf{u}^h\|_1 + \|p^h\|_0\right)\left(\|\mathbf{v}^h\|_1 + \|q^h\|_0\right) \tag{4.2}$$

for all $(\mathbf{u}^h,p^h)$ and $(\mathbf{v}^h,q^h)$ in $\mathbf{V}^h\times S^h$. We now prove the stability of the variational problem (3.8).

THEOREM 4.1. *Let* $(\mathbf{V}^h, S^h)$ *be one of the pairs* (2.9) *or* (2.10). *Then, there exists a positive constant* C *whose value is independent of* h *such that*

$$\sup_{(\mathbf{v}^h,q^h)\in\mathbf{V}^h\times S^h} \frac{\widetilde{Q}(\mathbf{u}^h,p^h;\mathbf{v}^h,q^h)}{\|\mathbf{v}^h\|_1 + \|q^h\|_0} \ge C\left(\|\mathbf{u}^h\|_1 + \|p^h\|_0\right) \quad \forall\,(\mathbf{u}^h,p^h)\in\mathbf{V}^h\times S^h\,. \tag{4.3}$$

Proof. We will construct a pair $(\widehat{\mathbf{v}}^h, \widehat{q}^h)$, such that

$$\widetilde{Q}(\mathbf{u}^h, p^h; \widehat{\mathbf{v}}^h, \widehat{q}^h) \geq C \left(\|\mathbf{u}^h\|_1 + \|p^h\|_0\right) \left(\|\widehat{\mathbf{v}}^h\|_1 + \|\widehat{q}^h\|_0\right) .$$

Setting $(\mathbf{v}^h, q^h) = (\mathbf{u}^h, -p^h)$ yields

$$\widetilde{Q}(\mathbf{u}^h, p^h; \mathbf{u}^h, -p^h) = \lambda \|\nabla \mathbf{u}^h\|_0^2 + \| \left(I - \Pi\right) p^h\|_0^2 .$$

For a given arbitrary but fixed pressure $p^h \in S^h$, let $\mathbf{w}$ and $\mathbf{w}^h$ be the functions that satisfy (2.12) and (2.13). Assume that $\mathbf{w}^h$ is normalized so that

$$\|\nabla \mathbf{w}^h\|_0 = \sqrt{\lambda}\, \|p^h\|_0 . \tag{4.4}$$

From (2.14) and (2.21) if $\Pi = \Pi_0$ and (2.17) and (2.22) if $\Pi = \Pi_1$ we have that

$$\int_\Omega p^h \nabla \cdot \mathbf{w}^h \, d\Omega \geq C_1 \|p^h\|_0^2 - C_2 \| \left(I - \Pi\right) p^h\|_0 \|p^h\|_0 .$$

Setting $(\mathbf{v}^h, q^h) = (-\alpha \mathbf{w}^h, 0)$, where α is a real, positive parameter, together with the last inequality and (4.4) yields

$$\begin{aligned}
\widetilde{Q}(\mathbf{u}^h, p^h; -\alpha \mathbf{w}^h, 0) &= -\alpha \int_\Omega \nabla \mathbf{u}^h \cdot \nabla \mathbf{w}^h \, d\Omega + \alpha \int_\Omega p^h \nabla \cdot \mathbf{w}^h \, d\Omega \\
&\geq -\alpha \|\nabla \mathbf{u}^h\|_0 \|\nabla \mathbf{w}^h\|_0 + \alpha \big(C_1 \|p^h\|_0^2 - C_2 \|(I - \Pi) p^h\|_0 \|p^h\|_0\big) \\
&\geq -\alpha \sqrt{\lambda} \|\nabla \mathbf{u}^h\|_0 \|p^h\|_0 + \alpha \big(C_1 \|p^h\|_0^2 - C_2 \|(I - \Pi) p^h\|_0 \|p^h\|_0\big) .
\end{aligned}$$

As a result, for $(\mathbf{v}^h, q^h) = (\mathbf{u}^h - \alpha \mathbf{w}^h, -p^h)$, we have the bound

$$\begin{aligned}
\widetilde{Q}(\mathbf{u}^h, p^h; \mathbf{u}^h - \alpha \mathbf{w}^h, -p^h) \geq \|\nabla \mathbf{u}^h\|_0^2 + \| \left(I - \Pi\right) p^h\|_0^2 + \alpha C_1 \|p^h\|_0^2 \\
- \alpha \sqrt{\lambda} \|\nabla \mathbf{u}^h\|_0 \|p^h\|_0 - \alpha C_2 \| \left(I - \Pi\right) p^h\|_0 \|p^h\|_0 .
\end{aligned}$$

Using the ε-inequality with $\varepsilon = C_1/2$, we have that

$$\sqrt{\lambda} \|\nabla \mathbf{u}^h\|_0 \|p^h\|_0 \leq \frac{\lambda}{C_1} \|\nabla \mathbf{u}^h\|_0^2 + \frac{C_1}{4} \|p^h\|_0^2$$

and

$$C_2 \| \left(I - \Pi\right) p^h\|_0 \|p^h\|_0 \leq \frac{C_2^2}{C_1} \| \left(I - \Pi\right) p^h\|_0^2 + \frac{C_1}{4} \|p^h\|_0^2 .$$

In combination with the earlier lower bounds, these inequalities lead to

$$\begin{aligned}
&\widetilde{Q}(\mathbf{u}^h, p^h; \mathbf{u}^h - \alpha \mathbf{w}^h, -p^h) \geq \\
&\quad \lambda \left(1 - \frac{\alpha}{C_1}\right) \|\nabla \mathbf{u}^h\|_0^2 + \frac{\alpha C_1}{2} \|p^h\|_0^2 + \left(1 - \frac{\alpha C_2^2}{C_1}\right) \| \left(I - \Pi\right) p^h\|_0^2 .
\end{aligned}$$

Choosing

$$\widehat{\alpha} = \min \left\{ \frac{C_1}{2}, \frac{C_1}{2C_2^2} \right\}$$

guarantees that

$$\left(1-\frac{\widehat{\alpha}}{C_1}\right)\geq\frac{1}{2}\quad\text{and}\quad\left(1-\frac{\widehat{\alpha}C_2^2}{C_1}\right)\geq\frac{1}{2}\,.$$

We now set

$$\widehat{\mathbf{v}}^h=\mathbf{u}^h-\widehat{\alpha}\mathbf{w}^h\qquad\text{and}\qquad\widehat{q}^h=-p^h\,.$$

It is then easy to see that

$$\begin{aligned}\widetilde{Q}(\mathbf{u}^h,p^h;\widehat{\mathbf{v}}^h,\widehat{q}^h)&\geq\frac{1}{2}\left(\lambda\|\nabla\mathbf{u}^h\|_0^2+\widehat{\alpha}C_1\|p^h\|_0^2+\|\left(I-\Pi\right)p^h\|_0^2\right)\\&\geq\frac{1}{6}\left(\sqrt{\lambda}\|\nabla\mathbf{u}^h\|_0+\sqrt{\widehat{\alpha}C_1}\|p^h\|_0+\|\left(I-\Pi\right)p^h\|_0\right)^2\\&\geq C\left(\|\nabla\mathbf{u}^h\|_0+\|p^h\|_0\right)^2,\end{aligned}$$

where the last bound follows from $(a+b+c)^2\leq 3(a^2+b^2+c^2)$. Finally, (4.4) implies that

$$\begin{aligned}\|\nabla\widehat{\mathbf{v}}^h\|_0+\|\widehat{q}^h\|_0&=\|\nabla(\mathbf{u}^h-\widehat{\alpha}\mathbf{w}^h)\|_0+\|p^h\|_0\leq\|\nabla\mathbf{u}^h\|_0+\widehat{\alpha}\|\nabla\mathbf{w}^h\|_0+\|p^h\|_0\\&\leq\|\nabla\mathbf{u}^h\|_0+\widehat{\alpha}\sqrt{\lambda}\|p^h\|_0+\|p^h\|_0\leq C\left(\|\nabla\mathbf{u}^h\|_0+\|p^h\|_0\right)\,,\end{aligned}$$

i.e., $(\widehat{\mathbf{v}}^h,\widehat{q}^h)$ is bounded by $(\mathbf{u}^h,p^h)$ in the norm of $\mathbf{H}_0^1(\Omega)\times L^2(\Omega)$. This proves the theorem. □

Together, (4.2) and (4.3) imply that (3.8) is a stable variational problem.

REMARK 1. *Because $\widetilde{Q}$ is symmetric,* (4.3) *is sufficient to establish weak coercivity of this form.*

REMARK 2. *The stabilized problem* (3.8) *is well-posed if* (4.2) *and* (4.3) *hold, i.e., if the bilinear form $\widetilde{Q}$ is continuous and weakly coercive. From the proof of Theorem* 4.1, *it is clear that weak coercivity only depends on Π having the appropriate range. The continuity of $\widetilde{Q}$, on the other hand, is impossible without assuming that Π itself is continuous.*

5. Error estimates. To prove convergence of stabilized solutions, the properties of Π must be augmented by an approximation hypothesis. We will assume that

$$\|(I-\Pi)p\|_0\leq Ch\|p\|_1\,. \tag{5.1}$$

for every $p\in H^1(\Omega)$.

THEOREM 5.1. *Let $(\mathbf{V}^h,S^h)$ denote one of the spaces* (2.9) *or* (2.10), *let $(\mathbf{u},p)$ be the solution of the Stokes problem* (2.4), *and let $(\mathbf{u}^h,p^h)\in\mathbf{V}^h\times S^h$ solve the stabilized mixed problem* (3.8), *where the operator Π defined in* (2.23) *satisfies* (4.1). *Then,*

$$\begin{aligned}&\|\mathbf{u}-\mathbf{u}^h\|_1+\|p-p^h\|_0\\&\qquad\leq C\Big(\inf_{q^h\in S^h}\|p-q^h\|_0+\inf_{\mathbf{v}^h\in\mathbf{V}^h}\|\mathbf{u}-\mathbf{v}^h\|_1+\|(I-\Pi)p\|_0\Big).\end{aligned} \tag{5.2}$$

Proof. Since $(\mathbf{V}^h, S^h)$ is a subspace of $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$, we have from (2.4) that

$$A(\mathbf{u}, \mathbf{v}^h) + B(p, \mathbf{v}^h) = F(\mathbf{v}^h) \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h$$
$$B(q^h, \mathbf{u}) = 0 \qquad \forall\, q^h \in S^h\,.$$

Subtracting these equations from (3.13)–(3.14) yields

$$A(\mathbf{u}^h - \mathbf{u}, \mathbf{v}^h) + B(p^h - p, \mathbf{v}^h) = 0 \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h$$
$$B(q^h, \mathbf{u}^h - \mathbf{u}) = G(p^h, q) \quad \forall\, q^h \in S^h\,,$$

or, equivalently,

$$\widetilde{Q}(\mathbf{u}^h - \mathbf{u}, p^h - p; \mathbf{v}^h, q^h) = G(p, q^h) \quad \forall\, (\mathbf{v}^h, q^h) \in \mathbf{V}^h \times S^h\,. \tag{5.3}$$

Let $(\mathbf{w}^h, r^h)$ be an arbitrary pair in $\mathbf{V}^h \times S^h$. We estimate the discrete error

$$\|\mathbf{u}^h - \mathbf{w}^h\|_1 + \|p^h - r^h\|_0$$

using the weak coercivity bound (4.3) and the error "orthogonality" (5.3):

$$\begin{aligned}
&C\left(\|\mathbf{u}^h - \mathbf{w}^h\|_1 + \|p^h - r^h\|_0\right)\\
&\quad\le \sup_{(\mathbf{v}^h, q^h)\in \mathbf{V}^h\times S^h} \frac{\widetilde{Q}(\mathbf{u}^h - \mathbf{w}^h, p^h - r^h; \mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 + \|q^h\|_0}\\
&\quad= \sup_{(\mathbf{v}^h, q^h)\in \mathbf{V}^h\times S^h} \frac{\widetilde{Q}(\mathbf{u}^h - \mathbf{u}, p^h - p; \mathbf{v}^h, q^h) + \widetilde{Q}(\mathbf{u} - \mathbf{w}^h, p - r^h; \mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 + \|q^h\|_0}\\
&\quad= \sup_{(\mathbf{v}^h, q^h)\in \mathbf{V}^h\times S^h} \frac{G(p, q^h) + \widetilde{Q}(\mathbf{u} - \mathbf{w}^h, p - r^h; \mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 + \|q^h\|_0}\,.
\end{aligned}$$

From (4.2) we have that

$$\widetilde{Q}(\mathbf{u} - \mathbf{w}^h, p - r^h; \mathbf{v}^h, q^h) \le C\left(\|\mathbf{u} - \mathbf{w}^h\|_1 + \|p - r^h\|_0\right)\left(\|\mathbf{v}^h\|_1 + \|q^h\|_0\right)$$

and from (4.1) we have that

$$G(p, q^h) \le C G(p, p)^{1/2} \|q^h\|_0\,.$$

As a result, there exists a positive constant C such that

$$\begin{aligned}
&C\left(\|\mathbf{u}^h - \mathbf{w}^h\|_1 + \|p^h - r^h\|_0\right)\\
&\le \sup_{(\mathbf{v}^h, q^h)\in \mathbf{V}^h\times S^h} \frac{G(p, p)^{1/2}\|q^h\|_0 + \left(\|\mathbf{u} - \mathbf{w}^h\|_1 + \|p - r^h\|_0\right)\left(\|\mathbf{v}^h\|_1 + \|q^h\|_0\right)}{\|\mathbf{v}^h\|_1 + \|q^h\|_0}\\
&\le G(p, p)^{1/2} + \left(\|\mathbf{u} - \mathbf{w}^h\|_1 + \|p - r^h\|_0\right) = \left(\|\mathbf{u} - \mathbf{w}^h\|_1 + \|p - r^h\|_0\right) + \|(I - \Pi)p\|_0\,.
\end{aligned}$$

To complete the proof, we use the triangle inequality to obtain

$$\begin{aligned}
&\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0\\
&\qquad\le \left(\|\mathbf{u} - \mathbf{w}^h\|_1 + \|p - r^h\|_0\right) + \left(\|\mathbf{u}^h - \mathbf{w}^h\|_1 + \|p^h - r^h\|_0\right)\\
&\qquad\le C\Big(\|\mathbf{u} - \mathbf{w}^h\|_1 + \|p - r^h\|_0 + \|(I - \Pi)p\|_0\Big)\,,
\end{aligned}$$

and then take the infimum over $\mathbf{w}^h \in \mathbf{V}^h$ and $r^h \in S^h$. □

Together with the assumption (5.1) this theorem can be used to show that solutions of (3.8) converge optimally with respect to the solution regularity.

COROLLARY 5.2. *Assume that* $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \times L_0^2(\Omega) \cap H^1(\Omega)$ *solves the Stokes problem* (2.1)–(2.2) *and that* $(\mathbf{u}^h, p^h)$ *is the solution of the stabilized mixed problem* (3.8), *where the operator* Π *defined in* (2.23) *satisfies* (4.1) *and* (5.1). *Then,*

$$\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 \leq Ch\left(\|u\|_2 + \|p\|_1\right) . \tag{5.4}$$

Proof. The assertion follows immediately from (5.2) using (1.4), (1.6), and (5.1). □

6. Implementation. Among the attractive features of our stabilization approach is the great flexibility in the definition of the stabilization term (3.1). The main prerequisite to achieve stabilization of the mixed method with the low-order finite element pairs (2.9) and (2.10) is to choose a Π with the appropriate range. The simplest way to accomplish this is to use standard finite element projection or interpolation operators. Then, the remaining assumptions about Π are easily verified.

From a practical viewpoint the main factors in the choice of Π are simplicity and locality, i.e., computation of its action must be done at the element level using only standard nodal data structures. With this in mind, a suitable choice of Π_0 to stabilize the lowest equal order pair (2.9) is a local L^2 projection operator. Given a function $q \in L^2(\Omega)$ we define $\Pi_0 : L^2(\Omega) \mapsto R_0$ by $\Pi_0 q = q^h \in R_0$ if and only if

$$\int_{\Omega_e} (\Pi_0 q - q)\, d\Omega_e = 0 \qquad \forall\, \Omega_e \in \mathcal{T}_h\,. \tag{6.1}$$

It is easy to see that

$$\Pi_0 q|_{\Omega_e} = \frac{1}{V(\Omega_e)} \int_{\Omega_e} q\, d\Omega_e$$

is the element average of q and that Π_0 satisfies both assumptions (4.1) and (5.1); see [18, p.102].

A suitable choice of Π_1 to stabilize the lowest order conforming pair (2.10) is a Clement-like interpolant; see [18, p.110]. Instead of using a projection onto a patch of elements that share the same node we choose to define our interpolant by using a projection onto the dual (or complementary) volume associated with each node. For piecewise constant pressures this choice leads to a particularly simple formula for the action of Π_1 that does not require explicit construction of a dual cell. Specifically, we define $\Pi_1 : L^2(\Omega) \mapsto R_1$ as follows. For a given node $\mathcal{N}_i$ in $\mathcal{T}_h$, let $\widehat{\Omega}_i$ denote its dual volume. Given a function $q \in L^2(\Omega)$, let q_i be the constant function on $\widehat{\Omega}_i$ that minimizes the functional

$$J_i(q) = \frac{1}{2} \int_{\widehat{\Omega}_i} (q_i - q)^2\, d\Omega\,; \tag{6.2}$$

then set

$$\Pi_1 q = \sum_{i=1}^{N_{nodes}} q_i N_i(\mathbf{x}) \in R_1\,, \tag{6.3}$$

where N_i denotes the nodal basis of R_1 and N_{nodes} is the number of nodes in $\mathcal{T}_h$. The action of the operator defined in (6.3) can be computed locally at the element level and has the same properties as the usual Clement interpolant, i.e., (4.1) and (5.1) are satisfied. For $q = q^h \in R_0$, the functional in (6.2) further simplifies to

$$J_i(q^h) = \sum_{\Omega_e \cap \Omega_i \neq 0} V_i(\Omega_e)(q_i - q_e^h)^2,$$

where q_e^h is the constant value of q^h on Ω_e and $V_i(\Omega_e)$ is the volume fraction of the element Ω_e that belongs to the dual cell $\widehat{\Omega}_i$ associated with node $\mathcal{N}_i$. For constant pressures, we can choose

$$V_i(\Omega_e) = V(\Omega_e)/n_e\,,$$

where n_e is the number of nodes in Ω_e. Minimization of J_i then yields the formula

$$q_i = \frac{\sum_{\Omega_e \in \Omega_i} V_i(\Omega_e) q_e^h}{\sum_{\Omega_e \in \Omega_i} V_i(\Omega_e)} = \sum_{\Omega_e \in \Omega_i} d_{ie} q_e^h\,,$$

i.e., the nodal values of $\Pi_1 q^h$ are area weighted averages of the surrounding constant pressure values of q^h.

The stabilized mixed problem gives rise to a linear system of algebraic equations with a matrix that has the form

$$\begin{bmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & -\mathbb{G} \end{bmatrix}. \tag{6.4}$$

The matrices $\mathbb{A}$ and $\mathbb{B}$ are assembled in the usual manner from the bilinear forms $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$, respectively, and $\mathbb{G}$ is a symmetric, semidefinite matrix generated at the element level from $G(\cdot,\cdot)$. The form of this matrix depends on the particular operator Π employed in the stabilization. However, computation of $\mathbb{G}$ is completely local and can be accomplished by augmenting the standard nodal assembly process by a few simple calculations. For example, the only information needed to compute $\mathbb{G}$ in the case of Π_1 is the area of each element. This information should be readily available during the standard assembly process; moreover, calculation of $\mathbb{G}$ is simple in comparison to the calculations required to determine $\mathbb{A}$ and $\mathbb{B}$.

It is also easy to see that $\mathbb{G}$ is a sparse matrix. In the case of $\Pi = \Pi_0$, its sparsity pattern is the same as for the standard nodal R_1 pressure mass matrix. In the case of $\Pi = \Pi_1$, the original mass matrix associated with piecewise constant pressures is diagonal while $\mathbb{G}$ is not. Nevertheless, the important point is that the action of $(I - \Pi)$ in both cases is obtained by multiplication of pressure degrees of freedom by a sparse matrix rather than by an inversion of a mass matrix as occurs in determining L^2 projections. As a result, in the context of iterative solution methods, our stabilization method is very efficient as it only requires one sparse matrix-vector multiply per iteration.

7. Numerical examples. In this section, we report on some numerical results obtained using the stabilized method (3.8). The main goal of these experiments is to verify the convergence rates of (5.2) for the low-order pairs (2.9) and (2.10) in two and three space dimensions. For each pair of spaces, we consider both simplicial and nonsimplicial partitions $\mathcal{T}_h$ of the computational domain into finite elements. Figure

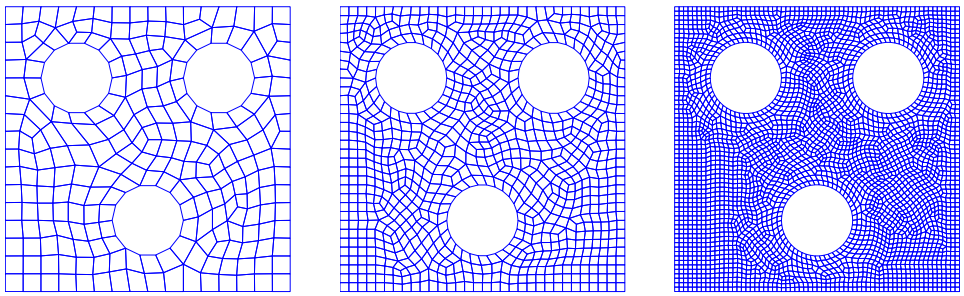

FIG. 7.1. *A sequence of refined non-uniform quadrilateral grids.*

7.1 shows an example of a non-simplicial sequence of grids used for a convergence study in two dimensions. The following error norms are used for the investigation of convergence rates:

$$e^h_{uL_2} = \|\mathbf{u}^h - \mathbf{u}\|_0 = \sqrt{\sum_{i=1}^{d} \int_\Omega (u_i^h - u_i)^2 d\Omega} \tag{7.1}$$

$$e^h_{uH_1} = \|\mathbf{u}^h - \mathbf{u}\|_1 = \sqrt{\sum_{i=1}^{d} \int_\Omega \nabla(u_i^h - u_i) \cdot \nabla(u_i^h - u_i) d\Omega} \tag{7.2}$$

$$e^h_{pL_2} = \|p^h - p\|_0 = \sqrt{\int_\Omega (p^h - p)^2 d\Omega}\,, \tag{7.3}$$

where d denotes the spatial dimension, u_i, $i = 1, \ldots, d$, denote the components of the vector $\mathbf{u}$, and $(\mathbf{u}^h, p^h)$ denotes the stabilized finite element approximation of the exact solution $(\mathbf{u}, p)$. To estimate convergence rates, we select a pair of smooth functions $(\mathbf{u}, p)$, with $\mathbf{u}$ solenoidal and p having zero mean, and evaluate the Stokes equations to generate the source term $\mathbf{f}$ and the boundary data. This synthetic data is then used by (3.8) to approximate the smooth exact solution on a sequence of grids.

The first example is for a unit square with[3] $\lambda = 1$ and the smooth exact solution

$$u_1 = x + x^2 - 2xy + x^3 - 3xy^2 + x^2 y \tag{7.4}$$

$$u_2 = -y - 2xy + y^2 - 3x^2 y + y^3 - xy^2 \tag{7.5}$$

$$p = xy + x + y + x^3 y^2 - 4/3\,. \tag{7.6}$$

The values of $\mathbf{u}$ on the boundary of the square are constrained to those given by (7.4) and (7.5). To remove the constant pressure mode from the numerical solution, the constraint

$$\int_\Omega p(\mathbf{x}) d\Omega = 0 \tag{7.7}$$

is also imposed. Results for stabilized triangular elements P_1-P_1 and P_1-P_0 and stabilized quadrilateral elements Q_1-Q_1 and Q_1-P_0 are shown in Figure 7.2. The $e^h_{uH_1}$ errors for the continuous pressure elements (P_1-P_1 and Q_1-Q_1) and the discontinuous

[3]The implementation of the new stabilized method for non-constant viscosity is straightforward by using, e.g., viscosity values at quadrature points.

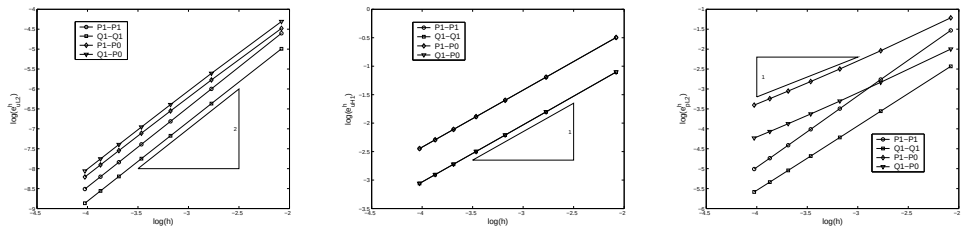

FIG. 7.2. *Errors for the first two-dimensional example (structured meshes).*

TABLE 7.1

Solution errors for triangular stabilized elements normalized with respect to results for the stable MINI element.

	P_1-P_1				P_1-P_0			
$1/h$	$e^h_{uL_2}$	$e^h_{uH_1}$	$e^h_{pL_2}$	e_{div}	$e^h_{uL_2}$	$e^h_{uH_1}$	$e^h_{pL_2}$	e_{div}
8	0.892	0.985	0.588	0.976	1.009	0.986	0.807	0.823
16	0.890	0.996	0.583	0.976	1.114	0.997	1.201	0.826
24	0.890	0.999	0.574	0.976	1.155	1.000	1.552	0.827
32	0.889	1.000	0.565	0.976	1.176	1.001	1.872	0.827
40	0.889	1.001	0.556	0.976	1.189	1.001	2.167	0.828
48	0.889	1.001	0.549	0.976	1.198	1.002	2.442	0.828
56	0.889	1.001	0.542	0.976	1.204	1.002	2.698	0.828

pressure elements (P_1-P_0 and Q_1-P_0) are nearly identical. Although not predicted by theory, the $e^h_{pL_2}$ line segment slopes for the continuous pressure elements exceed those of the discontinuous pressure elements. In all cases, the theoretical convergence rates are confirmed.

For purposes of comparison, Table 7.1 shows the P_1-P_1 and P_1-P_0 results of Figures 7.2 normalized with respect to those of the stable MINI element. Also shown in the table are the normalized values of the maximum error in the divergence within an element as defined by

$$e_{div} = \max_e \left| \int_{\Gamma_e} \mathbf{u} \cdot \mathbf{n} \, d\Gamma_e \right| , \tag{7.8}$$

where Γ_e is the boundary of element e and $\mathbf{n}$ is the unit outward normal of Γ_e. The normalized maximum errors in the divergence are close to the stable MINI element for both the P_1-P_1 and P_1-P_0 elements. The higher normalized values of $e^h_{pL_2}$ for the P_1-P_0 elements are consistent with previous comments regarding continuous and discontinuous pressure elements.

The second example uses the same exact solution, but now the square domain has three circular cutouts as shown in Figure 7.1. Note that it is necessary to adjust the constant value of 4/3 in (7.6) to satisfy (7.7). Meshes of triangles were obtained from the quadrilateral meshes by splitting each quadrilateral into two triangles. Plots of the error norms for the different element types are shown in Figure 7.3. In this figure, $h_e = 1/\sqrt{N_e}$ where N_e is the number of quadrilateral elements in the mesh. As expected, the error norms become smaller as the meshes are refined.

The final example is for a unit cube with $\lambda = 1$ and the smooth exact solution

$$u_1 = x + x^2 + xy + x^3 y \tag{7.9}$$

$$u_2 = y + xy + y^2 + x^2 y^2 \tag{7.10}$$

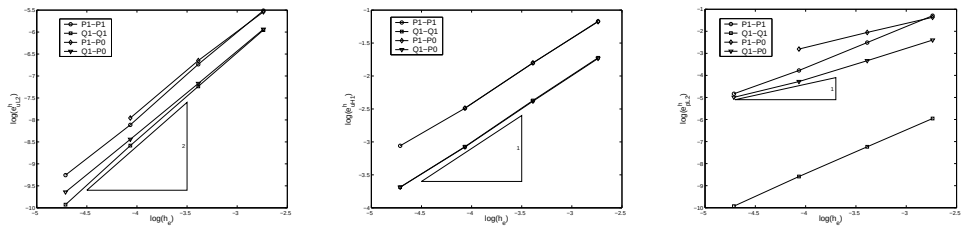

FIG. 7.3. *Errors for the second two-dimensional example (unstructured meshes).*

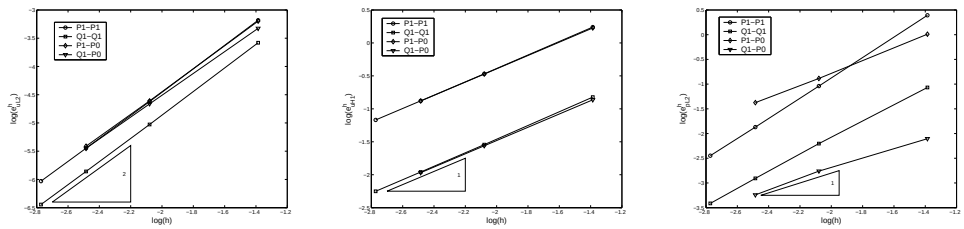

FIG. 7.4. *Errors for the three-dimensional example.*

$$u_3 = -2z - 3xz - 3yz - 5x^2yz \tag{7.11}$$

$$p = xyz + x^3y^3z - 5/32\,. \tag{7.12}$$

The hexahedral meshes have $(1/h)^3$ elements whereas the tetrahedral meshes have $6(1/h)^3$ elements. Plots of the error norms versus element length are shown in Figure 7.4. As was the case for the two-dimensional examples, the theoretical convergence rates are confirmed. Again, the $e^h_{pL_2}$ line segment slopes for the continuous pressure elements are larger than those for the discontinuous pressure elements.

For further examples and numerical studies, we refer to [14].

8. Conclusions. We have formulated a new approach to stabilization of low order velocity-pressure pairs for the incompressible Stokes equations. Central to our approach is the characterization of the LBB deficiency of the unstable pairs in terms of suitable operators, and their subsequent application in the formulation of a stabilized mixed variational equation. This characterization remains valid for a broad range of operators which makes our stabilization technique extremely flexible and leads to stabilized mixed methods with attractive computational properties. Most notably, our methods do not require selection of mesh dependent stabilization parameters, retain the symmetry of the original equations, and can be implemented at the element level with minimal additional cost. Numerical examples presented in this paper demonstrate the excellent stability and accuracy properties of the new methods.

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