ANALYSIS OF LEAST SQUARES FINITE ELEMENT METHODS FOR THE NAVIER-STOKES EQUATIONS

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This work was supported by the Air Force Office of Scientific Research under grants AFOSR F49620-93-1-0280 and AFOSR-93-1-0061

SUMMARY

Introduction

- Notation
- What is a least squares (LS) method
- Least squares strategy: how and why

LS for the Navier-Stokes Equations

- VVP equations in 2D and in 3D
- ADN infomercial
- Boundary conditions
- Modified equations
- The LS methods
- Implementation issues

Error Analysis

- Abstract approximation theory
- Application to LS methods

Sobolev Spaces

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \,|\, D^\alpha u \in L^p(\Omega) \;\forall |\alpha| \leq m\}$$

For example

$$L^2(\Omega) = \{u \,|\, \int_\Omega u^2 d\Omega < \infty\}$$

$$H^1(\Omega) = \{u \,|\, D^\alpha u \in L^2(\Omega), |\alpha| \leq 1\}$$

Norm for a function g belonging to $H^m(\Omega)$:

$$\|g\|_m^2 = \sum_{|\alpha| \leq m} \int_\Omega (D^\alpha g)^2 d\Omega$$

Sobolev spaces related to the Navier-Stokes equations:

$$L_0^2(\Omega) = \{u \in L^2(\Omega) \,|\, \int_\Omega u d\Omega = 0\}$$

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \,|\, u = 0 \;\text{ on } \Gamma\}$$

$$\tilde{H}^1(\Omega) = \{u \in H^1(\Omega) \,|\, \int_\Omega u d\Omega = 0\}$$

Curls and Vector Products.

Curl operator in 3D:

$$\mathbf{curl}\,\mathbf{u} = \nabla \times \mathbf{u}$$

Curl operator(s) in 2D:

$$\mathbf{curl}\,\omega = \begin{pmatrix} \omega_y \\ -\omega_x \end{pmatrix} \qquad \text{and} \qquad \mathrm{curl}\,\mathbf{u} = u_{2x} - u_{1y}\,.$$

"Vector product" in 2D:

$$\phi \longrightarrow (0, 0, \phi)$$
$$\mathbf{u} \longrightarrow (u_1, u_2, 0)$$
$$\mathbf{v} \longrightarrow (v_1, v_2, 0)$$

then we define:

$$\phi \times \mathbf{u} \to \phi \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$
$$\mathbf{u} \times \mathbf{v} \to u_1 v_2 - u_2 v_1$$

Vorticity in 3D/2D:

$$\omega = \mathbf{curl}\,\mathbf{u} \quad (= \nabla \times \mathbf{u})$$
$$\omega = \mathrm{curl}\,\mathbf{u} \quad (= \nabla \times \mathbf{u})$$

What is a Least Squares Method

Boundary value problem

$$\begin{cases} \mathcal{L}U = F & \text{in } \Omega \\ \mathcal{R}U = 0 & \text{on } \Gamma \end{cases}$$

where $\boldsymbol{\mathcal{L}} : \mathbf{X} \mapsto \mathbf{Y}$; $\mathbf{X}$ and $\mathbf{Y}$ are Hilbert spaces.

1. Least squares functional

$$J(U) = (\boldsymbol{\mathcal{L}}U - F, \boldsymbol{\mathcal{L}}U - F)_Y = \|\boldsymbol{\mathcal{L}}U - F\|_Y^2 \,.$$

2. Least squares principle

Seek $U \in \mathbf{X}$ such that $J(U) \le J(V)\ \forall V \in \mathbf{X}$.

3. Euler-Lagrange equation

$$\delta J(U) = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} J(U + \varepsilon V) = 0 \quad \forall V \in \mathbf{X}\,.$$

Equivalent variational problem

Seek $U \in \mathbf{X}$ such that
$Q(U;V) = \mathcal{F}(V)$ for all $V \in \mathbf{X}$.

Existence of minimizers.

Assume the *a priori* estimate

$$\|U\|_X \leq C\|\mathcal{L}U\|_Y \,.$$

Then $Q(\cdot;\cdot)$ is **coercive** on $\mathbf{X} \times \mathbf{X}$:

$$Q(U;U) = (\mathcal{L}U, \mathcal{L}U)_Y = \|\mathcal{L}U\|_Y^2 \geq C\|U\|_X^2 \,.$$

$\Longrightarrow$ Existence and uniqueness of the minimizer will follow from the Lax-Milgram Lemma, if one can establish an *a priori* estimate for the PDE.

$\boxed{\textbf{4. Discretization}}$

Choose $\mathbf{X}^h \subset \mathbf{X}$ and then solve the problem

Seek $U^h \in \mathbf{X}^h$ *such that*
$Q(U^h;V^h) = \mathcal{F}(V^h)$ *for all* $V^h \in \mathbf{X}^h$.

- Typically, $\mathbf{X}$ is a Sobolev space constrained by the boundary conditions.
- Discrete problem is equivalent to a linear system having *symmetric, positive definite* matrix.
- Approximations are optimaly accurate.

Least Squares Strategy: How and Why

1. Transformation of the original PDE or system of PDEs to a *first order system*:

$\Longrightarrow$ Discretization by C^0 finite elements may be possible.

$\Longrightarrow$ Direct and optimall approximations of physically important fields, e.g., vorticity, stresses.

2. Identification of spaces $\mathbf{X}$ and $\mathbf{Y}$ such that an *a priori* estimate holds and, formulation of the LS functional for the (first order) system:

$\Longrightarrow$ Existence and uniqueness of the minimizers

$\Longrightarrow$ Stability of the discretizations is guaranteed by the <u>inclusion</u> $\mathbf{X}^h \subset \mathbf{X}$ and **inf-sup** (LBB) type conditions are not required for $\mathbf{X}$

$\Longrightarrow$ Discretization results in linear systems with *symmetric, positive definite* matrices

$\Longrightarrow$ Discrete equations can be solved by robust iterative methods (e.g., CG methods)

$\Longrightarrow$ Assembly free methods are feasible.

Applications of least squares methods

- Least squares finite element methods are based on *minimization principle*:

  $\Longrightarrow$ very competitive when Galerkin formulation corresponds to a *saddle point* optimization, because inf-sup (LBB) condition is avoided.

- Boundary conditions can be imposed in a weak sense by including into the functional the term

$$\|\mathcal{R}U - G\|_{\Gamma}^{2}$$

  $\Longrightarrow$ Approximating functions need not satisfy the essential boundary conditions.

Examples of LS applications

- Stationary, incompressible flow
- Time dependent incompressible flow
- Convection-diffusion problems
- Purely hyperbolic problems

LS References

<u>General</u>

1970,73 - Bramble, Schatz (LS for $2m^{th}$ order BVP)

1973 - Baker (Simplified proofs for Bramble, et.al.)

1977 - Jesperson (LS for systems associated with elliptic PDE)

1979 - Glowinski et. al. (LS in H^{-1} norm)

1979 - Fix, Gunzburger, Nicolaides (LS for div - grad systems)

1979 - Wendland (LS for Petrovski elliptic systems)

1981 - Fix, Stephan (LS for domains with corners)

1985 - Aziz, Kellog, Stephens (LS for ADN systems)

1987 - Chang, Gunzburger (LSFEM for first order systems in 3-D)

1993 - Pehlivanov, Carey, Lazarov (LS for ODE's)

1994 - Lazarov, Manteufell, McCormick (FOSLS: H_{div} norms)

1994 - Bramble, Lazarov, Pasciak (LS in H^{-1} norm)

Convection-diffusion and hyperbolic problems

1992 - Chen (semidiscrete LS)

1994 - Jiang (iteratively reweighted LS)

1994 - Tsang (LS for advective transport)

LS for incompressible viscous flows

1979 - Glowinski et. al. (LS for primitive variable Navier-Stokes in H^{-1} norms)

1990 - Chang, Jiang (LS for VVP Stokes)

1990 - Chang, Povinelli (LS in 3D)

1992 - Lefebvre, Peraire and Morgan (LS for VVP Navier-Stokes, 3D)

1993 - Bochev, Gunzburger (LS for VVP Stokes)

1993 - Bochev, Gunzburger (LS for velocity-pressure-stress Stokes)

1994 - Bochev (LS for VVP Navier-Stokes)

1994 - Bochev, Gunzburger (WLS for VVP Navier-Stokes)

Velocity-Vorticity-Pressure Navier-Stokes Equations

Differential equations in 2D

$$\begin{aligned}
\nu\,\mathbf{curl}\,\omega + \mathbf{grad}\,r + \omega \times \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \to 2 \\
\mathrm{curl}\,\mathbf{u} - \omega &= 0 \quad \text{in } \Omega \to 1 \\
\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega \to 1
\end{aligned}$$

Differential equations in 3D

$$\begin{aligned}
\nu\,\mathbf{curl}\,\omega + \mathbf{grad}\,r + \omega \times \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \to 3 \\
\mathbf{curl}\,\mathbf{u} - \omega &= 0 \quad \text{in } \Omega \to 3 \\
\mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega \to 1\,.
\end{aligned}$$

Boundary conditions

$$\mathcal{R}U = 0 \quad \text{on } \Gamma$$

- $\mathbf{u}$ - velocity
- 2D: $\omega = \mathrm{curl}\,\mathbf{u}$; 3D: $\omega = \mathbf{curl}\,\mathbf{u}$ - vorticity
- $r = p + 1/2|\mathbf{u}|^2$ - total head, (p = pressure)

$$\int_\Omega r\, d\Omega = 0$$

- $\nu = 1/Re$ - kinematic viscosity
- $\mathbf{f}$ - body force

The ADN Theory

- Success of the LS method depends on the proper choice of the spaces $\mathbf{X}$ and $\mathbf{Y}$ (**coercivity**!)

- The spaces $\mathbf{X}$ and $\mathbf{Y}$ should be such that an *a priori* estimate holds for the linearized system

- The ADN theory identifies the spaces in which *a priori* estimates of the form $\|U\|_X \leq C\|\mathcal{L}U\|_Y$ hold for solutions of elliptic BVP

- The norms appearing in these estimates are chosen so that the operator $\mathcal{L}$ and boundary operator $\mathcal{R}$ satisfy a certain precise condition known as the *complementing condition*.

- The *complementing condition* guarantees that the boundary operator $\mathcal{R}$ is compatible with the operator $\mathcal{L}$ i.e., that the BVP is well-posed in the spaces $\mathbf{X}$ and $\mathbf{Y}$.

- Different boundary conditions may result in different a priori estimates, i.e., the choice of $\mathbf{X}$ and $\mathbf{Y}$ may depend on $\mathcal{R}$.

In two-dimensions:

- Four equations and unknowns
- Elliptic system of total order 4
- Needs 2 conditions on Γ; same as for the $(\mathbf{u}, p)$ Navier-Stokes in 2D: *can use the same* $\mathcal{R}$.

In three-dimensions

- Seven equations and unknowns;
- The VVP system cannot be elliptic in the sense of Agmon, Douglas and Nirenberg because $\det \mathcal{L}^P(\xi + \tau\xi') = 0$ will have a real root.

To derive a well-posed system:

- We add a seemingly redundant equation and a new "slack" variable ϕ:

$$\operatorname{div} \omega = 0 \quad \text{in } \Omega\,;$$

$$\mathbf{curl}\,\mathbf{u} - \omega + \mathbf{grad}\,\phi = 0 \quad \text{in } \Omega\,.$$

- Elliptic system of total order 8
- Needs 4 conditions on Γ; one more than the $(\mathbf{u}, p)$ Navier-Stokes in 3D: *must supplement* $\mathcal{R}$ *by an additional condition on* Γ*!*

Principal parts and a priori estimates

s_1	ω_y	r_x	0	0
s_2	$-\omega_x$	r_y	0	0
s_3	$-\omega$	0	$-u_{1y}$	u_{2x}
s_4	0	0	u_{1x}	u_{2y}
s/t	t_1	t_2	t_3	t_4

- $s_i \leq 0$ determine norms for the data
- $t_j \geq 0$ determine norms for the solution
- $\deg \mathcal{L}_{ij} \leq s_i + t_j$
- $\mathcal{L}^P = \{\mathcal{L}_{ij} \,|\, \deg \mathcal{L}_{ij} = s_i + t_j\}$

<u>The ADN a priori estimates</u>

- $\mathcal{L}^P$ is uniformly elliptic
- $\mathcal{L}^P$ satisfies the Supplementary Condition (2D)
- BCs satisfy the Complementing Condition

$$\|U\|_X = \sum_{i=1}^{n} \|v_j\|_{t_j} \leq C \sum_{i=1}^{n} \|f_i\|_{-s_i} = C\|\mathcal{L}U\|_Y$$

Classification of the Boundary Conditions

The principal part $\mathcal{L}^P$ of the VVP linearized equations is not unique! There are two sets of indices such that $\mathcal{L}_1^P$ and $\mathcal{L}_2^P$ are *uniformly elliptic*.

Type 1 BC: CC holds with

$$\mathcal{L}_1^P = \begin{pmatrix} \mathbf{curl}\,\omega & + & \mathbf{grad}\,r \\ & & \mathrm{curl}\,\mathbf{u} \\ & & \mathrm{div}\,\mathbf{u} \end{pmatrix}$$

$$\underbrace{s_1 = s_2 = 0}_{\textit{momentum}}, \quad s_3 = s_4 = 0 \;\; \text{in 2D}\,,$$

$$\underbrace{s_1 = \ldots = s_4 = 0}_{\textit{momentum and redundant}} \;; \quad s_5 = \ldots = s_8 = 0 \;\; \text{in 3D}\,.$$

Example: $\mathcal{R}U = (\mathbf{u}\cdot\mathbf{n}, r)$

Type 2 BC: CC holds with

$$\mathcal{L}_2^P = \begin{pmatrix} \mathbf{curl}\,\omega & + & \mathbf{grad}\,r \\ \boxed{-\omega} & + & \mathrm{curl}\,\mathbf{u} \\ & & \mathrm{div}\,\mathbf{u} \end{pmatrix}$$

$$\underbrace{s_1 = s_2 = 0}_{\textit{momentum}}, \quad s_3 = s_4 = -1 \;\; \text{in 2D}\,,$$

$$\underbrace{s_1 = \ldots = s_4 = 0}_{\textit{momentum and redundant}} \;; \quad s_5 = \ldots = s_8 = -1 \;\; \text{in 3D}\,.$$

Example: $\mathcal{R}U = \mathbf{u}$

The Modified Navier-Stokes Equations:

In two-dimensions

Let ω_0, r_0 and $\mathbf{u}_0$ solve

$$\begin{aligned} \mathbf{curl}\,\omega + \frac{1}{\nu}\mathbf{grad}\,r &= \frac{1}{\nu}\mathbf{f} \quad \text{in } \Omega \\ \operatorname{curl}\mathbf{u} - \omega &= 0 \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathcal{R}U &= 0 \quad \text{on } \Gamma \end{aligned}$$

Least squares methods will be formulated for

$$\begin{aligned} \mathbf{curl}\,\omega + \mathbf{grad}\,r + \nu^{-1}(\omega_0 \times \mathbf{u}_0) + \\ \nu^{-1}(\omega_0 \times \mathbf{u} + \omega \times \mathbf{u}_0 + \omega \times \mathbf{u}) &= 0 \quad \text{in } \Omega \\ \operatorname{curl}\mathbf{u} - \omega &= 0 \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathcal{R}U &= 0 \quad \text{on } \Gamma \end{aligned}$$

In three-dimensions

The "redundant" equation

$$\operatorname{div}\omega = 0$$

must be added for the *stability* of the method. The slack variable is identically zero and can be ignored

LS for the Navier-Stokes Equations

Least squares functional:

$$
\begin{aligned}
&J(\omega, \mathbf{u}, r) = \\
&= \frac{1}{2}\Big(\|\mathbf{curl}\,\omega + a(\omega, \mathbf{u}) + b(\omega, \mathbf{u}, \omega_0, \mathbf{u}_0) + \mathbf{grad}\, r\|_0^2 \\
&\quad + \|\mathrm{curl}\,\mathbf{u} - \omega\|_s^2 + \|\mathrm{div}\,\mathbf{u}\|_s^2\Big) \;,
\end{aligned}
$$

where $s = 0, 1$ for Type 1,2 BCs and

$$
\begin{aligned}
b(\omega, \mathbf{u}, \xi, \mathbf{v}) &= \frac{1}{\nu}\left(\omega \times \mathbf{u} + \xi \times \mathbf{v}\right) \\
a(\omega, \mathbf{u}) &= b(\omega_0, \mathbf{u}, \omega, \mathbf{u}_0)\,.
\end{aligned}
$$

Euler-Lagrange equation

Seek $(\omega, \mathbf{u}, r) \in \mathbf{X}_s$ *such that*

$$
\begin{aligned}
&Q((\omega, \mathbf{u}, r); (\xi, \mathbf{v}, q)) = \\
&= \big(\mathbf{curl}\,\omega + \mathbf{grad}\, r + a(\omega, \mathbf{u}) + b(\omega_0, \mathbf{u}_0, \omega, \mathbf{u}), \\
&\qquad \mathbf{curl}\,\xi + \mathbf{grad}\, q + a(\xi, \mathbf{v}) + b(\xi, \mathbf{u}, \omega, \mathbf{v})\big)_0 \\
&\quad + \left(\mathrm{curl}\,\mathbf{u} - \omega, \mathrm{curl}\,\mathbf{v} - \xi\right)_s \\
&\quad + \left(\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}\right)_s = 0 \qquad\quad \text{for all } (\xi, \mathbf{v}, q) \in \mathbf{X}_s\,.
\end{aligned}
$$

where

$$
\mathbf{X}_s = [H^1(\Omega) \times \tilde{H}^1(\Omega) \times H^{s+1}(\Omega)^2] \cap [\mathcal{R}U = 0]\,.
$$

Least Squares Finite Element Method

1. Choose the discrete space $\mathbf{X}_s^h \subset \mathbf{X}_s$:

2. Solve the Euler-Lagrange equation

$$\textit{Seek } U^h = (\omega^h, \mathbf{u}^h, r^h) \in \mathbf{X}_s^h \textit{ such that}$$

$$Q(U^h; V^h) = 0 \quad \forall V^h = (\xi^h, \mathbf{v}^h, q^h) \in \mathbf{X}_s^h \,.$$

Implementation Issues

The Euler-Lagrange equation is a nonlinear system that must be solved in an iterative manner.

Newton's method:

- Locally has quadratic convergence
- In a neighborhood of a minimizer the Hessian is symmetric and positive definite.

$\Longrightarrow$ Continuation methods are required to get an initial approximation inside the attraction ball:

- **Continuation along the constant:** simple, but cannot handle turning points
- **Continuation along the tangent:** can be made to handle turning points.

Navier-Stokes Equations: Advantages of the Least Squares Methods

- Approximating spaces are not subject to the **LBB** (**inf-sup**) condition

- All unknowns can be approximated by the *same finite element space*

- Newton linearization results in *symmetric, positive definite* linear systems, at least in the neighborhood of a solution:

  $\Longrightarrow$ Using a properly implemented continuation (with respect to the Reynolds number) techniques, a solution method can be devised that will only encounter symmetric, positive definite linear systems in the solution process

  $\Longrightarrow$ Robust iterative methods can be used

  $\Longrightarrow$ A solution method that is *assembly free* even at an element level can be devised

- No *artificial boundary conditions* for ω need be introduced at boundaries at which $\mathbf{u}$ is specified

- Accurate vorticity approximations are obtained.

Error Analysis of the Least Squares Method: Abstract Approximation Theory

Abstract problem (Girault, Raviart, 1984)

$$F(\lambda, U) \equiv U + T \cdot G(\lambda, U) = 0\,,$$

where $\Lambda \subset R$ is compact interval; $\mathbf{X}$ and $\mathbf{Y}$ are Banach spaces and $T \in L(\mathbf{Y}, \mathbf{X})$.

Regular branch of solutions

Assume that $\{(\lambda, U(\lambda)\,|\,\lambda \in \Lambda\}$ is such that

$$F(\lambda, U(\lambda)) = 0 \quad \text{for } \lambda \in \Lambda\,.$$

1. The set $\{(\lambda, U(\lambda)\,|\,\lambda \in \Lambda\}$ is called *branch of solutions* if the map $\lambda \rightarrow U(\lambda)$ is a continuous function from Λ into $\mathbf{X}$

2. The set $\{(\lambda, U(\lambda)\,|\,\lambda \in \Lambda\}$ is called *regular branch* if $D_U F(\lambda, U(\lambda))$ is an isomorphism from $\mathbf{X}$ into $\mathbf{X}$ for all $\lambda \in \Lambda$.

$\boxed{\textbf{Discretizations}}$

- Choose a discrete subspace $\mathbf{X}^h \subset \mathbf{X}$;
- Choose $T_h \in L(\mathbf{Y}, \mathbf{X}^h)$ to be an approximating operator for the linear part T of F.
- Then, consider the approximate problem

$$F^h(\lambda, U^h) \equiv U^h + T_h \cdot G(\lambda, U^h) = 0\,.$$

- Approximation in F^h is introduced by approximating only the linear operator T:

  $\Longrightarrow$ F^h has the same differentiability properties as the nonlinear map F.

Abstract approximation result

We make the followng assumptions:

A1. $\{(\lambda, \tilde{U}(\lambda)) | \lambda \in \Lambda\}$ is a branch of regular solutions;

A2. G is a C^2 mapping $G : \Lambda \times \mathbf{X} \mapsto \mathbf{Y}$;

A3. All second derivatives $D_U^2 G$ are bounded on all bounded subsets of $\Lambda \times \mathbf{X}$;

A4. There exists a space $\mathbf{Z} \subset \mathbf{Y}$, with a continuous imbedding, such that

$$D_U G(\lambda, U) \in L(\mathbf{X}, \mathbf{Z}) \ \ \forall U \in \mathbf{X}\,;$$

A5. The operator T_h satisfies conditions

$$\begin{aligned} \lim_{h \to 0} \|(T - T_h) g\|_X \ &= \ 0 \ \ \forall g \in \mathbf{Y}\,; \\ \lim_{h \to 0} \|T - T_h\|_{L(Z,X)} \ &= \ 0\,. \end{aligned}$$

Then, for h sufficiently small there exists a unique C^2 function $\lambda \mapsto U^h \in \mathbf{X}^h$, s.t. $\{(\lambda, U^h(\lambda)) | \lambda \in \Lambda\}$ is a branch of regular solutions of $F^h(\lambda, U^h) = 0$ and

$$\|U^h - U\|_X \le C \|(T - T_h) \cdot G(\lambda, U)\|_X \quad \forall \lambda \in \Lambda$$

Application to the least squares method

$$Q((\omega,\mathbf{u},r);(\xi,\mathbf{v},q))_s =$$
$$= (\mathbf{curl}\,\omega + \mathbf{grad}\,r, \mathbf{curl}\,\xi + \mathbf{grad}\,q)_0$$
$$+ (\mathrm{curl}\,\mathbf{u} - \omega, \mathrm{curl}\,\mathbf{v} - \xi)_s + (\mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v})_s\ .$$

$$\mathbf{Y}_s = H^{-1}(\Omega) \times H^{-1}(\Omega) \times H^{-1-s}(\Omega)^2$$

The operator T/T_h:

$\forall g \in \mathbf{Y}_s\,; Tg = U \in \mathbf{X}_s$ if and only if U solves

Seek $U \in \mathbf{X}_s$ *such that*
$Q(U;V)_s = (g,V) \quad \forall V \in \mathbf{X}_s\,.$

For T_h take $U^h, V^h \in \mathbf{X}_s^h$. T and T_h are the Stokes LS solution operator and its discretization.

The operator G:

$\forall U \in \mathbf{X}_s\,; G(\lambda, U) = g \in \mathbf{Y}_s$ if and only if

$$(\mathbf{curl}\,\omega + \mathbf{grad}\,r + a(\omega,\mathbf{u}) + b(\omega,\mathbf{u},\omega_0,\mathbf{u}_0),$$
$$b(\xi,\mathbf{u}_0,\xi,\mathbf{u}) + b(\omega_0,\mathbf{v},\omega,\mathbf{v}))_0$$
$$+ (a(\omega,\mathbf{u}) + b(\omega,\mathbf{u},\omega_0,\mathbf{u}_0), \mathbf{curl}\,\xi + \mathbf{grad}\,q)_0$$
$$= (g_1,\xi)_0 + (g_2,\mathbf{v})_0 + (g_3,q)_0 \quad \forall(\xi,\mathbf{v},q) \in \mathbf{X}_s\,.$$

With the identification

$$U = (\omega, \mathbf{u}, r),\ \ U^h = (\omega^h, \mathbf{u}^h, r^h)\ \ \text{and}\ \ \lambda = \frac{1}{\nu}$$

the Euler-Lagrange equation and its discretization can be cast into the canonical forms

$$U + T \cdot G(\lambda, U) = 0\ \ \text{and}\ \ U^h + T_h \cdot G(\lambda, U^h) = 0$$

Concerning the error estimates one can show that the assumptions [A1.] - [A5.] hold. As a result one can prove the following:

Theorem 1 *Assume that* $\{(\lambda, \tilde{U}(\lambda)) | \lambda \in \Lambda\}$ *is a regular branch of (sufficiently smooth) solutions. Then, for h sufficiently small, the discrete Euler-Lagrange equation has a unique branch* $\{(\lambda, U^h(\lambda)) | \lambda \in \Lambda\}$ *of regular solutions such that*

$$\begin{aligned} \|\omega(\lambda) - \omega^h(\lambda)\|_1 \ &+\ \|\mathbf{u}(\lambda) - \mathbf{u}^h(\lambda)\|_{1+s} \\ &+\ \|r(\lambda) - r^h(\lambda)\|_1 \\ &\leq\ Ch\left(\|\omega(\lambda)\|_2 + \|\mathbf{u}(\lambda)\|_{2+s} + \|r(\lambda)\|_2\right) \end{aligned}$$

where $s = 0$ *for Type 1 and* $s = 1$ *for Type 2 BCs.*

Table 1: Boundary Conditions

	Boundary conditions	3D	2D	Type
BC1	Velocity	$\mathbf{u}$	$\mathbf{u}$	2
	Slack variable	ϕ	-	
BC1A	Velocity	$\mathbf{u}$	$\mathbf{u}$	2
	Normal vorticity	$\omega \cdot \mathbf{n}$	-	
BC2	Pressure	p	p	
	Normal velocity	$\mathbf{u} \cdot \mathbf{n}$	$\mathbf{u} \cdot \mathbf{n}$	1
	Normal vorticity	$\omega \cdot \mathbf{n}$	-	
	Slack variable	ϕ	-	
BC2A	Pressure	p	p	not well
	Normal velocity	$\mathbf{u} \cdot \mathbf{n}$	$\mathbf{u} \cdot \mathbf{n}$	posed in 3D
	Tangential vorticity	$\mathbf{n} \times \omega \times \mathbf{n}$	-	1 in 2D
BC3	Pressure	p	p	1 in 2D
	Tangential velocity	$\mathbf{n} \times \mathbf{u} \times \mathbf{n}$	$\mathbf{u} \cdot \mathbf{t}$	2 in 3D
	Slack variable	ϕ	-	
BC3A	Pressure	p	p	
	Tangential velocity	$\mathbf{n} \times \mathbf{u} \times \mathbf{n}$	$\mathbf{u} \cdot \mathbf{t}$	1
	Normal vorticity	$\omega \cdot \mathbf{n}$	-	
BC4	Normal velocity	$\mathbf{u} \cdot \mathbf{n}$	$\mathbf{u} \cdot \mathbf{n}$	
	Tangential vorticity	$\mathbf{n} \times \omega \times \mathbf{n}$	ω	1
	Slack variable	ϕ	-	
BC5	Tangential velocity	$\mathbf{n} \times \mathbf{u} \times \mathbf{n}$	$\mathbf{u} \cdot \mathbf{t}$	1
	Tangential vorticity	$\mathbf{n} \times \omega \times \mathbf{n}$	ω	
BC6	Vorticity	ω	ω	not well
	Pressure	p	p	posed