A Discourse on Saddle-Point Problems, Their Regularization, Stabilization and Finite Element Solution

December 3, 2003 /9:00am

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Thanks to:
T. Barth, M. Gunzburger, R. Lehoucq, J. Shadid

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An Overview

- **Babuska Theory:** solvability and stability of general linear systems
- **Origins of saddle-point problems**
- **Structure of saddle-point problems**
  - Geometrical
  - Variational
- **Brezzi Theory:** solvability and stability of SP linear systems
- **Regularization and stabilization of SP problems**
  - Reasons to stabilize
  - Modification of Lagrangians
  - Modifications of optimality equations
  - Residual and non-residual stabilization

Resources

- **Saddle-point problems**
  - Brezzi, RAIRO B-R2 (1974)
  - Bramble, Math. Comp. 37 (1981)
  - Boland, Nicolaides, SINUM 20 (1983)
- **Mixed methods**
  - Brezzi, Fortin, Mixed FEM, Springer (1991)
  - Brezzi, Bathe, CMAME 82, (1990)
  - Arnold, Numer. Math. 37 (1981)
- **Stabilization**
  - Hughes, Franca, Balestra, CMAME 59 (1986)
  - Hughes, Franca CMAME 65 (1987)
  - Brezzi, Douglas, Numer. Math. 53 (1988)
  - Douglas, Wang, Math. Comp. 52 (1989)
  - Silvester, Kechkar, CMAME 79 (1990)
  - Codina, Blasco, CMAME 182 (2000)

Finite Element Modeling

Variational principle

seek $u \in V$ such that $B(u,v) = F(v) \quad \forall v \in V$

Discrete variational principle $V^h \subset V$

seek $u_h \in V_h$ such that $B(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

FEM = Variational principle + Finite Dimensional Subspace (Quasi Projection)

A sequence of linear systems

seek $u_h \in V_h \equiv \mathbf{R}^n$ such that $K_h u_h = F_h$

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Babuska Theory: Solvability and Stability for General Sequences of Linear Systems

A single linear system:

$$Ku = F$$

A sequence of linear systems

$$K_h u_h = F_h$$

Solvability

$$Ku = 0 \;\Rightarrow\; u \equiv 0$$

$$K_h u_h = 0 \;\Rightarrow\; u_h \equiv 0$$

Stability

$$\frac{\|\Delta u\|}{\|u\|} \le \|K\|\|K^{-1}\|\frac{\|\Delta F\|}{\|F\|}$$

$$\frac{\|\Delta u_h\|}{\|u_h\|} \le \|K_h\|\|K_h^{-1}\|\frac{\|\Delta F_h\|}{\|F_h\|}$$

Standard condition number not appropriate for assessing stability for sequences of linear systems:

$$\|K\|\|K^{-1}\| = \frac{\lambda_{\max}}{\lambda_{\min}} \;\longrightarrow\; O(h^{-2})$$ for 2nd order PDE's

Domain and Range of K_h

Standard condition number viewpoint on K_h:

$$K_h \in \mathbf{R}^{n\times n} \quad \Rightarrow \quad K_h : \underbrace{\left(\mathbf{R}^n, \|\cdot\|\right)}_{\text{DOMAIN}} \to \underbrace{\left(\mathbf{R}^n, \|\cdot\|\right)}_{\text{RANGE}}$$

We must remember that while $K_h \in \mathbf{R}^{n\times n}$ **for a fixed** h**,**

$$K_h \xrightarrow{h\to 0} \mathcal{L}; \qquad \mathcal{L} : \underbrace{\left(D, \|\cdot\|_D\right)}_{\text{DOMAIN}} \to \underbrace{\left(R, \|\cdot\|_R\right)}_{\text{RANGE}}$$

$$K_h \in \mathbf{R}^{n\times n} \quad \Rightarrow \quad K_h : \underbrace{\left(\mathbf{R}^n, \|\cdot\|_D\right)}_{\text{DOMAIN}} \to \underbrace{\left(\mathbf{R}^n, \|\cdot\|_R\right)}_{\text{RANGE}}$$

Range and domain must be measured differently!

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Choosing the right norms

Domain (solution) norm:

$$B(u,v) \le k\|u\|_D \|v\|_D \quad \Rightarrow \quad v_h^T K_h u_h \le k\|u_h\|_D \|v_h\|_D$$

Easily identified from **continuity** of the bilinear form

Range (data) norm: complete sets of homeomorphisms (ADN-1962)

$$-\Delta : H^k(\Omega) \to H^{k-2}(\Omega)$$

$$\|u\|_1 \le k\|-\Delta u\|_{-1}$$

$$\underbrace{H^1(\Omega)}_{\text{SOLUTION}} \leftarrow \underbrace{L^2(\Omega)}_{\text{PIVOT}} \rightarrow \underbrace{H^{-1}(\Omega)}_{\text{DATA}}$$

$$\|u\|_k \le k\|-\Delta u\|_{k-2}$$

$$\underbrace{H^k(\Omega)}_{\text{SOLUTION}} \leftarrow \underbrace{H^{k-1}(\Omega)}_{\text{PIVOT}} \rightarrow \underbrace{H^{k-2}(\Omega)}_{\text{DATA}}$$

Solution and **Data** spaces are **dual** relative to a **pivot** space on a **Hilbert scale**

Stability bound

Domain: $\left(\mathbf{R}^n, \|\cdot\|_D\right)$ $\quad \|v_h\|_D \rightarrow$ implied by continuity

Range: $\left(\mathbf{R}^n, \|\cdot\|_R\right)$ $\quad \|f_h\|_R \rightarrow \|f_h\|_{D*} = \sup_{\mathbf{v}} \dfrac{v_h^T f_h}{\|v_h\|_D}$ Dual norm!

Operator norms:

$$\|K_h\| = \sup_{v_h} \frac{\|K_h v_h\|_R}{\|v_h\|_D}$$

$$\|K_h^{-1}\| = \sup_{v_h} \frac{\|K_h^{-1} v_h\|_D}{\|v_h\|_R}$$

Stability bound

$$\frac{\|\Delta u_h\|_D}{\|u_h\|_D} \le \|K_h\| \|K_h^{-1}\| \frac{\|\Delta F_h\|_R}{\|F_h\|_R}$$

Stability Conditions

Lemma

$$\|K_h\| \le k \qquad \left\|K_h^{-1}\right\|^{-1} = \inf_{u_h} \sup_{v_h} \frac{v_h^T K_h u_h}{\|v_h\|_D \|u_h\|_D}$$

Theorem (Babuska, Necas)

The sequence of linear systems is **uniformly stable** in *h*

$$v_h^T K_h u_h \le k \|v_h\|_D \|u_h\|_D \qquad \textbf{continuity}$$

$$\inf_{u_h} \sup_{v_h} \frac{v_h^T K_h u_h}{\|v_h\|_D \|u_h\|_D} \ge \gamma > 0 \qquad \textbf{coercivity}$$

where *k* and γ are independent of *h*.

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Algebraic interpretation

Equivalent form

$$\left.\begin{array}{l}\inf\limits_{u_h}\sup\limits_{v_h}\dfrac{v_h^T K_h u_h}{\|v_h\|_D \|u_h\|_D} \geq \gamma \\ \\ \|u_h\|_D^2 = u_h^T S_h u_h\end{array}\right\} \Rightarrow \underbrace{\min_{u_h}\max_{v_h}\frac{v_h^T K_h u_h}{\left(v_h^T S_h v_h\right)^{1/2}\left(u_h^T S_h u_h\right)^{1/2}}}_{\sigma_1(K_h, S_h)} \geq \gamma$$

Weak coercivity is equivalent to having the smallest generalized singular value of K_h bounded away from zero, **independently of the mesh size** *h:*

$$\sigma_1\left(K_h, S_h\right) \geq \gamma$$

The continuous case

$$B(u,v) \le k\|u\|\|v\|$$

$$\sup_{v} \frac{B(u,v)}{\|v\|} \ge \gamma \|u\|$$

$$\sup_{u} \frac{B(u,v)}{\|u\|} > 0$$

$\exists! u$ such that $B(u,v) = F(v) \quad \forall v \in V$

$$\|u\| \le \frac{k}{\gamma}\|F\|$$

Discrete + Continuous inf-sup conditions

$\Downarrow$

Stability and convergence of finite element solutions:

$$\|u - u_h\| \le \left(1 + \frac{k}{\gamma}\right) \inf_{v_h} \|u - v_h\|$$

Origins of Saddle-Point Problems

Physical models

Kinematic relation

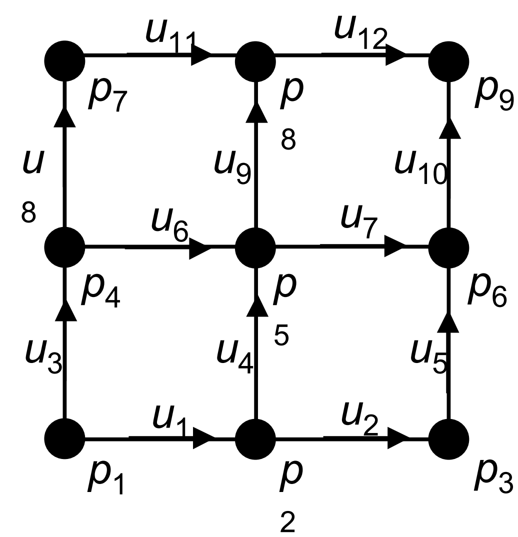
$$u_1 = p_2 - p_1$$
$$u_2 = p_3 - p_2$$
$$u_3 = p_4 - p_1$$
$$u_4 = p_5 - p_2$$
$$u_5 = p_6 - p_3$$
$$u_6 = p_5 - p_4$$
$$u_7 = p_6 - p_5$$
$$u_8 = p_7 - p_4$$
$$u_9 = p_8 - p_5$$
$$u_{10} = p_9 - p_6$$
$$u_{11} = p_8 - p_7$$
$$u_{12} = p_9 - p_8$$

Constitutive equation

$$v_i = \rho_i u_i$$

Continuity relation

$$-v_1 - v_3 = 0$$
$$+v_1 - v_2 - v_4 = 0$$
$$+v_2 - v_5 = 0$$
$$+v_3 - v_6 - v_8 = 0$$
$$+v_4 + v_6 - v_7 - v_9 = 0$$
$$+v_7 + v_5 - v_{10} = 0$$
$$+v_8 - v_{11} = 0$$
$$+v_9 + v_{11} - v_{12} = 0$$
$$+v_{10} + v_{12} = 0$$



$p \rightarrow$ "pressure"

$u \rightarrow$ "velocity"

$\rho \rightarrow$ "density"

$v \rightarrow$ "flow"

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Matrix Form

Kinematic	Constitutive	Continuity
$u + B^T p = g$	$u = Av$	$Bu = f$

$$
\begin{pmatrix}
-1 & 1 & & & & & & & \\
 & -1 & 1 & & & & & & \\
-1 & & & 1 & & & & & \\
 & -1 & & & 1 & & & & \\
 & & -1 & & & 1 & & & \\
 & & & -1 & 1 & & & & \\
 & & & & -1 & 1 & & & \\
 & & & -1 & & & 1 & & \\
 & & & & -1 & & & 1 & \\
 & & & & & -1 & & & 1 \\
 & & & & & & -1 & 1 & \\
 & & & & & & & -1 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
\frac{1}{\rho_i} & & \\
 & \ddots & \\
 & & \frac{1}{\rho_i}
\end{pmatrix}
$$

$$
\begin{pmatrix}
-1 & & -1 & & & & & & & & & \\
1 & -1 & & -1 & & & & & & & & \\
 & 1 & & & -1 & & & & & & & \\
 & & 1 & & & -1 & & -1 & & & & \\
 & & & 1 & & 1 & -1 & & -1 & & & \\
 & & & & 1 & & 1 & & & -1 & & \\
 & & & & & & & 1 & & & -1 & \\
 & & & & & & & & 1 & & 1 & -1 \\
 & & & & & & & & & 1 & & 1
\end{pmatrix}
$$

$$
\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} g \\ f \end{pmatrix}
$$

$$
-BA^{-1}B^T p = f - BA^{-1} g
$$

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PDE Examples

	Potential flow	Thermal diffusion	Electro statics	Linear elasticity	Electrical network
p	Pressure	Temperature	Potential	Displacement	Potential
u	Velocity	Heat flux	Electric field	Strain	Voltage
A^{-1}	Permeability	Thermal conductivity	Conductivity Ohm's law	Compliance Hook's law	Conductivity Ohm's law
v	Flow rate	Heat flow	Current	Stress	Current
f	Fluid Source	Heat Source	Source Current	Applied load	Applied current
g	N/A	Heat battery	Battery	N/A	Battery

Geometrical structure

Recall the discrete network of pipes…

Constitutive

Kinematic

Continuity

$$
\begin{aligned}
u_1 &= p_2 - p_1 \\
u_2 &= p_3 - p_2 \\
u_3 &= p_4 - p_1 \\
u_4 &= p_5 - p_2 \\
u_5 &= p_6 - p_3 \\
u_6 &= p_5 - p_4 \\
u_7 &= p_6 - p_5 \\
u_8 &= p_7 - p_4 \\
u_9 &= p_8 - p_5 \\
u_{10} &= p_9 - p_6 \\
u_{11} &= p_8 - p_7 \\
u_{12} &= p_9 - p_8
\end{aligned}
$$

$$v_i = \rho_i u_i$$

$$
\begin{aligned}
-v_1 - v_3 &= 0 \\
+v_1 - v_2 - v_4 &= 0 \\
+v_2 - v_5 &= 0 \\
+v_3 - v_6 - v_8 &= 0 \\
+v_4 + v_6 - v_7 - v_9 &= 0 \\
+v_7 + v_5 - v_{10} &= 0 \\
+v_8 - v_{11} &= 0 \\
+v_9 + v_{11} - v_{12} &= 0 \\
+v_{10} + v_{12} &= 0
\end{aligned}
$$

- Kinematic and continuity equations depend only on "network topology" (**incidence matrices**!)
- **Metric** properties are introduced by

This distinct pattern appears over and over in physical models (Tonti, 1974).

Mathematical tools for the study of such patterns are provided by differential forms on manifolds

(Hyman, Scovel, Shashkov 1987-, Bossavit 1989, Nicolaides, 1989-, Demkowicz, 1999, Hiptmair,2000)

A crash course in differential forms

De Rham Cohomology (exact sequence)

Primal

$$W^0 \xrightarrow[d]{\nabla} W^1 \xrightarrow[d]{\nabla\times} W^2 \xrightarrow[d]{\nabla\bullet} W^3$$

$\updownarrow \qquad \updownarrow$ **Hodge** $\ast$ $\updownarrow \qquad \updownarrow$

$$\tilde{W}^3 \xleftarrow[d]{\nabla\bullet} \tilde{W}^2 \xleftarrow[d]{\nabla\times} \tilde{W}^1 \xleftarrow[d]{\nabla} \tilde{W}^0$$

Dual

Factorization diagram

$$\begin{array}{ccccc} & W^{k-1} & \leftarrow \ast_\mu \rightarrow & \tilde{W}^{n-k+1} & \\ d^{k-1} & \downarrow & & \uparrow & d^{n-k} \\ & W^k & \leftarrow \ast_\varepsilon \rightarrow & \tilde{W}^{n-k} & \end{array}$$

topology topology

Exactness: $dd = 0$

Poincare Lemma $d\omega^k = 0 \Rightarrow \omega^k = d\omega^{k-1}$

$\Rightarrow$ Existence of potentials:

$$\nabla\psi = 0 \quad \Rightarrow \quad \psi = Const$$
$$\nabla\times\mathbf{E} = 0 \quad \Rightarrow \quad \mathbf{E} = \nabla\varphi$$
$$\nabla\cdot\mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla\times\mathbf{A}$$

"All" 2nd order PDE's

K=1

$$\begin{array}{ccccc} & a & \leftarrow \ast_\mu \rightarrow & -\alpha & \\ d^0 = \nabla & \downarrow & & \uparrow & \nabla\cdot = d^2 \\ & -b & \leftarrow \ast_\varepsilon \rightarrow & \beta & \end{array}$$

$$\nabla a = -b \qquad \nabla\cdot\beta = -\alpha$$
$$\alpha = \ast_\mu a \qquad \beta = \ast_\varepsilon b$$
$$-\nabla\cdot\varepsilon\nabla a + \mu a = f$$

Variational Structure

Complementary energy of the system

$$G(v) = \frac{1}{2} v^T A v - v^T g$$

Admissible states of the system minimize energy

$$\min_v G(v) \quad \text{subject to } Bv = g$$

$$\downarrow$$

$$L(v,q) = G(v) + q^T (Bv - g) \rightarrow \sup_q \inf_v L(v,q) \rightarrow \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

Elimination of *q*

$$J(v) = \sup_q L(v,q)$$

primal problem

Elimination of *v*

$$J^*(q) = \inf_v L(v,q)$$

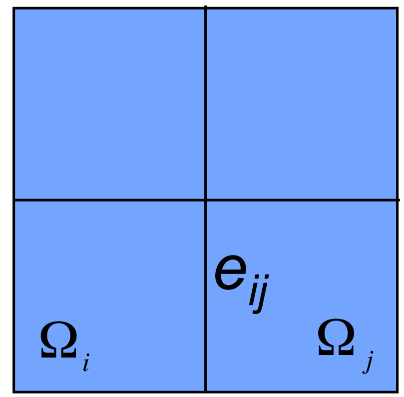
dual problem

Other sources of saddle-point problems

Domain decomposition, Hybrid methods (DG)

$$J(q) = \frac{1}{2}\int_{\Omega} |\nabla q|^2 \, d\Omega - \int_{\Omega} fq \, d\Omega$$

$$J(q) = \sum_{\Omega_i} \left(\frac{1}{2}\int_{\Omega_i} |\nabla q|^2 \, d\Omega - \int_{\Omega_i} fq \, d\Omega \right)$$



Optimization problem

$\min\limits_{q \in X} J(q)$ subject to $q \in H^1(\Omega)$

$$X = \prod_i H^1(\Omega_i)$$

Note that $q \in H^1(\Omega) \quad \Leftrightarrow \quad q_i = q_j$ in $H^{1/2}(e_{ij})$

$$\inf_q \sup_\lambda = \sum_{\Omega_i} \left(\frac{1}{2}\int_{\Omega_i} |\nabla q|^2 \, d\Omega - \int_{\Omega_i} fq \, d\Omega - \int_{e_{ij}} \lambda (u_i - u_j) \, d\Gamma \right)$$

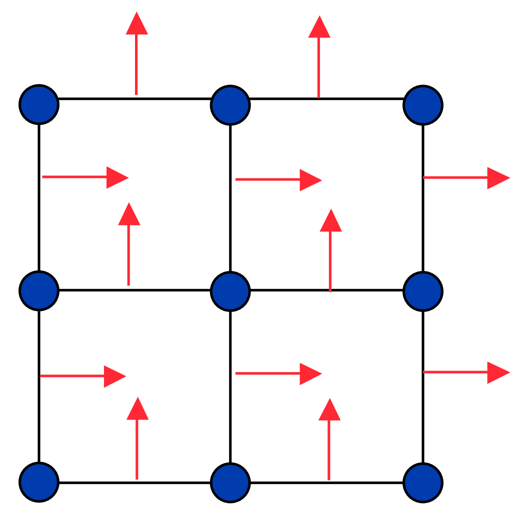
Constrained Interpolation

Given: weakly divergence free **nodal** velocity

$$\int_{\Omega} q \nabla \cdot u_{nodal} \, d\Omega = 0 \quad \forall q \in S^h$$

Wanted: divergence free **RT** velocity

$$\nabla \cdot v_{RT} = 0$$



Constrained optimization problem

$$\min_{v_{RT}} J(v_{RT}) \text{ subject to } \nabla \cdot v_{RT} = 0$$

$$J(v_{RT}) = \frac{1}{2} \| v_{RT} - u_{nodal} \|_o^2$$

$$\sup_q \inf_{v_{RT}} \frac{1}{2} \| v_{RT} - u_{nodal} \|_o^2 - \int_{\Omega} q \nabla \cdot v_{RT} \, d\Omega$$

"Darcy flow"

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Brezzi Theory: Solvability and Stability for Saddle-Point Linear Systems

Specialization of the Babuska theory for

$$K_h u_h = F_h \quad \rightarrow \quad \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} v_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ g_h \end{pmatrix}$$

$$K_h \in \mathbf{R}^{(n+m)\times(n+m)}$$
$$A_h \in \mathbf{R}^{n\times n}$$
$$B_h \in \mathbf{R}^{m\times n}$$

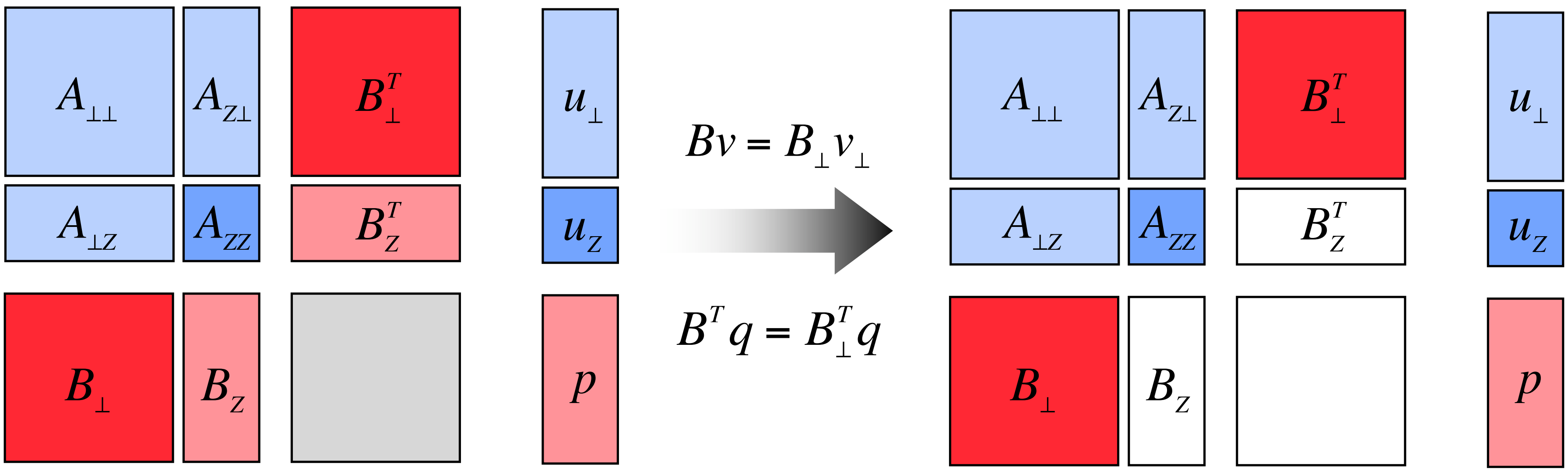
Goal: **solvability** and **stability** conditions in terms of the submatrices A_h and B_h

A critical space $Z = \{v \in \mathbf{R}^n \,|\, Bv = 0\} = \mathrm{Ker}(B)$ $\quad \mathbf{R}^n = Z \oplus Z^\perp$

$$v \in \mathbf{R}^n \quad \rightarrow \quad v = v_Z + v_\perp$$

$$v_Z = (\underbrace{0,\ldots\ldots,0}_{n-k}, v_Z^1,\ldots,v_Z^k) \qquad v_\perp = (v_Z^1,\ldots,v_Z^{n-k}\underbrace{0,\ldots\ldots,0}_{k})$$

Problem structure



$$
\begin{aligned}
A_{\perp\perp} u_{\perp} + A_{Z\perp} u_{Z} + B_{\perp}^{T} p &= f_{\perp} \\
A_{\perp Z} u_{\perp} + A_{ZZ} u_{Z} &= f_{Z} \\
B_{\perp} u_{\perp} &= g
\end{aligned}
$$

$\Rightarrow$

Solvability & stability: depend on A_{ZZ} and $B_{\perp}$

Arnold, 1981

Solvability

Solvability conditions

$$\left.\begin{array}{ll} B_\perp^T p = f_\perp & \text{solvable for } \forall f_\perp \\ A_{ZZ} u_Z = f_Z & \text{solvable for } \forall f_Z \\ B_\perp u_\perp = g & \text{solvable for } \forall g \end{array}\right\} \Rightarrow \left\{\begin{array}{l} B_\perp : Z^\perp \to \mathbf{R}^m \\ A_{ZZ} : Z \to \mathbf{R}^m \end{array}\right. \text{ are invertible}$$

Solvability in terms of *A* and *B*

$$B^T p = 0 \qquad\qquad \Rightarrow \quad p = 0$$

$$v_Z A u_Z = 0 \quad \forall v_Z \in Z \quad \Rightarrow \quad u_Z = 0$$

Stability

Domain and range norms

$$A_h \in \mathbf{R}^{n\times n} \quad \Rightarrow \quad A_h : \underbrace{\left(\mathbf{R}^n, \|\cdot\|_{DA}\right)}_{\text{DOMAIN}} \rightarrow \underbrace{\left(\mathbf{R}^n, \|\cdot\|_{RA}\right)}_{\text{RANGE}} \qquad \|A_h\| = \sup_{v_h} \frac{\|A_h v_h\|_{RA}}{\|v_h\|_{DA}}$$

$$B_h \in \mathbf{R}^{n\times m} \quad \Rightarrow \quad B_h : \underbrace{\left(\mathbf{R}^n, \|\cdot\|_{DB}\right)}_{\text{DOMAIN}} \rightarrow \underbrace{\left(\mathbf{R}^m, \|\cdot\|_{RB}\right)}_{\text{RANGE}} \qquad \|B_h\| = \sup_{q_h} \frac{\|B_h q_h\|_{RB}}{\|q_h\|_{DB}}$$

A priori bound

$$\|u_h\|_{DA} + \|p_h\|_{DB} \le C\left(\|A_{\perp Z}\|, \|A_{ZZ}^{-1}\|, \|B_{\perp}^{T}\|, \|B_{\perp}^{-1}\|\right)\left(\|f_h\|_{RA} + \|g_h\|_{RB}\right)$$

Stability $\Leftrightarrow$ $\|A_{\perp Z}\|, \|A_{ZZ}^{-1}\|, \|B_{\perp}^{T}\|, \|B_{\perp}^{-1}\|$ **bounded uniformly in** ***h***

The Brezzi theory: stability conditions

Continuity

$$v_h^T A_h u_h \le k_A \|u_h\|_{DA} \|v_h\|_{DA} \quad \Rightarrow \quad \|A_{\perp Z}\| \le k_A$$

$$v_h^T B_h q_h \le k_B \|v_h\|_{DA} \|q_h\|_{DB} \quad \Rightarrow \quad \|B_\perp^T\| \le k_B$$

Coercivity

$$\inf_{u_h \in Z} \sup_{v_h \in Z} \frac{v_h^T A_h u_h}{\|u_h\|_{DA} \|v_h\|_{DA}} \ge \gamma_A \quad \Rightarrow \quad \|A_{ZZ}^{-1}\| \le \frac{1}{\gamma_A}$$

$$\inf_{q_h} \sup_{v_h} \frac{v_h^T B_h q_h}{\|q_h\|_{DB} \|v_h\|_{DA}} \ge \gamma_B \quad \Rightarrow \quad \|B_\perp^{-1}\| \le \frac{1}{\gamma_B}$$

The sequence of saddle-point problems is stable iff

$$k_A, k_B, \gamma_A, \gamma_B$$

are independent of *h*

Comments and remarks

- There are **two** (2) inf-sup conditions - one for A and one for B!
- Together they imply an inf-sup condition for the big matrix K
- A is often SPD on the **kernel** (Darcy flow)
- A may be SPD on the **whole space** (Stokes equations)!!!

  $\Rightarrow$ The inf-sup condition on A is often overlooked and Brezzi conditions are identified solely with the second inf-sup condition!

- May lead to serious **embarrassment**, both conditions are needed for uniform stability!

For an algebraic interpretation of the second inf-sup condition talk to Rich or see e.g. Silvester, Wathen et.al.

The roles of geometrical and variational structures

Variational

- **Operator**-centric point of view
  - **Problem** = operator equation on function spaces
  - **Discretization** = operator equation + functional approximation
- Stability conditions
- Error estimates

stability conditions not constructive -
do not reveal structure of stable discretizations

Geometrical

- **Topology**-centric point of view
  - **Problem** = equilibrium relation on manifolds
  - **Discretization** = equilibrium relation + manifold approximation
- Forces **physically compatible** discretization patterns
- Leads to identification of stable **functional** approximations

Stabilization of saddle-point problems

Why stabilize? **Conformity $\not\Rightarrow$ stability & accuracy**

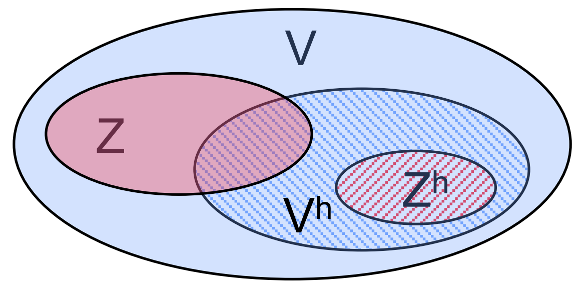
FE spaces must be compatible

- -Stable pairs too complicated or not available (3D mixed elasticity)
- -Unequal order approximation and/or different grids
- -Less uniformity in data structures

Indefinite algebraic problems

- -more difficult to solve iteratively

Compatibility impinges on efficiency!



Goals of stabilization

Stability for arbitrary conforming spaces (including equal order)

$\Rightarrow$ More regular data structures (improved parallel efficiency)

Improve the type of the algebraic problem

$\Rightarrow$ Allow application of more efficient iterative solvers

$\Rightarrow$ Allow definition of better preconditioners

Stabilization ≠ regularization

Regularization: modification of the original equation that

- *makes it **"easier" to solve** while requiring compatible spaces*

Stabilization: modification of the original equation that

- *makes it **stable** for any conforming spaces*
- *maintains the asymptotic order of the stable discretization*

Stabilization and regularization look alike and are easy to confuse

$$K_h = \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{A}_h & \tilde{B}_h^T \\ \tilde{C}_h & \tilde{D} \end{pmatrix} = \tilde{K}_h$$

Modification of the **Lagrangian**
Modification of the **equations**

However, stabilization ensures that the sequence $\tilde{K}_h u_h = F_h$ is stable:

$$v_h^T \tilde{K}_h u_h \le k \|v_h\|_D \|u_h\|_D \quad \& \quad \inf_{u_h} \sup_{v_h} \frac{v_h^T \tilde{K}_h u_h}{\|v_h\|_D \|u_h\|_D} \ge \gamma > 0$$

Top 4 Modified Lagrangians

$$J(v)+(Bv,q)-\|Av+B^Tq-f\|^2$$

$$J(v)+(Bv,q)-\|Av+B^Tq-f\|^2+\|Bv\|^2$$

$$Au+B^Tp=f$$

← *Optimality system* →

$$Bu\qquad=0$$

$$-\frac{2}{r}\|q\|^2$$

$$J(v)+(Bv,q)$$

$$\frac{r}{2}\|Bv\|$$

$$J(v)+(Bv,q)-\frac{2}{r}\|q\|^2$$

Lagrange multipliers

$$J(v)+(Bv,q)+\frac{r}{2}\|Bv\|^2$$

$\min\limits_v J(v)$ subject to $Bv=0$

$$q=\frac{r}{2}Bv$$

penalty

$$q=0$$

$$J(v)+\frac{r}{2}\|Bv\|^2$$

Quiz: Which are stabilized and which are regularized?

$J(v)+(Bv,q)+\frac{r}{2}\|Bv\|^2$	Augmented Lagrangian
$J(v)+\frac{r}{2}\|Bv\|^2$	Classical penalty
$J(v)+(Bv,q)-\frac{2}{r}\|q\|^2$	Penalized Lagrangian
$J(v)+(Bv,q)-\|Av+B^Tq-f\|^2+\|Bv\|^2$	Full Least-squares
$J(v)+(Bv,q)-\|Av+B^Tq-f\|^2$	Least-squares

The regularized methods:

$J(v)+\left(Bv,q\right)+\frac{r}{2}\lVert Bv\rVert^2$	Augmented Lagrangian	$\begin{pmatrix} A_h + rB_h^T B_h & B_h^T \\ B_h & 0 \end{pmatrix}$
$J(v)+\frac{r}{2}\lVert Bv\rVert^2$	Classical penalty	$A_h + rB_h^T B_h$
$J(v)+\left(Bv,q\right)-\frac{2}{r}\lVert q\rVert^2$	Penalized Lagrangian	$\begin{pmatrix} A_h & B_h^T \\ B_h & -\varepsilon G_h \end{pmatrix}$

Augmented and Penalized Lagrangian are related to classical penalty!

As $\varepsilon \to 0$ classical penalty converges to the original problem $\to \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}$

$\Rightarrow$ **Unstable unless A_h and B_h generated by stable spaces!**

Classical penalty **is not** a **stabilization** procedure but rather a **solution method** to uncouple the variables

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Augmented Lagrangian vs. Penalty

Augmented Lagrangian - modification by the constraint

$$J(v) + \left(Bv, q\right) + \frac{r}{2}\|Bv\|^2$$

- Constraint equation **retained**: $Bv=0$!
  - $\Rightarrow$ There's no penalty error and r does not have to approach ∞
  - $\Rightarrow$ Stable pairs required: constraint not relaxed!
- Problem still **indefinite** but better conditioned than penalty!
- Example of ***consistent regularization***:

  goal is to make the equations easier to solve by iterative methods (see Glowinski et.al. 1988) but without incurring a penalty error.
- Unlike penalty, AL cannot uncouple the variables!

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The stabilized methods

$$J(v) + (Bv,q) - \|Av + B^T q - f\|_\tau^2 + \|Bv\|_\mu^2$$

$$\begin{pmatrix} A_h - \tau A_h^T A_h + \mu B_h^T B_h & -\tau A_h^T B_h^T + B_h^T \\ -\tau B_h A_h + B_h & -\tau B_h B_h^T \end{pmatrix}$$

$$J(v) + (Bv,q) - \|Av + B^T q - f\|_\tau^2$$

$$\begin{pmatrix} A_h - \tau A_h^T A_h & -\tau A_h^T B_h^T + B_h^T \\ -\tau B_h A_h + B_h & -\tau B_h B_h^T \end{pmatrix}$$

- Constraint equation **relaxed**: $Bv \neq 0$!

  ⇒ Potentially stable for all conforming pairs

  ⇒ May require tunable parameters (the magic τ) for stability:

$$u_h^T \tilde{K}_h u_h \geq \gamma \|u_h\|_D^2 \quad \text{or} \quad \inf_{u_h} \sup_{v_h} \frac{v_h^T \tilde{K}_h u_h}{\|v_h\|_D \|u_h\|_D} \geq \gamma > 0$$

- Example of ***consistent stabilization***:

  Modification is effected by terms that vanish on the exact solution and so the formal order of accuracy is retained

Modification of the equations

- Modification effected at the equations level
- May be designed to act only on the constraint equation
- Resulting problem may not be related to optimization
- Stabilized problem not symmetric even if original was

$$
\begin{aligned}
A_h u_h + B_h^T p_h &= f_h \\
B_h u_h &= \tau B_h \left(A_h u_h + B_h^T p_h - f_h \right)
\end{aligned}
\qquad
\begin{pmatrix} A_h & B_h^T \\ -\tau A_h B_h + B_h & -\tau B_h B_h^T \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ -\tau B_h f_h \end{pmatrix}
$$

$$
\begin{aligned}
A_h u_h + B_h^T p_h &= f_h \\
B_h u_h &= 0
\end{aligned}
$$

Original equation

$$
\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix}
$$

All stabilized methods considered so far are examples of **residual based stabilization**

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Examples of stabilized methods

Stokes problem

$$a(u_h,v_h)+b(p_h,v_h)=(f,v_h) \quad \forall v_h \in V_h$$
$$b(q_h,u_h)=0 \quad \forall q_h \in S_h$$

$$a(u,v)=\mu\int_\Omega \nabla u\cdot\nabla v\,d\Omega$$
$$b(q,v)=-\int_\Omega q\nabla\cdot v\,d\Omega$$

Residual based stabilization

$$Q_h^\gamma(u_h,p_h;v_h,q_h)=a(u_h,u_h)+B(p_h,v_h)+\gamma B(q_h,v_h)$$ Galerkin term

Stabilizing term $-\sum_K \delta h^2\int_K(-\Delta u_h+\nabla p_h-f)\cdot(\alpha(-\Delta v_h)+\gamma\nabla q_h)dK$

$$Q_h^\pm(u_h,p_h;u_h,q_h)=a(u_h,v_h)+b(p_h,v_h)\pm b(q_h,u_h)$$
$$-\sum_K \delta h^2\int_K(-\Delta u_h+\nabla p_h-f)\cdot\begin{cases}(\pm\nabla q_h)\\(-\Delta v_h\pm\nabla q_h)\\(+\Delta v_h\pm\nabla q_h)\end{cases}dK$$

SGLS
GLS
RGLS

- Consistency requires at least **quadratic** spaces

Hughes, Franca, Brezzi, Douglas…1986-

Non-residual based stabilization

Absolutely stable Pressure-Poisson method

Replaces Laplace operator by a discrete operator

$$-\Delta_h : V_h \to V_h$$

$$\tilde{u}_h = -\Delta_h u_h \Leftrightarrow \left(\nabla \tilde{u}_h - \nabla u, \nabla \tilde{v}_h\right) = 0 \ \forall \tilde{v}_h \in V_h$$

Stabilized problem

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

$$b(q_h, u_h) = \alpha\left(-\Delta_h u_h + \nabla p_h - f, \nabla q_h\right) \qquad \forall q_h \in S_h$$

- Stable and optimally accurate for any parameter value
- Almost a residual-based stabilization
- Avoids problems with computation of -Δ for P1-P1, Q1-Q1

Bochev, Gunzburger SINUM (2004)

Non-residual based stabilization

Pressure Gradient Projection method (PGP)

Relaxes constraint by the difference between ∇p and its projection

$$Q : L^2(\Omega)^d \rightarrow V_h$$

$$\tilde{u}_h = Qu \Leftrightarrow (\tilde{u}_h - u, \tilde{v}_h) = 0 \ \forall \tilde{v}_h \in V_h$$

Stabilized problem

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

$$b(q_h, u_h) = \alpha\big((Q - I)\nabla p_h, \nabla q_h\big) \qquad \forall q_h \in S_h$$

- Motivated by fractional step methods for transient Navier-Stokes
- Projects onto continuous velocity space $\Rightarrow$ global problem
- Parameter dependent

Codina, Blasco, CMAME 143 (1997)

Non-residual based stabilization

Pressure Polynomial Projection method (PPP)

Relaxes constraint by the difference between *p* and its projection

$$Q : L^2(\Omega) \rightarrow \lfloor S_h \rfloor$$

$$q_h = Qq \Leftrightarrow (q_h - q, q_h) = 0 \ \forall q_h \in \lfloor S_h \rfloor$$

Stabilized problem

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

$$b(q_h, u_h) = \boxed{((Q - I)p_h, q_h)} \qquad \forall q_h \in S_h$$

- Motivated by velocity-pressure mismatch penalization
- Projects onto discontinuous pressure space ⇒ local problems
- Parameter independent

Dohrmann, Bochev, submitted

Non-residual based stabilization

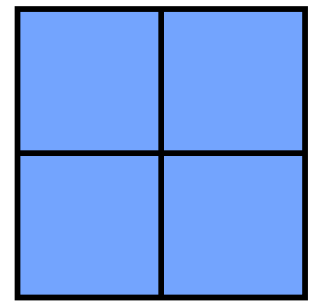
Pressure Jump Stabilization

Relaxes constraint by the jump of the pressure field

Global PJ Stabilized problem

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

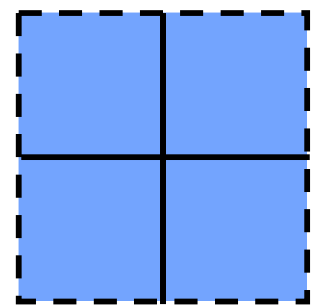
$$b(q_h, u_h) = \alpha h \sum_e \int_e [p_h][q_h] dS \qquad \forall q_h \in S_h$$



Local PJ Stabilized problem

$$a(u_h, v_h) + b(p_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

$$b(q_h, u_h) = \alpha h \sum_{patch} \sum_e \int_e [p_h][q_h] dS \qquad \forall q_h \in S_h$$



patch

- GPJ forces pressure into H^1 - not the natural pressure space
- LPJ forces pressure into H^1 on the patch only $\Rightarrow$ L^2 in Ω

Silvester, Kechkar CMAME 79 (1990)

Concluding remarks (Cliff Notes on stabilization)

Regularization vs. stabilization

Goal of regularization is to make the problem more convenient to solve, not to circumvent the inf-sup conditions.

Goal of stabilization is to circumvent inf-sup conditions and make the problem stable for any conforming spaces

Modification of Lagrangians vs. equations

Modification of Lagrangians preserves symmetry
Equations remain connected to optimization

Modification of equations is more flexible
Connection with optimization is lost

Question

Is there any intrinsic value and potential benefit to solver design when problems are connected to optimization?

Residual vs. non-residual (Cliff Notes on stabilization)

Non-residual stabilization:

Avoids computation of second order derivatives

Works better for the lowest order nodal elements (P1-P1 & Q1-Q1)

Parameter dependence can be avoided

Must be developed on a case by case basis

Residual stabilization:

Retains the order of the Galerkin method

Easy to develop for any problem - just add residuals…

Can be very sensitive to parameter choice

May require computation of higher order derivatives

Question

Non-residual stabilization causes less change to the mixed matrix. Can this be beneficial to solvers?