A Multiscale Discontinuous Galerkin Method

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Nomenclature

$\mathcal{T}_h(\Omega)$ → Partition of Ω into finite elements *K*

$V(\cdot)$ → The set of all vertices of an element or a collection of elements

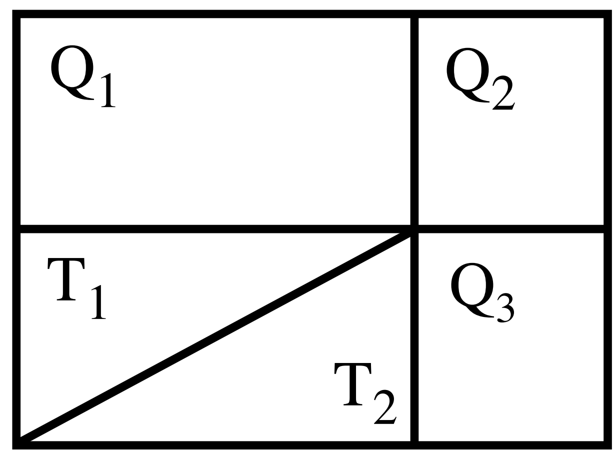
$E(\cdot)$ → The set of all edges of an element or a collection of elements

Γ_h → The set of all edges = $E(\mathcal{T}_h(\Omega))$

Γ_h^0 → The set of internal edges = $E(\mathcal{T}_h(\Omega / \partial\Omega))$

K → A generic finite element: *T* or Q

$\hat{K}$ → A generic reference element



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The Discontinuous Space

Reference element space

$$S^{p(\hat{K})}\left(\hat{K}\right) \rightarrow \text{ polynomials of degree} \leq p(\hat{K})$$

Local element space:

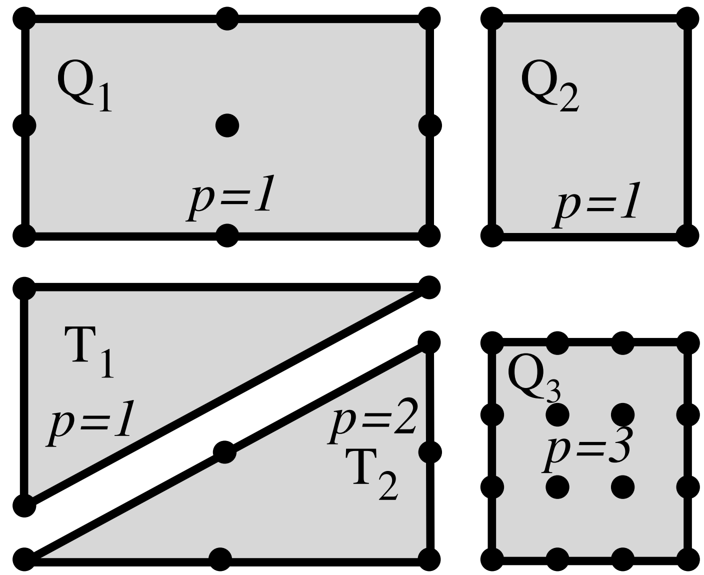
$$S^{p(K)}\left(K\right) \rightarrow \text{ image of the reference space}$$

Discontinuous space:

$$\Phi_h(\Omega) = \left\{\varphi_h \in L^2(\Omega) | \varphi_h|_K \in S^{p(K)}(K)\ \forall K \in \mathcal{T}_h\right\}$$

Hierarchical basis

$$\phi_h = \sum_{\mathbf{v}} v_{\mathbf{v}} V_{\mathbf{v}}\left(\mathbf{x}\right) + \sum_{\mathbf{e}} e_{\mathbf{e}} E\left(\mathbf{x}\right) + \sum_{k} b_k B_k\left(\mathbf{x}\right)$$



Formal union of the local spaces, polynomial degrees not constrained by continuity across element boundaries.

The (Minimal) Continuous Space

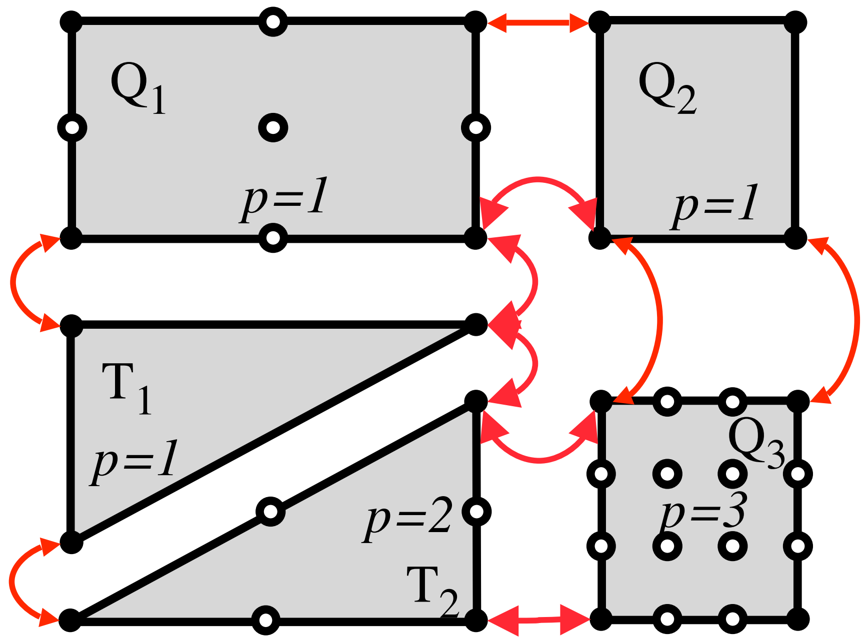
The continuous space

- Required for the **additive decomposition** step in VMS
- Defined with respect to the **same partition** of Ω into finite elements
- Completely **unrelated** to the discontinuous space

The minimal continuous space

- Used here for simplicity
- Spanned by vertex shape functions
- H^1 conforming

$$\overline{\Phi}_h(\Omega) = \left\{ \overline{\varphi}_h \in H^1(\Omega) | \overline{\varphi}_{h|K} \in S^1(K)\ \forall K \in \mathcal{T}_h' \right\}$$



Basis

$$\overline{N}_i(\mathbf{v}_j) = \delta_{ij} \qquad \overline{N}_{\mathbf{v}|K} = V_{\mathbf{v}}$$

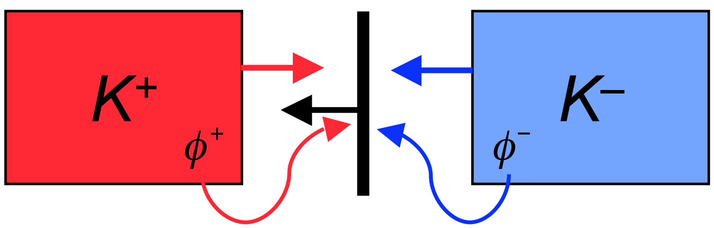
Orientations, Jumps and Averages

Orientation

- Elements oriented as sources
- Edges oriented by choosing a normal

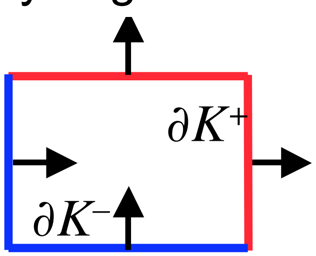
Upwind (-) & downwind (+) elements

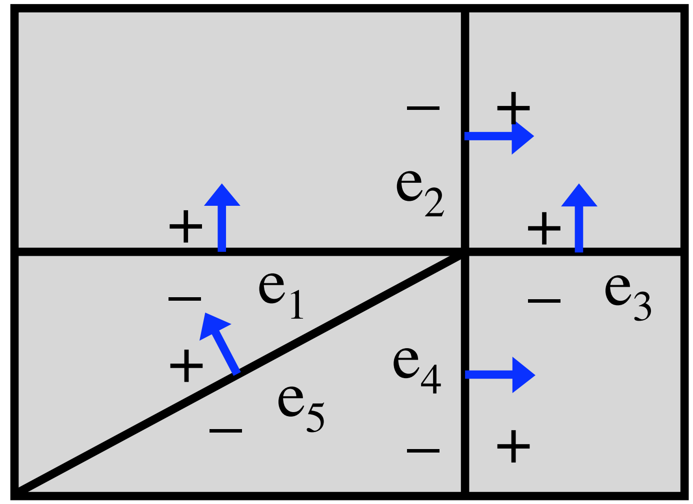
- Determined by shared edge orientation



Inflow & outflow element boundary

- Determined by edge orientations





Jumps and averages

$$\langle \phi \rangle = \frac{1}{2}\left(\phi^+ + \phi^-\right) \qquad [\phi] = \phi^+ \mathbf{n}^+ + \phi^- \mathbf{n}^-$$

$$\langle \mathbf{u} \rangle = \frac{1}{2}\left(\mathbf{u}^+ + \mathbf{u}^-\right) \qquad [\mathbf{u}] = \mathbf{u}^+ \cdot \mathbf{n}^+ + \mathbf{u}^- \cdot \mathbf{n}^-$$

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Variational Multiscale DG approach

Model BVP

$$\mathcal{L}(\mathbf{x},D)\phi = f \quad \text{in } \Omega \quad \text{and} \quad \mathcal{R}(\mathbf{x},D)\phi = g \quad \text{on } \Gamma$$

Donor DG formulation

$$B_{DG}(\phi_h,\psi_h) = F_{DG}(\psi_h) \quad \forall \psi_h \in \Phi_h(\Omega)$$

$\Phi_h(\Omega)$ DG space

$$T : \overline{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega)$$

Interscale operator

$$B_{DG}(T\overline{\phi}_h, T\overline{\psi}_h) = F_{DG}(T\overline{\psi}_h) \quad \forall \overline{\psi}_h \in \overline{\Phi}_h(\Omega)$$

$\overline{\Phi}_h(\Omega)$ C^0 space

VMS-DG formulation

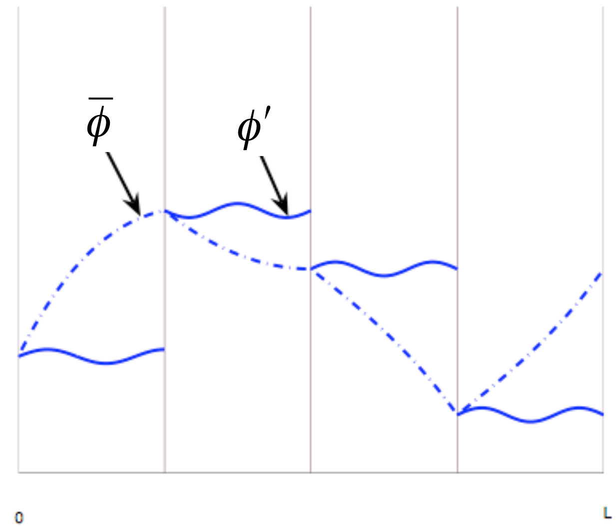
Interscale operator

The key to VMS-DG is the definition of T

– T must **reflect** PDE's structure

– T must be **locally** defined (for efficiency)

We use VMS to derive the local problems



Additive decomposition

$$\forall \phi \in \Phi_h(\Omega) \mapsto \phi = \bar{\phi} + \phi'$$

$\bar{\phi} \in \bar{\Phi}_h(\Omega)$ **Coarse scale**

$\phi' \in \Phi_h(\Omega)$ **Fine scale**

VMS equations

$$B_{DG}(\bar{\phi}_h, \bar{\psi}_h) + B_{DG}(\phi'_h, \bar{\psi}_h) = F_{DG}(\bar{\psi}_h) \quad \forall \bar{\psi}_h \in \bar{\Phi}_h(\Omega)$$ ← **Coarse** scale equation

$$B_{DG}(\phi'_h, \psi'_h) + B_{DG}(\bar{\phi}_h, \psi'_h) = F_{DG}(\psi'_h) \quad \forall \psi'_h \in \Phi_h(\Omega)$$ ← **Fine** scale equation

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The local problem

Assume:

$$B_{DG}(\phi_h,\psi_h) = \sum_{K} B_K(\phi_h,\psi_h) \qquad \leftarrow \text{ element form}$$

$$+ \sum_{\mathbf{e}\in\Gamma_h} B_\Gamma(\phi_h,\psi_h) \qquad \leftarrow \text{ boundary form}$$

$$+ \sum_{\mathbf{e}\in\Gamma_h^0} B_{\mathbf{e}}\left(\left\{\phi_h^-,\phi_h^+\right\},\left\{\psi_h^-,\psi_h^+\right\}\right) \qquad \leftarrow \text{ edge form}$$

Element equations coupled through the edge form

To derive the local problem

1. Treat coarse scale function as data for the fine scale equation
2. Restrict fine scale equation to an element K

$$B_{DG}(\phi_h',\psi_h') = F_{DG}(\psi_h') - B_{DG}(\bar{\phi}_h,\psi_h') \equiv R_K(\bar{\phi}_h) \quad \forall \psi_h' \in S^{p(K)}(K)$$

3. Uncouple fine scale equation from adjacent elements

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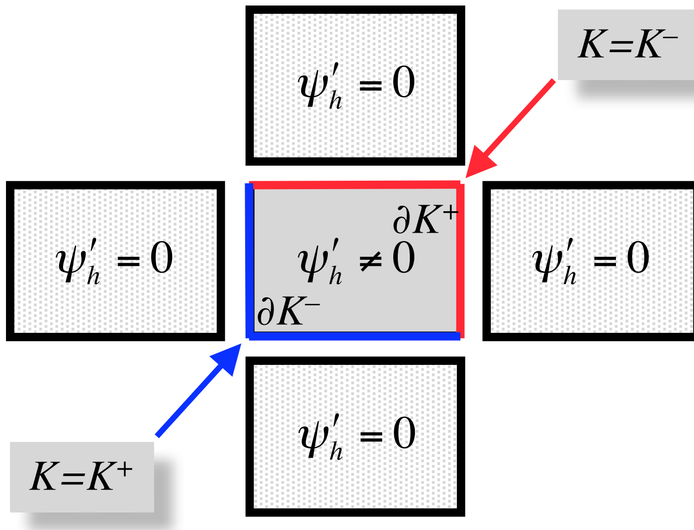
Restriction to an element

States of the restricted weight function

$$(\psi'_h)^+ = \begin{cases} \psi'_h & \text{for } \mathbf{e} \in \partial K^- \\ 0 & \text{for } \mathbf{e} \in \partial K^+ \end{cases} \qquad (\psi'_h)^- = \begin{cases} 0 & \text{for } \mathbf{e} \in \partial K^- \\ \psi'_h & \text{for } \mathbf{e} \in \partial K^+ \end{cases}$$



$$(\psi'_h)^+ = \chi(\partial K^-)\psi'_h \qquad (\psi'_h)^- = \chi(\partial K^+)\psi'_h$$

States of the coarse scale solution

$$\bar{\phi}_h \in C^0 \quad \Rightarrow \quad (\bar{\phi}_h)^+ = (\bar{\phi}_h)^- = \bar{\phi}_h$$

Coarse scale residual

$$R_K(\bar{\phi}_h) = F_{DG}(\psi'_h) - B_K(\bar{\phi}_h, \psi'_h) - B_\Gamma(\bar{\phi}_h, \psi'_h) - \sum_{\mathbf{e} \in E(K)} B_{\mathbf{e}}\left(\{\bar{\phi}_h, \bar{\phi}_h\}, \{\chi(\partial K^+)\psi'_h, \chi(\partial K^-)\psi'_h\}\right)$$

Uncoupling from adjacent elements

Coupled fine scale equation on *K*

$$B_K(\phi_h',\psi_h') + B_\Gamma(\phi_h',\psi_h') + \sum_{\mathbf{e}\in E(K)} B_{\mathbf{e}}\left(\left\{(\phi_h')^+,(\phi_h')^-\right\},\left\{\chi(\partial K^+)\psi_h',\chi(\partial K^-)\psi_h'\right\}\right) = R_K(\overline{\phi}_h)$$

Redefine edge form in terms of the element fine scale function only

$$B_{\mathbf{e}}\left(\left\{(\phi_h')^+,(\phi_h')^-\right\},\left\{\chi(\partial K^+)\psi_h',\chi(\partial K^-)\psi_h'\right\}\right) \rightarrow B_{\mathbf{e}}'\left(\left\{\phi',\phi'\right\},\left\{\chi(\partial K^+)\psi_h',\chi(\partial K^-)\psi_h'\right\}\right)$$

Uncoupled fine scale equation on *K*

$$B_K'(\phi_h',\psi_h') \equiv B_K(\phi_h',\psi_h') + B_\Gamma(\phi_h',\psi_h') + \sum_{\mathbf{e}\in E(K)} B_{\mathbf{e}}'\left(\left\{\phi_h'\right\},\left\{\psi_h'\right\}\right) = R_K(\overline{\phi}_h)$$

Interscale operator

$$B_K'(\phi_h',\psi_h') = R_K(\overline{\phi}_h) \quad \Rightarrow \quad T_K : \overline{\Phi}_h(\Omega) \mapsto S^{p(K)}(K) \quad \Rightarrow \quad \boxed{\begin{array}{c} T : \overline{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega) \\ T_{|K} = T_K \end{array}}$$

Application to scalar advection-diffusion

Boundary value problem

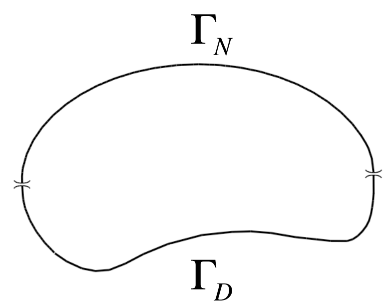
$$\nabla \cdot \left(F_a + F_d\right) = f \quad \text{in } \Omega$$

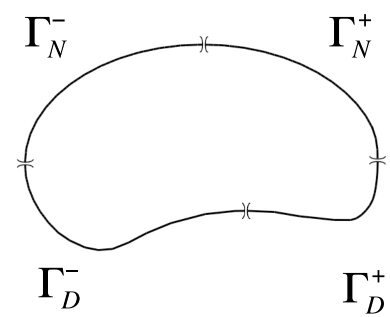
$$\phi = g \quad \text{on } \Gamma_D$$

$$\left(F_a + F_d\right) \cdot \mathbf{n} = h^- \quad \text{on } \Gamma_N^-$$

$$\left(F_d\right) \cdot \mathbf{n} = h^+ \quad \text{on } \Gamma_N^+$$

$$\left(\chi(\Gamma_N^-)F_a + F_d\right) \cdot \mathbf{n} = h \quad \text{on } \Gamma_N$$





Flux

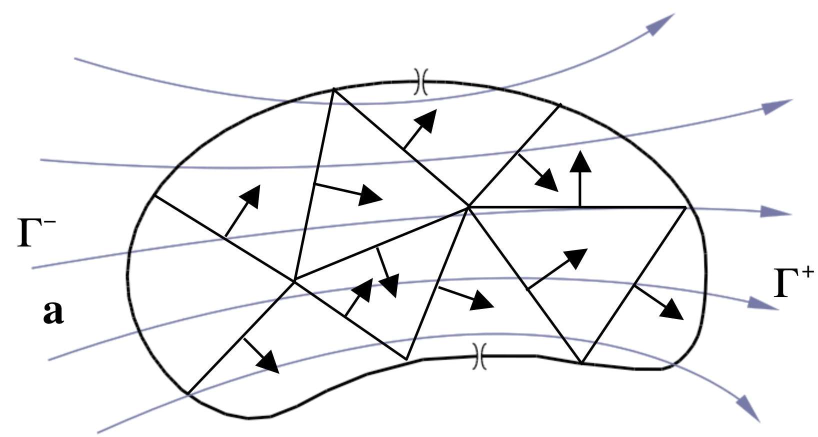
$F_a = \mathbf{a}\phi$ advective

$F_d = -\kappa \nabla \phi$ diffusive

Edge orientation

By advective direction

$\Rightarrow$ Upwind & downwind elements



A donor DG formulation

Residual formulation

$$0=\sum_K -\int_K \left(F^K\cdot\nabla\psi+f\psi_K\right)dx+\int_{\Gamma_N}\left(\chi(\Gamma_N^+)F_a\cdot\mathbf{n}-h\right)\psi d\Gamma+\int_{\Gamma_D}(F\cdot\mathbf{n})\psi_K d\Gamma$$

$$+\,\varepsilon\int_{\Gamma_D}(\phi-g)W(\psi)d\Gamma$$ ← Weak Dirichlet condition

$$+\sum_{\mathbf{e}\in\Gamma_h^0}\int_{\mathbf{e}}\mathcal{F}_b\left(\phi^+;\phi^-\right)\left(\psi^+-\psi^-\right)d\Gamma$$ ← Flux balance

$$+\sum_{\mathbf{e}\in\Gamma_h^0}\int_{\mathbf{e}}\mathcal{F}_c\left(\psi^+;\psi^-\right)\left(\phi^+-\phi^-\right)d\Gamma$$ ← Weak continuity

$$+\sum_{\mathbf{e}\in\Gamma_h^0}\int_{\mathbf{e}}\alpha[\phi][\psi]d\Gamma$$ ← Least-squares stabilization

coupling terms

Numerical fluxes

$$\mathcal{F}_b\left(\phi^+;\phi^-\right)=F_b^h\left(\phi^+;\phi^-\right)\cdot\mathbf{n}=\left(s_{11}F_a^h+s_{12}F_d^h\right)\cdot\mathbf{n}$$

$$\mathcal{F}_c\left(\phi^+;\phi^-\right)=F_c^h\left(\phi^+;\phi^-\right)\cdot\mathbf{n}=\left(s_{21}F_a^h+s_{22}F_d^h\right)\cdot\mathbf{n}$$

← Approximate true flux

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Weak form components

Equivalent form of flux terms

$$\int_{\mathbf{e}} \mathcal{F}_b(\phi^+;\phi^-)(\psi^+ - \psi^-)\,d\Gamma = \int_{\mathbf{e}} F_b^h(\phi^+;\phi^-)[\psi]\,d\Gamma$$

$$\int_{\mathbf{e}} \mathcal{F}_c(\psi^+;\psi^-)(\phi^+ - \phi^-)\,d\Gamma = \int_{\mathbf{e}} F_c(\psi^+;\psi^-)[\phi]\,d\Gamma$$

DG bilinear form components

$$B_K(\phi,\psi) = -\int_K F^K \cdot \nabla\psi\,dx$$ ← volume

$$B_\Gamma(\phi,\psi) = \int_{\Gamma_N} \left(\chi(\Gamma_N^+)F_a \cdot \mathbf{n}\right)\psi\,d\Gamma + \int_{\Gamma_D} (F\cdot\mathbf{n})\psi_K\,d\Gamma + \varepsilon\int_{\Gamma_D} \phi W(\psi)\,d\Gamma$$ ← boundary

$$B_{\mathbf{e}}\left(\{\phi^+,\phi^-\},\{\psi^+,\psi^-\}\right) = \sum_{\mathbf{e}\in E(K)} \int_{\mathbf{e}} \boxed{F_b(\phi^+;\phi^-)[\psi]} + \boxed{F_c(\psi^+;\psi^-)[\phi]} + \boxed{\alpha[\phi][\psi]}\,d\Gamma$$ ← edge

Coupling terms

Numerical flux & weight functions

Flux	DG-A	DG-B
$F_b^h\left(\phi^+;\phi^-\right)$	$F_a\left(\phi^-\right)+\left\langle F_d\left(\phi\right)\right\rangle$	$F_a\left(\phi^-\right)+F_d\left(\phi^-\right)$
$F_c\left(\psi^+;\psi^-\right)$	$s\left\langle F_d\left(\psi\right)\right\rangle$	$sF_d\left(\psi^-\right)$
$W\left(\psi\right)$	$\psi+sF_d\left(\psi^-\right)\cdot\mathbf{n}$	

DG-A $F_a\left(\phi^-\right)+\left\langle F_d\left(\phi\right)\right\rangle$ ← Advective flux is **upwinded**, diffusive is **averaged**

DG-B $F_a\left(\phi^-\right)+F_d\left(\phi^-\right)$ ← Total flux is **upwinded**

DG formulation (see *D. Arnold at al, SINUM 2000)*

s=1	skew-symmetric	excellent stability properties
s=0	neutral	requires LS stabilization
s=-1	symmetric	adjoint consistent, optimal L^2 rates

Hughes, Scovazzi, Bochev, Buffa, submitted to CMAME (2005)

Specialization of the edge form

DG-A

$$B_{\mathbf{e}}\left(\{\phi^+,\phi^-\},\{\psi^+,\psi^-\}\right)=\sum_{\mathbf{e}\in E(K)}\int_{\mathbf{e}}\alpha[\phi][\psi]d\Gamma$$

$$+\int_{\mathbf{e}}\underbrace{\left(\mathbf{a}\phi^- - \kappa\nabla\left(\phi^+ + \phi^-\right)/2\right)}_{F_b^h}\cdot[\psi] - \underbrace{s\left(\kappa\nabla\left(\psi^+ + \psi^-\right)/2\right)}_{F_c^h}\cdot[\phi]d\Gamma$$

DG-B

$$B_{\mathbf{e}}\left(\{\phi^+,\phi^-\},\{\psi^+,\psi^-\}\right)=\sum_{\mathbf{e}\in E(K)}\int_{\mathbf{e}}\alpha[\phi][\psi]d\Gamma$$

$$+\int_{\mathbf{e}}\underbrace{\left(\mathbf{a}\phi^- - \kappa\nabla\phi^-\right)}_{F_b^h}\cdot[\psi] - \underbrace{s\left(\kappa\nabla\psi^-\right)}_{F_c^h}\cdot[\phi]d\Gamma$$

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Local conservation properties of DG-B

Consider an internal element *T* and

$$\psi_h = \begin{cases} 1 \text{ on } K = T \\ 0 \text{ on } K \neq T \end{cases}$$

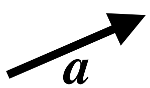
Approximately conservative with LS

$$\int_\Omega f + \sum_{\mathbf{e}\in\partial T} \int_{\mathbf{e}} \left(\mathbf{a}\phi_h^- - \kappa\nabla\phi_h^-\right)\cdot\mathbf{n} = -\int_{\mathbf{e}} \alpha\left[\phi_h^-\right]\cdot\mathbf{n}$$

$$\int_\Omega f + \sum_{\mathbf{e}\in\partial T} \int_{\mathbf{e}} \left(\mathbf{a}\phi_h^- - \kappa\nabla\phi_h^-\right)\cdot\mathbf{n} = 0$$

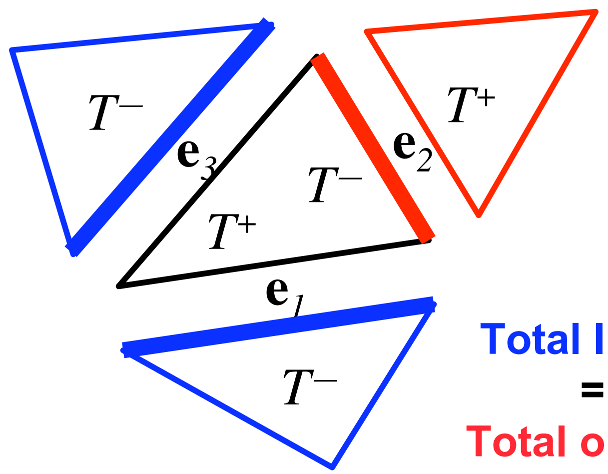
Upwinding of total flux in DG-B

Locally conservative without LS



$$\mathbf{a}\phi_h^- - \kappa\nabla\phi_h^- = \mathbf{a}\phi_h^C - \kappa\nabla\phi_h^C$$
on $\mathbf{e}_1 \cup \mathbf{e}_3$

Inflow flux
on contiguous elements



$$\mathbf{a}\phi_h^- - \kappa\nabla\phi_h^- = \mathbf{a}\phi_h^T - \kappa\nabla\phi_h^T$$
on $\mathbf{e}_2$

Outflow flux
on the given element

Total Inflow
=
Total outflow

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The interscale operator

To obtain *T* we start with the coupled element equation

$$B_K(\phi'_h,\psi'_h)+B_\Gamma(\phi'_h,\psi'_h)+$$
$$\sum_{\mathbf{e}\in E(K)}\int_{\mathbf{e}} F_b\left((\phi'_h)^+;(\phi'_h)^-\right)\cdot\underbrace{\mathbf{n}_K\psi'_h}_{[\psi'_h]}+F_c\big(\underbrace{\chi(\partial K^-)\psi'_h}_{(\psi'_h)^+};\underbrace{\chi(\partial K^+)\psi'_h}_{(\psi'_h)^-}\big)[\phi'_h]+\alpha[\phi'_h]\cdot\underbrace{\mathbf{n}_K\psi'_h}_{[\psi'_h]}\,d\Gamma=R_K(\bar{\phi}_h)$$

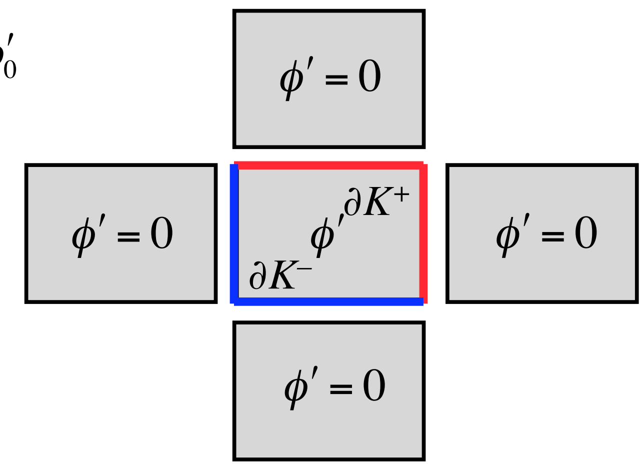
and redefine the states, jump and average of the fine scale $\phi'\in S^{p(K)}(K)$ as follows:

To compute states and jump extend by zero: $\phi'\rightarrow\phi'_0$

$$(\phi'_0)^+=\chi(\partial K^-)\phi'_0 \quad\Rightarrow\quad (\phi')^+=\chi(\partial K^-)\phi'$$

$$(\phi'_0)^-=\chi(\partial K^+)\phi'_0 \quad\Rightarrow\quad (\phi')^-=\chi(\partial K^+)\phi'$$

$$[\phi'_0]=\mathbf{n}^+\chi(\partial K^-)\phi'_0+\mathbf{n}^-\chi(\partial K^+)\phi'_0 \quad\Rightarrow\quad [\phi']=\mathbf{n}_K\phi'$$



To compute edge averages extend by continuity: $\phi'\rightarrow\phi'_\infty$

$$\langle\phi'_\infty\rangle=\frac{1}{2}(\phi'_\infty+\phi'_\infty)=\phi'_\infty \quad\Rightarrow\quad \langle\phi'\rangle=\phi'$$

Redefined edge form

Flux	DG-A	DG-B
$F_b^h(\phi)$	$F_a\left(\chi(\partial K^+)\phi\right) + F_d(\phi)$	$F_a\left(\chi(\partial K^+)\phi\right) + F_d\left(\chi(\partial K^+)\phi\right)$
$F_c(\psi)$	$sF_d(\psi)$	$sF_d\left(\chi(\partial K^+)\psi\right)$

DG-A

$$B'_{\mathbf{e}}\left(\{\phi'\},\{\psi'\}\right) = \sum_{\mathbf{e}\in E(K)} \int_{\mathbf{e}} \underbrace{\left(\mathbf{a}\chi(\partial K^+)\phi' - \kappa\nabla\phi'\right)}_{F_b^h} \cdot \mathbf{n}_K\psi' - \underbrace{s\kappa\nabla\psi'}_{F_c^h} \cdot \mathbf{n}_K + \underbrace{\alpha\phi'\psi'}_{LS}\, d\Gamma$$

DG-B

$$B'_{\mathbf{e}}\left(\{\phi'\},\{\psi'\}\right) = \sum_{\mathbf{e}\in E(K)} \int_{\mathbf{e}} \underbrace{\chi(\partial K^+)\left(\mathbf{a}\phi' - \kappa\nabla\phi'\right)}_{F_b^h} \cdot \mathbf{n}_K\psi' - \underbrace{s\chi(\partial K^+)\kappa\nabla\psi'}_{F_c^h} \cdot \mathbf{n}_K + \underbrace{\alpha\phi'\psi'}_{LS}\, d\Gamma$$

The new edge forms involve values of the fine scale ϕ' from one element only

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The local problems

DG-A

$$B_K(\phi',\psi') + B_\Gamma(\phi',\psi') + \sum_{\mathbf{e}\in E(K)} \int_{\mathbf{e}} \left(\mathbf{a}\chi(\partial K^+)\phi' - \kappa\nabla\phi'\right)\cdot\mathbf{n}_K\psi' - s\kappa\nabla\psi'\cdot\mathbf{n}_K + \alpha\phi'\psi'\, d\Gamma = R_K(\bar{\phi})$$

DG-B

$$B_K(\phi',\psi') + B_\Gamma(\phi',\psi') + \sum_{\mathbf{e}\in E(K)} \int_{\mathbf{e}} \chi(\partial K^+)\left(\mathbf{a}\phi' - \kappa\nabla\phi'\right)\cdot\mathbf{n}_K\psi' - s\chi(\partial K^+)\kappa\nabla\psi'\cdot\mathbf{n}_K + \alpha\phi'\psi'\, d\Gamma = R_K(\bar{\phi})$$

PDE form of the local problem

$$\nabla\cdot F(\phi') = f - \nabla\cdot F(\bar{\phi}) \quad \text{in } K$$

Residual driven!

$\phi' = 0$ on ∂K^- | $\phi' = 0$ on ∂K

advective ← limit → diffusive

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VMS-DG Revisited

Interscale operator

$$B'_K(\phi'_h,\psi'_h) = R_K(\bar{\phi}_h) \qquad T_K : \overline{\Phi}_h(\Omega) \mapsto S^{p(K)}(K) \quad \rightarrow$$

$$T : \overline{\Phi}_h(\Omega) \mapsto \Phi_h(\Omega)$$

$$T_{|K} = T_K$$

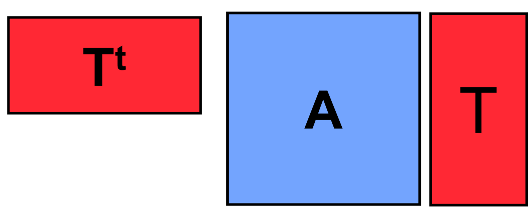
VMS-DG formulation

$$B_{DG}(T\bar{\phi}_h, T\bar{\psi}_h) = F_{DG}(T\bar{\psi}_h) \quad \forall \bar{\psi}_h \in \overline{\Phi}_h(\Omega)$$

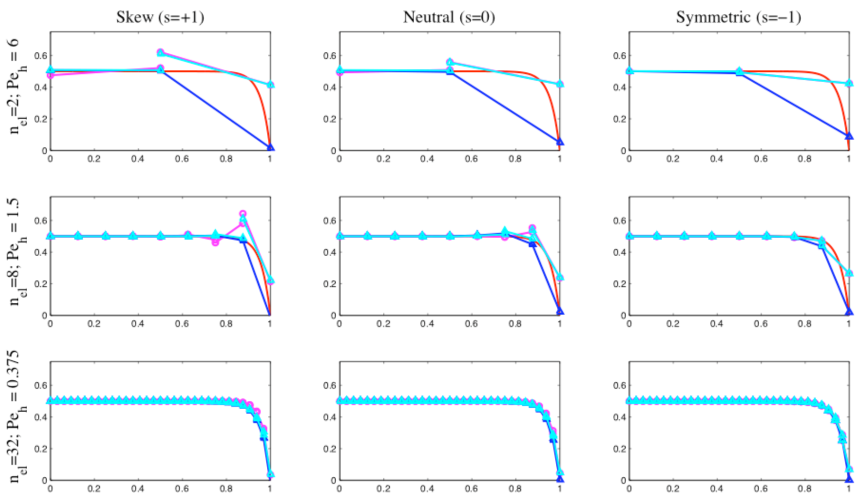
Algebraic form

$$B_{DG}(\phi_h,\psi_h) = F_{DG}(\psi_h) \quad \rightarrow \quad \mathbf{Ax} = \mathbf{f} \qquad \mathbf{A} \in \mathbf{R}^{n\times n}; \quad n = \dim(\Phi_h)$$

$$B_{DG}(T\bar{\phi}_h, T\bar{\psi}_h) = F_{DG}(T\bar{\psi}_h) \quad \rightarrow \quad \mathbf{T}^t\mathbf{ATx} = \mathbf{T}^t\mathbf{f} \qquad \mathbf{T} \in \mathbf{R}^{n\times m}; \quad m = \dim(\overline{\Phi}_h)$$



Numerical examples 1D

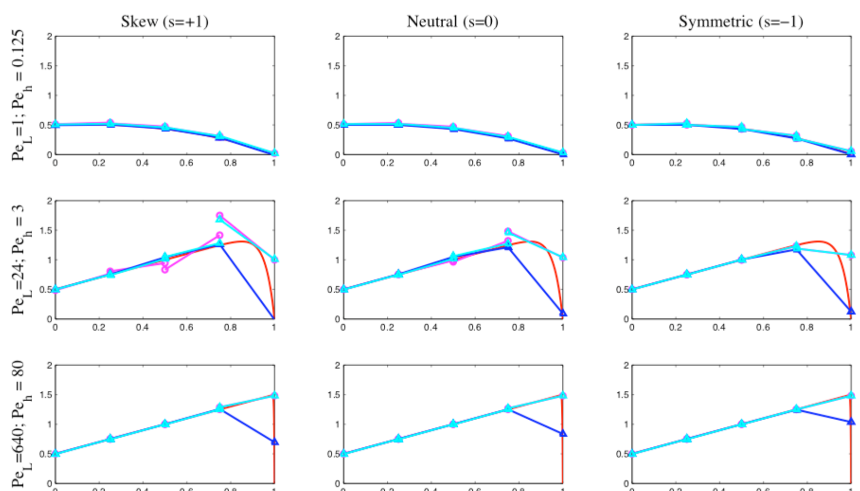


f=0

Solution plots 2, 8 and 32 elements

Red = exact
Blue = MDG (continuous)
L-blue = MDG (discontinuous)
Magenta = Donor DG

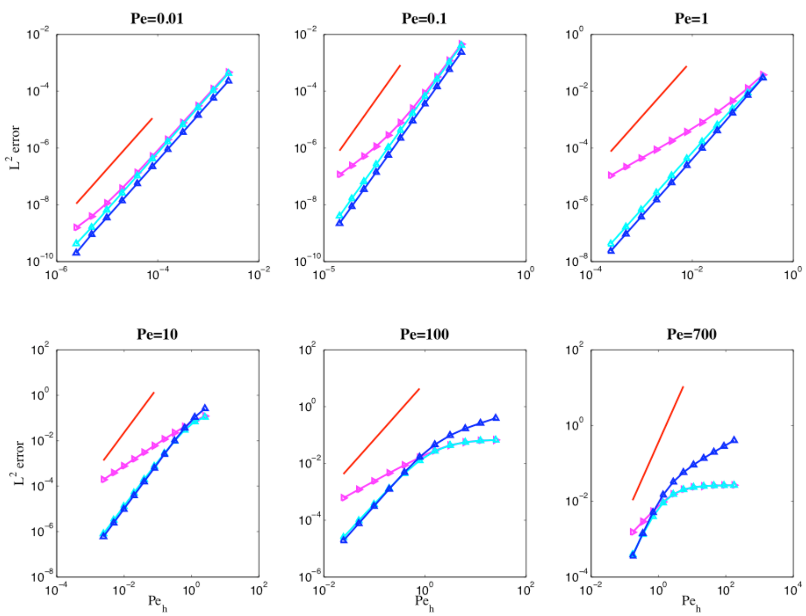
f=1



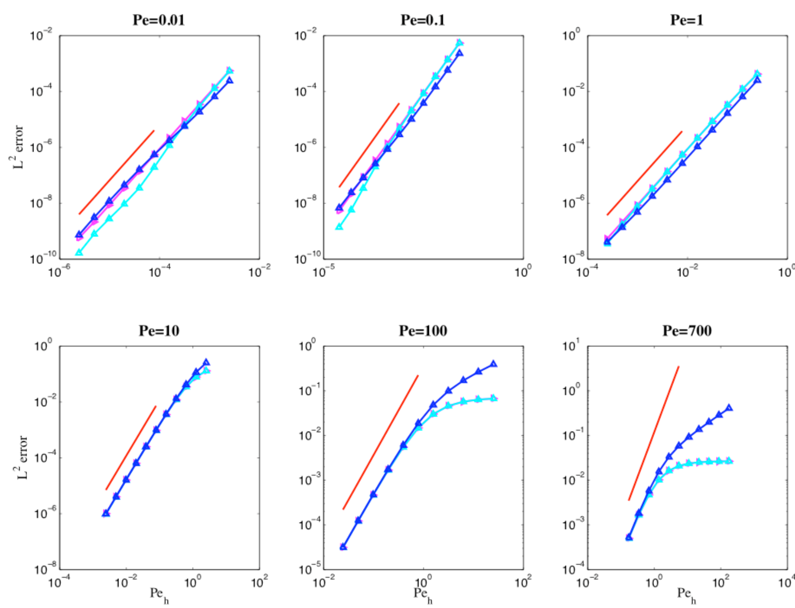
VMS-MDG is very close to the DG solution, while costs much less to compute.

Convergence rates

neutral s=0



symmetric s=-1

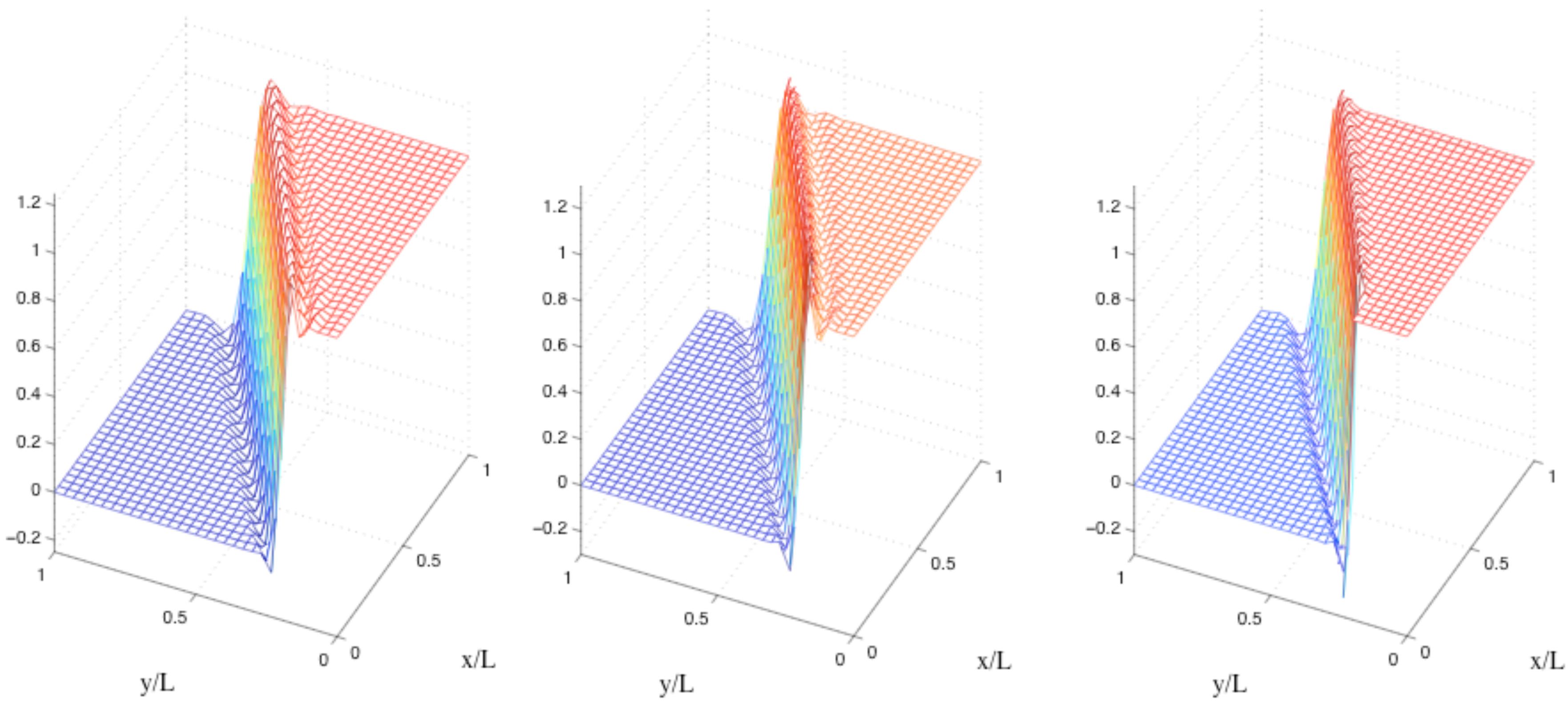


Skew similar to neutral

For some donor DG formulations VMS-MDG converges better than the donor DG itself.

Numerical examples in 2D

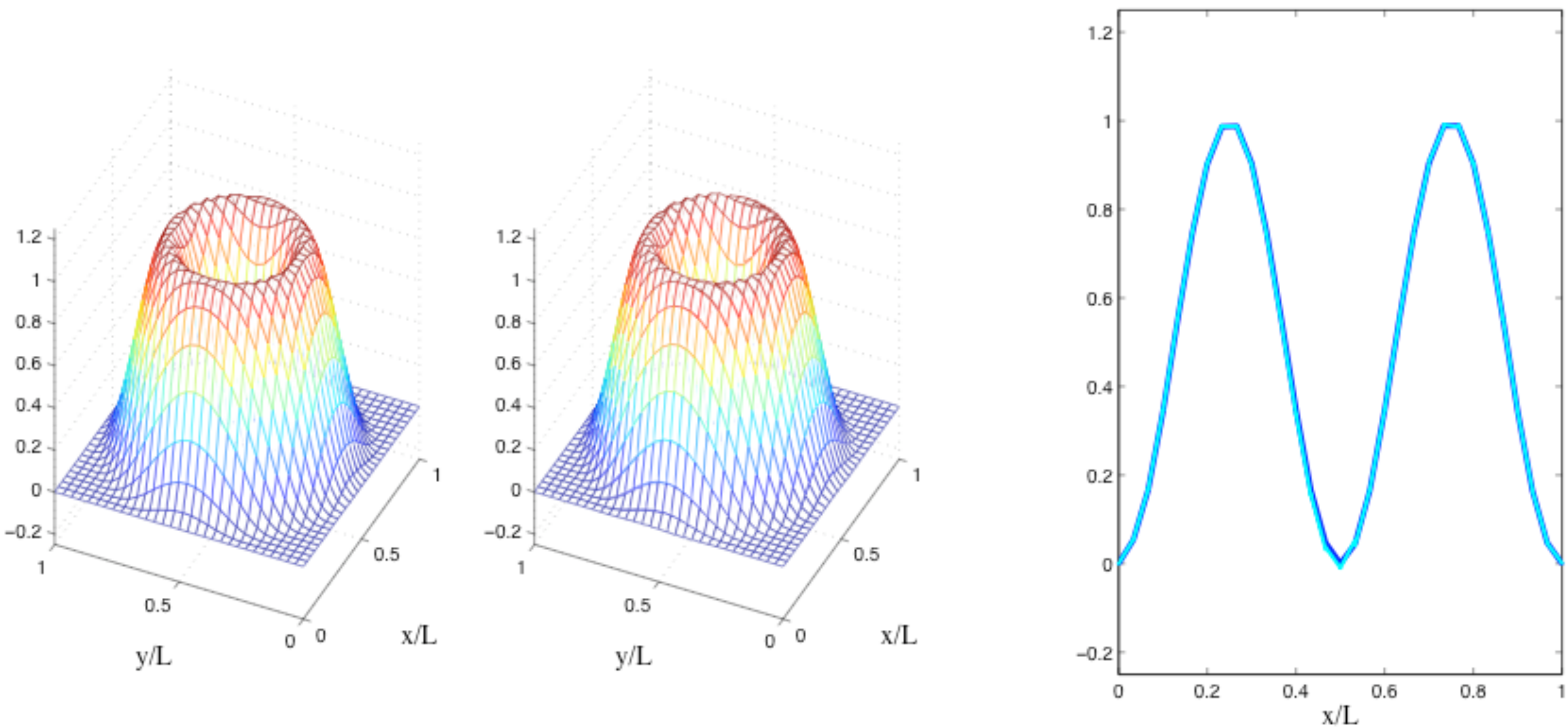
Advection skew to the mesh



$\overline{\phi}_h$ $\qquad$ $\phi_h = \overline{\phi}_h + \phi_h'$ $\qquad$ ϕ_h^{DG}

Numerical examples 2D

Rotating cone



Conclusions

We proposed a new VMS-DG method that combines advantages of DG with the more efficient computational structure of the continuous Galerkin method.

VMS-DG

- ❑ Solves a long-standing problem with proliferation of DoFs in DG methods
- ❑ Provides a template for “clever”, problem-dependent local condensation
- ❑ Local problems are residual-driven
- ❑ Attains and even somewhat improves upon both DG and continuous Galerkin

Future research directions

- ❑ The local problem
  - – Adding discontinuity capturing operators
  - – Connection of with Riemann solvers
- ❑ The donor DG method