Mimetic Least Squares Principles for Electromagnetics

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Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy’s National Nuclear Security Administration under contract DE-AC04-94AL85000.
1. What is a mimetic discretization
2. An algebraic topology framework
3. Direct and conforming discretizations

Mimetic Least-Squares Principles for Electromagnetics

What are LSP and the reasons to use them.

Mimetic LSP, their conservative properties and interpretation as realizations of weak material laws

A new interpretation of LSP: Realizations of a discrete Hodge * operator

Mixed, Galerkin and Least-Squares methods for 2nd order problems share a common ancestor: the 4-field principle

Mac Hyman, Misha Shashkov
Mathematical Modeling and Analysis T-7
Los Alamos National Laboratory

Max Gunzburger
CSIT Florida State University
Least-Squares 101

\[ Lu = f \quad \text{in} \quad \Omega \]
\[ Ru = h \quad \text{on} \quad \Gamma \]

\[
\min_{u \in X} J(u; f, h) = \frac{1}{2} \left( \| Lu - f \|_{X, \Omega}^2 + \| Ru - h \|_{Y, \Gamma}^2 \right)
\]

\[
(Lu, Lv)_\Omega + (Ru, Rv)_\Gamma = (f, Lu)_\Omega + (h, Rv)_\Gamma
\]

**Top 3 reasons people want to do least squares:**

😊 Using \( C^0 \) nodal elements

😊 Avoiding inf-sup conditions

😊 Solving SPD systems

**don’t want to do least squares:**

😊 Conservation

😊 Conservation

😊 Conservation

**We will show that**

➢ Using **nodal elements** is not necessarily the best choice in LSFEM, and so it is arguably the **least-important advantage** attributed to least-squares methods

➢ By using other elements least-squares acquire **additional** conservation properties

➢ Surprisingly, this kind of least-squares turns out to be **related** to mixed methods
Compatibility matters even for least-squares: magnetic diffusion

\[ \text{Ker}(\text{curl}) = \{0\} \]

\[ \nabla \times \frac{1}{\mu} B = \sigma E \quad \text{Ampere} \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \quad \text{Faraday} \]
Further evidence that compatibility matters

Lax-Wendroff theorem:

If approximation computed by a conservative and consistent method converges, then the limit is a weak solution of the conservation law.

Grid decomposition property (Fix, Gunzburger, Nicolaides, 1978)

A discrete Hodge decomposition property is necessary and sufficient for stable and accurate LSFEM discretization of the Kelvin principle.

Staggered FD and FV (MAC, Yee’s FDTD, Box integration)

Conservation requires placing different variables at different grid locations so as to achieve a discrete Stokes theorem.

Discrete models must reflect mathematical structure of continuum models
How to achieve compatibility?

Compatible discretization requires:

- Mathematical tools to **discover** and **encode** structure of PDEs
- A discrete framework that **mimics** that structure: mutually consistent notions of
  - Discrete vector calculus, Hodge theory, entropy condition, conservation...

In most physical models

- **Fields** are observed **indirectly** by measuring **global** quantities (flux, circulation, etc)
- **Physical laws** are **relationships** between **global** quantities (conservation, equilibrium)

Differential forms provide the tools to **encode** such relationships

- **Integration:** an abstraction of the **measurement** process
- **Differentiation:** gives rise to **local invariants**
- **Poincare Lemma:** expresses **local geometric** relations
- **Stokes Theorem:** gives rise to **global relations**
Algebraic topology approach

Algebraic topology provides the tools to mimic the structure

- **Computational grid** is algebraic topological complex
- **k-forms** are encoded as k-cell quantities (k-cochains)
- **Derivative** is provided by the coboundary
- **Inner product** induces combinatorial Hodge theory
- **Singular cohomology** preserved by the complex

Framework for mimetic discretizations (Bochev, Hyman, IMA Proceedings)

- Translation: Fields $\rightarrow$ forms $\rightarrow$ cochains
- Basic mappings: reduction and reconstruction
  - Combinatorial operations: induced by reduction map
  - Natural operations: induced by reconstruction map
  - Derived operations: induced by natural operations

Differential Forms

**Smooth differential forms** \( \Lambda^k(\Omega) : x \rightarrow \omega(x) \in \Lambda^k(T_x\Omega) \)

**DeRham complex** \( \mathbb{R} \rightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \rightarrow 0 \)

**Metric conjugation** \( \ast : \Lambda^k(T_x\Omega) \rightarrow \Lambda^{n-k}(T_x\Omega) \iff \omega \ast \xi = (\omega, \xi)_x \omega_n \)

**\(L^2\) inner product on \( \Lambda^k(\Omega) \)** \( (\omega, \xi)_\Omega = \int_\Omega (\omega, \xi)_x \omega_n \Rightarrow (\omega, \xi)_\Omega = \int_\Omega \omega \ast \xi \)

**Codifferential** \( d^* : \Lambda^{k+1}(\Omega) \rightarrow \Lambda^k(\Omega) \iff (d\omega, \xi)_\Omega = (\omega, d^*\xi)_\Omega \)

**Hodge Laplacian** \( \Delta : \Lambda^k(\Omega) \rightarrow \Lambda^k(\Omega) \rightarrow \Delta = dd^* + d^*d \)

**Completion of \( \Lambda^k(\Omega) \)** \( \Lambda^k(L^2, \Omega) \)

**Sobolev spaces** \( \Lambda^k(d, \Omega) = \{ \omega \in \Lambda^k(L^2, \Omega) \mid d\omega \in \Lambda^{k+1}(L^2, \Omega) \} \)
Chains and cochains

Computational grid = Chain complex

\[ \partial : C_k \rightarrow C_{k-1} \]
\[ \partial \partial = 0 \]

\[ C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} C_3 \]

Field representation = Cohomology complex

\[ C^k = L(C_k, R) = C_k^* \]
\[ \langle \sigma^i, \sigma_j \rangle = \delta_{ij} \]

\[ \delta : C^k \rightarrow C^{k+1} \]
\[ \langle \omega, \partial \eta \rangle = \langle \delta \omega, \eta \rangle \]

\[ \delta \delta = 0 \]

\[ C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} C^3 \]

\[ 0 = \partial \partial K^3 \]
\[ K^1 \xrightarrow{\delta} \partial K^1 \xrightarrow{\delta} \delta \delta K^1 = 0 \]
Basic mappings

Reduction

\[ \mathcal{R} : \Lambda^k \left( L^2, \Omega \right) \rightarrow C^k \]

Natural choice

\[ \langle \mathcal{R} \omega, \sigma \rangle = \int_{\Omega} \omega \]

\[ \text{DeRham map} \]

\[ \mathcal{R} d = \delta \mathcal{R} \]

Proof

\[ \langle \delta \mathcal{R} \omega, c \rangle = \langle \mathcal{R} \omega, \partial c \rangle = \int_{\Omega} \omega = \int_{\Omega} d \omega = \langle \mathcal{R} d \omega, c \rangle \]

Reconstruction

\[ \mathcal{I} : C^k \rightarrow \Lambda^k \left( L^2, \Omega \right) \]

No natural choice

\[ \mathcal{R} \mathcal{I} = \text{id} \]

\[ \mathcal{I} \mathcal{R} = \text{id} + O(h^s) \]

\[ \ker \mathcal{I} = 0 \]

Conforming

\[ \mathcal{I} : C^k \rightarrow \Lambda^k \left( d, \Omega \right) \]

\[ \mathcal{I} d = \delta \mathcal{I} \]
Combinatorial operations

Discrete derivative

**Forms** are dual to **manifolds** \( \langle d\omega, \Omega \rangle = \langle \omega, \partial \Omega \rangle \)

**Cochains** are dual to **chains** \( \langle \delta a, \sigma \rangle = \langle a, \partial \sigma \rangle \)

δ approximates d on cochains

Discrete integral

\[ \int_{\sigma} a = \langle a, \sigma \rangle \]

Stokes theorem

\[ \langle \delta a, \sigma \rangle = \langle a, \partial \sigma \rangle \]

\[ \delta \delta = 0 \]

\[ Rd = \delta R \]
# Natural and derived operations

<table>
<thead>
<tr>
<th>Natural</th>
<th>Inner product</th>
<th>((a,b)_x = (Ia, Ib)_x)</th>
<th>((a,b)_\Omega = \int (a,b)<em>x \omega_n = (Ia, Ib)</em>\Omega)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Wedge product</td>
<td>(\wedge : C^k \times C^l \mapsto C^{k+l})</td>
<td>(a \wedge b = \mathcal{R}(Ia \wedge Ib))</td>
</tr>
<tr>
<td>Derived</td>
<td>Adjoint derivative</td>
<td>(\delta^* : C^{k+1} \mapsto C^k)</td>
<td>((\delta^* a,b)<em>\Omega = (a,\delta b)</em>\Omega)</td>
</tr>
</tbody>
</table>

Provides a second set of **grad**, **div** and **curl** operators. Scalars encoded as 0 or 3-forms, vectors as 1 or 2-forms, derivative choice depends on encoding.

**Discrete Laplacian**

\[
D : C^k \mapsto C^k \quad D = \delta^* \delta + \delta \delta^*
\]

Derived operations are necessary to avoid **internal inconsistencies** between the discrete operations: \(I\) is only **approximate inverse** of \(\mathcal{R}\) and natural operations will clash.

**Example**

**Natural adjoint**

\[
d^* = (-1)^k \ast d^* \quad \Rightarrow \delta^* = (-1)^k \mathcal{R} \ast d \ast I
\]

\(I\) must be regular and \((\delta^* a,b)_\Omega = (a,\delta b)_\Omega + O(h^s) \Rightarrow \delta^* \) not true adjoint
Mimetic properties (I)

**Discrete Poincare lemma** (existence of potentials in contractible domains)

\[
\begin{align*}
  d\omega_k &= 0 & \Rightarrow & & \omega_k &= d\omega_{k+1} \\
  \delta c^k &= 0 & \Rightarrow & & c^k &= \delta c^{k+1}
\end{align*}
\]

**Discrete Stokes Theorem**

\[
\begin{align*}
  \langle d\omega_{k-1}, c_k \rangle &= \langle \omega_{k-1}, \partial c_k \rangle \\
  \langle \delta c^{k-1}, c_k \rangle &= \langle c^{k-1}, \partial c_k \rangle
\end{align*}
\]

**Discrete “Vector Calculus”**

\[
\begin{align*}
  dd &= 0 & \delta\delta &= \delta*\delta* = 0 \\
  \omega \wedge \eta &= (-1)^k \eta \wedge \omega & a \wedge b &= (-1)^k b \wedge a \\
  d(\omega \wedge \eta) &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta & \delta(a \wedge b) &= \delta a \wedge b + (-1)^k a \wedge \delta b & \text{(Regular I)}
\end{align*}
\]

Any feature of the continuum system that is implied by differential forms calculus is inherited by the discrete model

Called **mimetic** property by Hyman and Scovel (1988)
Mimetic properties (II)

Inner product induces combinatorial Hodge theory on cochains

Co-cycles of \((\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3)\) \(\xrightarrow{R}\) co-cycles of \((C^0, C^1, C^2, C^3)\)

\[ d\omega = 0 \implies \delta R\omega = 0 \]

Discrete Harmonic forms

\[ H^k(\Omega) = \{\eta \in \Lambda^k(\Omega) \mid d\eta = d^*\eta = 0\} \quad H^k(K) = \{c^k \in C^k \mid \delta c^k = \delta^* c^k = 0\} \]

Discrete Hodge decomposition

\[ \omega = d\rho + \eta + d^*\sigma \quad a = \delta b + h + \delta^* c \]

Theorem

\[ \dim \ker(\Delta) = \dim \ker(D) \]

Remarkable property of the mimetic \(D\) - kernel size is a topological invariant!
Discrete $\ast$ operation

Natural definition

$$\ast_N : C^k \to C^{n-k} \quad \ast_N = \mathcal{R} \ast I$$

Derived definition

$$\ast_D : C^k \to C^{n-k} \quad \int_\Omega a \wedge \ast_D b = (a,b)_\Omega \quad \text{mimics} \quad (\omega, \xi)_\Omega = \int_\Omega \omega \wedge \ast \xi$$

Theorem

$$\ast_N \mathcal{R}^h = \mathcal{R} \ast \omega^h \quad \forall \omega^h \in \text{Range}(\mathcal{I}\mathcal{R}) \quad \text{CDP on the range}$$

$$\int_\Omega \mathcal{I}\mathcal{R}(\mathcal{I}a \wedge I \ast_D b) = \int_\Omega (\mathcal{I}a \wedge I b) \quad \text{Weak CDP}$$

$$\int_\Omega b \wedge \ast_N b = (a,b)_\Omega + O(h^s) \quad \ast_N = \ast_D + O(h^s)$$
Problems with the discrete $*$

Action of $*$ must be coordinated with the other discrete operations

<table>
<thead>
<tr>
<th></th>
<th>($\cdot,\cdot$)</th>
<th>$\wedge$</th>
<th>$\delta^*$</th>
<th>$K$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$*_N$</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>✓</td>
<td>—</td>
</tr>
<tr>
<td>$*_D$</td>
<td>✓</td>
<td>✓</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Analytic $*$ is a local, invertible operation $\Rightarrow$ positive diagonal matrix

$\dim C^k \neq \dim C^{n-k}$ $\Rightarrow$ $*_{N}: C^k \mapsto C^{n-k}$ cannot be a square matrix!

Construction of $*$ is nontrivial task unless primal-dual grid is used!
Implications

A consistent discrete framework requires a choice of a primary operation either $\ast$ or $(\cdot,\cdot)$ but not both.

A *discrete* $\ast$ is the primary concept in Hiptmair (2000), Bossavit (1999)

- Inner product derived from discrete $\ast$
- Used in **explicit discretization of material laws**

The **natural inner product** is the primary operation in our approach

- **Sufficient** to give rise to combinatorial Hodge theory on cochains
- **Easier** to define than a discrete $\ast$ operation
- **Incorporate** material laws in the natural inner product, or
- **Enforce** material laws **weakly** (justified by their approximate nature)
## Algebraic equivalents

<table>
<thead>
<tr>
<th>Operation</th>
<th>Matrix form</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$D_k$</td>
<td>${-1,0,1}$</td>
</tr>
<tr>
<td>$(\cdot,\cdot)$</td>
<td>$M_k$</td>
<td>SPD</td>
</tr>
<tr>
<td>$a_1 \wedge b_1$</td>
<td>$\sum W_{11}$</td>
<td>Skew symm.</td>
</tr>
<tr>
<td>$a_1 \wedge b_2$</td>
<td>$\sum W_{12}$</td>
<td>$W_{12}^T=W_{21}$</td>
</tr>
<tr>
<td>$b_2 \wedge a_1$</td>
<td>$\sum W_{21}$</td>
<td></td>
</tr>
<tr>
<td>$\delta^*$</td>
<td>$M_k^{-1} D_k^T M_{k+1}$</td>
<td>rectangular</td>
</tr>
<tr>
<td>$\mathcal{D}$</td>
<td>$M_k^{-1} D_k^T M_{k+1} D_k + D_{k-1} M_k^{-1} D_{k-1}^T M_k$</td>
<td>square</td>
</tr>
<tr>
<td>$*_D$</td>
<td>$W_{12}(*_Da)=M_3a$</td>
<td>pair</td>
</tr>
</tbody>
</table>
Reconstruction and natural inner products

<table>
<thead>
<tr>
<th>Co-volume</th>
<th>Mimetic</th>
<th>Whitney</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Co-volume Diagram" /></td>
<td><img src="image2" alt="Mimetic Diagram" /></td>
<td><img src="image3" alt="Whitney Diagram" /></td>
</tr>
</tbody>
</table>

\[ \mathbf{M} = \begin{pmatrix} h_1 h_1^\perp \\ h_2 h_2^\perp \\ h_3 h_3^\perp \end{pmatrix} \]

\[ \mathbf{M} = \begin{pmatrix} \frac{V_2}{\sin^2 \phi_2} + \frac{V_3}{\sin^2 \phi_3} \\ \frac{V_3 \cos \phi_2}{\sin^2 \phi_3} \\ \frac{V_2 \cos \phi_2}{\sin^2 \phi_2} \end{pmatrix} \begin{pmatrix} \frac{V_3 \cos \phi_3}{\sin^2 \phi_3} \\ \frac{V_1}{\sin^2 \phi_1} + \frac{V_3}{\sin^2 \phi_3} \\ \frac{V_2 \cos \phi_2}{\sin^2 \phi_2} \end{pmatrix} \begin{pmatrix} \frac{V_2 \cos \phi_2}{\sin^2 \phi_2} \\ \frac{V_1}{\sin^2 \phi_1} + \frac{V_2}{\sin^2 \phi_2} \end{pmatrix} \]

\[ \mathbf{\delta}^* = \begin{pmatrix} \ldots & \ldots & \ldots \\ \ldots & (w_{ij}, w_{kl}) & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix} \]

**local** | **non-local** | **non-local**

Hyman, Scovel (1988)

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Mimetic discretization: translation to forms

1st order PDE with material laws
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \mathbf{J} = \sigma \mathbf{E} \]
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad \mathbf{B} = \mu \mathbf{H} \]

1st order PDE with codifferentials
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ \frac{1}{\sigma} \nabla \times \frac{1}{\mu} \mathbf{B} = \mathbf{E} \]

2nd order PDE
\[ \nabla \times \frac{1}{\sigma} \nabla \times \frac{1}{\mu} \mathbf{B} = -\frac{\partial \mathbf{B}}{\partial t} \]
\[ d * \sigma^{-1} d * \mu^{-1} b = -d_t b \]

**NOTE**: we could have eliminated the **primal** pair \((\mathbf{E}, \mathbf{B})\) and obtain the last two equations in terms of the **dual** pair \((\mathbf{H}, \mathbf{J})\).

\[ \text{domain} \quad \text{curl} \quad \text{range} \]

\[ (e, b) \]

\[ (h, j) \]
Direct and conforming mimetic models

\[ \delta \delta^* b^2 = -\delta_i b^2 \]
\[ e^1 \in C^2; \quad b^2 \in C^2 \]
\[ \delta e^1 = -\delta_i b^2 \]
\[ e^1 = \delta^* b^2 \]

Direct

\[ d \ast_{\sigma^{-1}} d \ast_{\mu^{-1}} b = -d_i b \]

Conforming

\[ de = -d_i b \]
\[ e = \ast_{\sigma^{-1}} d \ast_{\mu^{-1}} b \]

\[ \left( db_h^2, \hat{db}_h^2 \right)_\Omega = \left( -d_i b_h^2, \hat{b}_h^2 \right)_\Omega \]
\[ e_h^1 \in \Lambda^1(d,K); \quad b_h^2 \in \Lambda^2(d,K) \]
\[ de_h^1 = -d_i b_h^2 \]
\[ \left( e_h^1, \hat{e}_h^1 \right)_\Omega = \left( b_h^2, \hat{e}_h^1 \right)_\Omega \]

**Theorem** (Bochev & Hyman)

Assume that \( \mathcal{I} \) is *regular* reconstruction operator. Then, the **direct** and the **conforming** mimetic methods are completely equivalent.
Mimetic models with weak material laws

Translate to an equivalent 4-field constrained optimization problem

\[ \begin{align*}
    de &= -d_ib^{*}_{\sigma^{-1}} j = e \\
    dh &= j^{*}_{\mu^{-1}} b = h
\end{align*} \]

\[ \min \frac{1}{2} \left( \| \sqrt{\sigma} \left( j^{*}_{\sigma^{-1}} - e \right) \|_2^2 + \| \mu \left( b^{*}_{\mu^{-1}} - h \right) \|_2^2 \right) \]

subject to \( de = -d_ib \) and \( dh = j \)

Discretize in time

\[ \min \frac{1}{2} \left( \| \sqrt{\sigma} \left( j_h - e_h^{*} \right) \|_2^2 + \| \mu \gamma \left( b_h^{*} - h_h^{1} \right) \|_2^2 \right) \]

subject to \( de_h^1 = -\gamma \left( b_h^2 - \overline{b}_h^2 \right) \) and \( dh_h^1 = j_h^2 \)

Discretize in space (fully mimetic)

\[ \min \frac{1}{2} \left( \| \sqrt{\sigma} \left( j^2_h - e^1_h \right) \|_2^2 + \| \mu \gamma \left( b^2_h - h^1_h \right) \|_2^2 \right) \]

subject to \( \delta e^1 = -\gamma \left( b^2 - \overline{b}^2 \right) \) and \( \delta h^1 = j^2 \)

Conforming

- Does not require a primal-dual grid complex
- Explicit discretization of material laws is avoided
- Construction of a discrete \( \ast \) operation not required

Direct

Advantages

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So, where are the least-squares?

We start from the (fully) mimetic discrete 4-field principle

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} \left( \sigma^{-1} j_h^2 - e_h^1 \right) \right\|^2 + \left\| \mu \gamma \left( \mu^{-1} b_h^2 - h_h^1 \right) \right\|^2 \right) \quad \text{subject to} \quad de_h^1 = -\gamma \left( b_h^2 - \overline{b}_h^2 \right) \quad \text{and} \quad dh_h^1 = j_h^2
\]

Properties of discrete spaces imply that constraints can be satisfied exactly. 
⇒ we can eliminate the variables in the ranges of the differential operators:

\[
de_h^1 = -\gamma \left( b_h^2 - \overline{b}_h^2 \right) \Rightarrow b_h^2 = \overline{b}_h^2 - \gamma^{-1} de_h^1 \Rightarrow \mu^{-1} b_h^2 - h_h^1 = \mu^{-1} \overline{b}_h^2 - (\mu \gamma)^{-1} de_h^1 - h_h^1
\]

\[
dh_h^1 = j_h^2 \Rightarrow j_h^2 = dh_h^1 \Rightarrow \sigma^{-1} j_h^2 - e_h^1 = \sigma^{-1} dh_h^1 - e_h^1
\]

The constrained 4 field principle reduces to the unconstrained (least-squares) problem

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} \left( \sigma^{-1} dh_h^1 - e_h^1 \right) \right\|^2 + \left\| \sqrt{\mu \gamma} \left( (\mu \gamma)^{-1} de_h^1 + h_h^1 - \mu^{-1} b_h^2 \right) \right\|^2 \right)
\]

⇒ LSP is equivalent to a mimetic discretization of the 4-field principle
Properties

Theorem 1 The mimetic LSP has the following properties:

It is **conservative** in the sense that there exist \( b_h^2, j_h^2 \subseteq \Lambda^2(d,K) \) such that the pairs \( (e_h^1, b_h^2), (h_h^1, j_h^2) \) solve the **discrete eddy current equations**

\[
de_h^1 = -\gamma(b_h^2 - \overline{b_h^2}) \quad \text{and} \quad dh_h^1 = j_h^2
\]

The least-squares solution \( (e_h^1, b_h^2), (h_h^1, j_h^2) \) has the discrete **involutions**

\[
db_h^2 = 0 \quad \text{and} \quad dj_h^2 = 0
\]

The least-squares solution converges optimally.

**Duality is fundamental to Maxwell’s equations**

- \((e,j)\) provide two **complementary** viewpoints on **electric fields** (line vs. surface)
- \((h,b)\) provide two **complementary** viewpoints on **magnetic fields** (line vs. surface)
- In the continuum world the two viewpoints **can coexist** at the same point in space
- In the discrete world the two viewpoints **cannot coexist**:
  - **Line fields** are encoded by **circulations** \( \Rightarrow \) live on edges
  - **Surface fields** are encoded by **fluxes** \( \Rightarrow \) live on faces
Least-squares “as usual” vs. the 4-field principle

4-field variational principle

(E,B),(H,J) mimetic discretization

Elimination of range vars (B,J)

Mimetic least-squares in (E,H)

Completely equivalent

Eddy current eqs with material laws

Elimination of some variables

A least-squares principle

Conforming least-squares

Variables used

In the domain of curl

\[
\begin{align*}
\nabla \times E &= -\mu \dot{H} \\
\nabla \times H &= \sigma E \\
\n\nabla \times \sigma^{-1} J &= -\mu \dot{H} \\
\n\nabla \times H &= J \\
\n\n\nabla \times E &= -\dot{B} \\
\n\n\nabla \times \mu^{-1} B &= \sigma E
\end{align*}
\]

\[
\begin{align*}
\| \nabla \times E + \mu \dot{H} \|^2 + \| \nabla \times H - \sigma E \|^2 \\
\| \nabla \times \sigma^{-1} J + \mu \dot{H} \|^2 + \| \nabla \times H - J \|^2 \\
\| \nabla \times E + \dot{B} \|^2 + \| \nabla \times \mu^{-1} B - \sigma E \|^2
\end{align*}
\]
What can go wrong?

Mimetic least-squares are a result of a fully mimetic discretization of the 4-field principle followed by elimination of range variables (B,J)

⇒ Duality of viewpoints is inherited and represented in mimetic least-squares by (E,H)

Elimination followed by a least-squares principle may obscure the origin of the fields:
For example, the primal first-order system gives rise to a functional in (E,B)

\[ \| \nabla \times E + B \|^2 + \| \nabla \times \mu^{-1}B - \sigma E \|^2 \]

where the primal range variable B is also asked to be in the domain of curl (the home of the other primal variable E), i.e., B has to be simultaneously a surface and a line field. Recall that in the discrete world a vector field can be either surface or line field but not both!
Attempts to fix this problem by using C^0 elements and adding a div B=0 term completely miss the point that div B=0 is an involution rather than a constraint!

The inadequacy of C^0 spaces is further obscured by the fact that coercivity of the LSP is inherited on any proper subspace of H(curl), and so it seems OK to use such spaces. However this introduces spurious modes because C^0 elements cannot approximate well the (large) nullspace of the curl!
Where are the mixed methods?

A **fully mimetic** discretization of the semidiscrete 4-field principle

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} (\sigma^{-1} j - e) \right\|^2 + \left\| \mu \gamma (\mu^{-1} b - h) \right\|^2 \right) \quad \text{subject to} \quad de = -\gamma (b - \bar{b}) \quad \text{and} \quad dh = j
\]

uses mimetic approximations for both the **primal** and the **dual** variables:

\[
\Lambda^1(d,K) \times \Lambda^2(d,K) \iff \left( e_h^1, b_h^2 \right) \quad \Leftarrow \quad (e,b); (h,j) \quad \Rightarrow \quad \left( h_h^1, j_h^2 \right) \implies \Lambda^1(d,K) \times \Lambda^2(d,K)
\]

and **reduces to a mimetic least-squares**. However, we can apply mimetic discretization to just one of the two pairs of variables, either the primal or the dual:

\[
\Lambda^1(d,K) \times \Lambda^2(d,K) \iff \left( e_h^1, b_h^2 \right) \quad \Leftarrow \quad (e,b) \quad \Rightarrow \quad \left( e_h^2, b_h^1 \right) \implies \Lambda^2(d,K) \times \Lambda^1(d,K)
\]

A **primal mimetic** method

\[
\Lambda^2(d,K) \times \Lambda^1(d,K) \iff \left( h_h^2, j_h^1 \right) \quad \Leftarrow \quad (h,j) \quad \Rightarrow \quad \left( h_h^1, j_h^2 \right) \implies \Lambda^1(d,K) \times \Lambda^2(d,K)
\]

A **dual mimetic** method

It turns out that this leads to methods of the mixed type!
The primal mimetic method

We start from the primal mimetic discrete 4-field principle

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} \left( \sigma^{-1} j^1 - e^1 \right) \right\|^2 + \left\| \sqrt{\mu} \gamma \left( \mu^{-1} b^2_h - h^2_h \right) \right\|^2 \right) \quad \text{subject to} \quad de^1_h = -\gamma \left( b^2_h - \overline{b}^2_h \right) \quad \text{and} \quad d^* h^2_h = j^1_h
\]

Clearly, the minimum is achieved when \( j^1_h = \sigma e^1_h \) and \( h^2_h = \mu^{-1} b^2_h \). After the dual variables are eliminated from the constraints we are left with the discrete equations

\[
d\hat{e}^1_h = -\gamma \left( b^2_h - \overline{b}^2_h \right) \quad \text{and} \quad \mu_{\hat{b}^2_h} = \sigma e^1_h
\]

Using that \( \left( d^* h^2_h, \hat{e}^1_h \right) = \left( h^2_h, \hat{e}^1_h \right) + \left\langle h^1_h, \hat{e}^1_h \right\rangle \Gamma_2 \quad \forall \ \hat{e}^1_h \in \mathcal{L}^1(d,K) \) gives the **mixed problem**

\[
de^1_h = -\gamma \left( b^2_h - \overline{b}^2_h \right) \quad \text{and} \quad \mu^{-1} b^2_h, d\hat{e}^1_h \right\rangle + \left\langle h^1_h, \hat{e}^1_h \right\rangle = \left( \sigma e^1_h, \hat{e}^1_h \right) \quad \forall \ \hat{e}^1_h \in \mathcal{L}^1(d,K)
\]

The **range** variable can be **eliminated** to obtain a **Rayleigh-Ritz** type equation

\[
\gamma \left( \sigma e^1_h, \hat{e}^1_h \right) + \left\langle \mu^{-1} de^1_h, d\hat{e}^1_h \right\rangle = \gamma \left\langle h^1_h, \hat{e}^1_h \right\rangle \Gamma_1 + \gamma \left( \mu^{-1} \overline{b}^2_h, d\hat{e}^1_h \right) \quad \forall \ \hat{e}^1_h \in \mathcal{L}^1(d,K)
\]

It is a fully discrete version of the equivalent, **second order** eddy current equation

\[
\sigma \dot{E} + \nabla \times \mu^{-1} \nabla \times E = 0
\]
The three methods: summary

**Fully mimetic**

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} (\sigma^{-1} j^2_h - e^1_h) \right\|^2 + \left\| \sqrt{\mu} \gamma (\mu^{-1} b^2_h - h^1_h) \right\|^2 \right) \quad \text{subject to} \quad de^1_h = -\gamma (b^2_h - \bar{b}^2_h) \quad \text{and} \quad dh^1_h = j^2_h
\]

**Primal mimetic**

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} (\sigma^{-1} dh^1_h - e^1_h) \right\|^2 + \left\| \sqrt{\mu} \gamma (\mu^{-1} h^1_h + b^1_h - \mu^{-1} \bar{b}^2_h) \right\|^2 \right)
\]

\[
d^1 e^1_h = -\gamma (b^2_h - \bar{b}^2_h) \quad \text{and} \quad \left( \mu^{-1} b^2_h, d\hat{e}^1_h \right) + \left( h^1_h, \hat{e}^1_h \right)_{\Gamma_1} = \left( \alpha e^1_h, \hat{e}^1_h \right) \quad \forall \, \hat{e}^1_h \in \Lambda^1(d,K)
\]

\[
\gamma (\alpha e^1_h, \hat{e}^1_h) + \left( \mu^{-1} d e^1_h, \hat{e}^1_h \right) = \gamma \left( h^1_h, \hat{e}^1_h \right)_{\Gamma_1} + \gamma \left( \mu^{-1} \bar{b}^2_h, d\hat{e}^1_h \right) \quad \forall \, \hat{e}^1_h \in \Lambda^1(d,K)
\]

\[
\sigma \dot{\mathbf{E}} + \nabla \times \mu^{-1} \nabla \times \mathbf{E} = 0
\]

**Dual mimetic**

\[
\min \frac{1}{2} \left( \left\| \sqrt{\sigma} (\sigma^{-1} j^2_h - e^2_h) \right\|^2 + \left\| \sqrt{\mu} \gamma (\mu^{-1} b^1_h - h^1_h) \right\|^2 \right) \quad \text{subject to} \quad d^2 e^2_h = -\gamma (b^1_h - \bar{b}^1_h) \quad \text{and} \quad dh^1_h = j^2_h
\]

\[
d^1 h^1_h = j^2_h \quad \text{and} \quad \left( \sigma^{-1} j^2_h, d\hat{h}^1_h \right) + \left( e^1_h, \hat{h}^1_h \right)_{\Gamma_2} = -\gamma \left( \mu h^1_h + \bar{b}^1_h, \hat{h}^1_h \right) \quad \forall \, \hat{h}^1_h \in \Lambda^1(d,K)
\]

\[
\gamma (\mu h^1_h, \hat{h}^1_h) + \left( \sigma^{-1} d h^1_h, d\hat{h}^1_h \right) = -\left( e^1_h, \hat{h}^1_h \right)_{\Gamma_2} + \gamma \left( b^1_h, \hat{h}^1_h \right) \quad \forall \, \hat{h}^1_h \in \Lambda^1(d,K)
\]

\[
\mu \dot{\mathbf{H}} + \nabla \times \sigma^{-1} \nabla \times \mathbf{H} = 0
\]
A surprise!

**Theorem 2** Let \((e_h^1, b_h^2), (h_h^1, j_h^2)\) be the mimetic least-squares solution. Then
\[(e_h^1, b_h^2)\] is the solution of the primal mimetic method
If \(b(x,0) = 0\), or we solve in frequency domain, we also have that
\[(h_h^1, j_h^2)\] is the solution of the dual mimetic method

This means, mimetic LS is equivalent to simultaneous solution of the primal and dual methods
Proof

The first order necessary condition for the least-squares principle is
\[
\left(\sigma^{-1/2} dh^1_h - \sigma^{-1/2} e^1_h, \sigma^{-1/2} \hat{d}^1_h - \sigma^{-1/2} \hat{e}^1_h\right) + \left((\mu\gamma)^{-1/2} de^1_h + (\mu\gamma)^{1/2} h^1_h, (\mu\gamma)^{-1/2} \hat{d}^1_h + (\mu\gamma)^{1/2} \hat{h}^1_h\right)
\]
\[
= \left(\mu^{-1}(\mu\gamma)^{1/2} b^2_h, (\mu\gamma)^{-1/2} de^1_h + (\mu\gamma)^{1/2} \hat{h}^1_h\right)
\]

Expand each term

\[
\left(\sigma^{-1/2} dh^1_h - \sigma^{-1/2} e^1_h, \sigma^{-1/2} \hat{d}^1_h - \sigma^{-1/2} \hat{e}^1_h\right) = (\sigma e^1_h, \hat{e}^1_h) + (\sigma^{-1} dh^1_h, \hat{d}^1_h) - (dh^1_h, \hat{e}^1_h) - (e^1_h, \hat{d}^1_h)
\]

\[
(\mu\gamma)^{-1/2} de^1_h + (\mu\gamma)^{1/2} h^1_h, (\mu\gamma)^{-1/2} \hat{d}^1_h + (\mu\gamma)^{1/2} \hat{h}^1_h) = \gamma\left(\mu h^1_h, \hat{h}^1_h\right) + \gamma^{-1}\left(\mu^{-1} de^1_h, \hat{d}^1_h\right) + (de^1_h, \hat{h}^1_h) + (h^1_h, \hat{d}^1_h)
\]

The least-squares optimality system **uncouples** into two independent equations

\[
\gamma\left(\sigma e^1_h, \hat{e}^1_h\right) + (\mu^{-1} de^1_h, \hat{d}^1_h) = \gamma\left(h^1_h, \hat{e}^1_h\right)_{\Gamma_2} + \gamma\left(\mu^{-1} b^2_h, \hat{d}^1_h\right) \quad \forall \hat{e}^1_h \in \Lambda^1(d,K) \quad \text{Primal mimetic}
\]

\[
\gamma\left(\mu h^1_h, \hat{h}^1_h\right) + (\sigma^{-1} dh^1_h, \hat{d}^1_h) = -\left(e^1_h, \hat{h}^1_h\right)_{\Gamma_1} + \gamma\left(b^2_h, \hat{h}^1_h\right) \quad \forall \hat{h}^1_h \in \Lambda^1(d,K)
\]

If \(b(x,0)=0\), or in frequency domain, then the 2nd LS equation is identical to

\[
\gamma\left(\mu h^1_h, \hat{h}^1_h\right) + (\sigma^{-1} dh^1_h, \hat{d}^1_h) = -\left(e^1_h, \hat{h}^1_h\right)_{\Gamma_1} + \gamma\left(b^1_h, \hat{h}^1_h\right) \quad \forall \hat{h}^1_h \in \Lambda^1(d,K) \quad \text{Dual mimetic}
\]
Compatibility matters even for least-squares!

Nodal $C^0$ LSFEM solution
$\text{Ker(curl)} = \{0\}$

Mimetic LSFEM solution
$\text{Ker(curl)} = \{\text{grad } p\}$
Conclusions

Even in least-squares: 

Compatibility pays and there’s no free lunch

References


Related work

