Mimetic Least Squares Principles for Electromagnetics

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Mimetic Least-Squares Principles for Electromagnetics

What are LSP and the reasons to use them.

Mimetic LSP, their conservative properties and interpretation as realizations of weak material laws

1. What is a mimetic discretization
2. An algebraic topology framework
3. Direct and conforming discretizations

A new interpretation of LSP: Realizations of a discrete Hodge * operator

Mixed, Galerkin and Least-Squares methods for 2nd order problems share a common ancestor: the 4-field principle

Least-Squares 101

$\mathcal{L}u = f$ in Ω

$\mathcal{R}u = h$ on Γ

$$\min_{u\in X} J(u;f,h) \equiv \frac{1}{2}\left(\|\mathcal{L}u - f\|^2_{X,\Omega} + \|\mathcal{R}u - h\|^2_{Y,\Gamma}\right)$$

$$(\mathcal{L}u,\mathcal{L}v)_\Omega + (\mathcal{R}u,\mathcal{R}v)_\Gamma = (f,\mathcal{L}u)_\Omega + (h,\mathcal{R}v)_\Gamma$$

$$\mathbf{Au} = \mathbf{b}$$

Top 3 reasons people

want to do least squares:

- ☺ **Using C^0 nodal elements**
- ☺ **Avoiding inf-sup conditions**
- ☺ **Solving SPD systems**

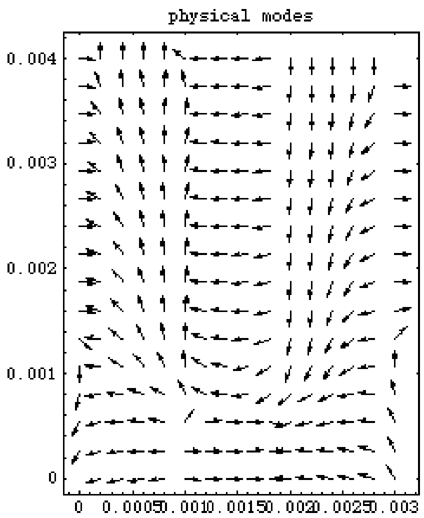
don't want to do least squares:

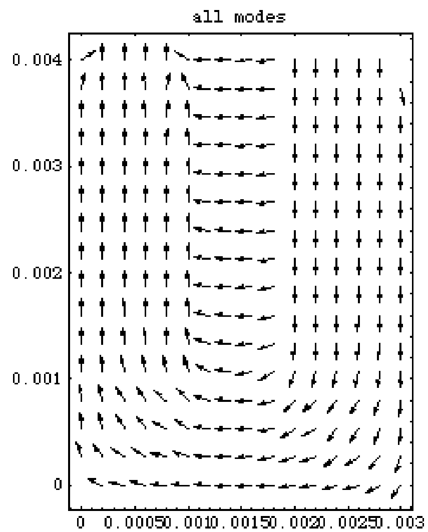
- ☹ **Conservation**
- ☹ **Conservation**
- ☹ **Conservation**

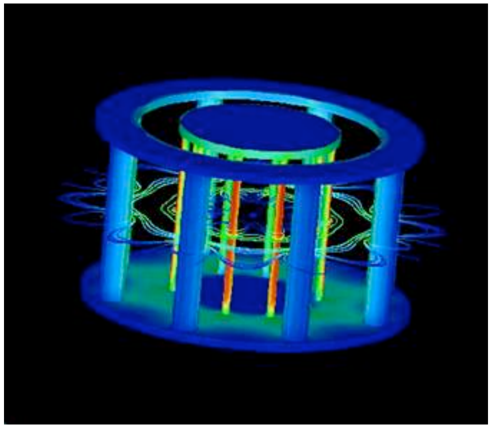
We will show that

- Using **nodal elements** is not necessarily the best choice in LSFEM, and so it is arguably the **least-important advantage** attributed to least-squares methods
- By using **other** elements least-squares acquire **additional** conservation properties
- Surprisingly, this kind of least-squares turns out to be **related** to mixed methods

Compatibility matters even for least-squares: magnetic diffusion







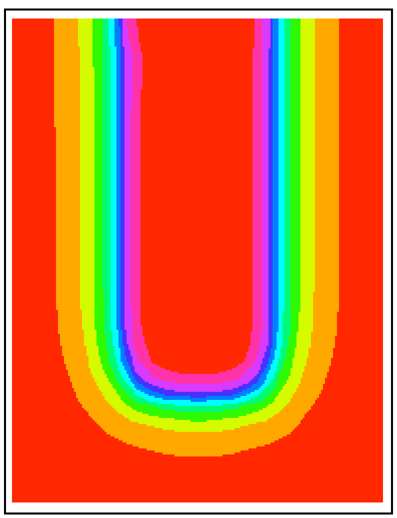
LSFEM solution using C^0 elements

Ker(curl)={0}

$$\nabla \times \frac{1}{\mu}\mathbf{B} = \sigma\mathbf{E} \quad \text{Ampere}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{Faraday}$$

B



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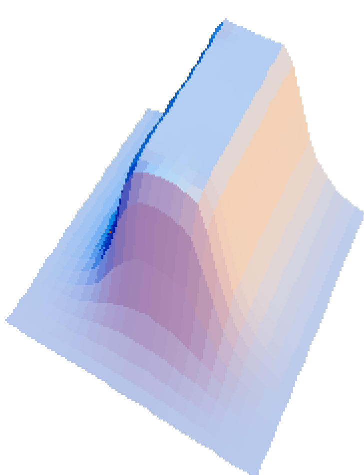

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Further evidence that compatibility matters

Lax-Wendroff theorem:

If approximation computed by a **conservative** and **consistent** method converges, then the limit is a **weak solution** of the conservation law.

Grid decomposition property (Fix, Gunzburger, Nicolaides, 1978)

A **discrete Hodge decomposition** property is **necessary** and **sufficient** for stable and accurate LSFEM discretization of the Kelvin principle.

Staggered FD and FV (MAC, Yee's FDTD, Box integration)

Conservation requires placing **different variables** at **different grid locations** so as to achieve a **discrete Stokes** theorem.

Discrete models must reflect mathematical structure of continuum models

How to achieve compatibility?

Compatible discretization requires:

- Mathematical tools to **discover** and **encode** structure of PDEs
- A discrete framework that **mimics** that structure: mutually consistent notions of
  - Discrete vector calculus, Hodge theory, entropy condition, conservation…

In most physical models

- **Fields** are observed **indirectly** by measuring **global** quantities (flux, circulation, etc)
- **Physical laws** are **relationships** between **global** quantities (conservation, equilibrium)

Differential forms provide the tools to encode such relationships

- **Integration:** an abstraction of the *measurement* process
- **Differentiation:** gives rise to *local invariants*
- **Poincare Lemma:** expresses *local geometric* relations
- **Stokes Theorem:** gives rise to *global relations*

Algebraic topology approach

Algebraic topology provides the tools to mimic the structure

- **Computational grid** is algebraic topological complex
- ***k*-forms** are encoded as *k*-cell quantities (*k*-cochains)
- **Derivative** is provided by the coboundary
- **Inner product** induces combinatorial Hodge theory
- **Singular cohomology** preserved by the complex

Framework for mimetic discretizations (*Bochev, Hyman, IMA Proceedings)*

- **Translation:** Fields → forms → cochains
- **Basic mappings:** **reduction** and **reconstruction**
  - **Combinatorial operations:** induced by **reduction** map
  - **Natural operations:** induced by **reconstruction** map
  - **Derived operations:** induced by **natural** operations

Branin (1966), Dodzuik (1976), Hyman & Scovel (1988-92), Nicolaides (1993), Dezin (1995), Shashkov (1990-), Mattiussi (1997), Schwalm (1999), Teixeira (2001)

Differential Forms

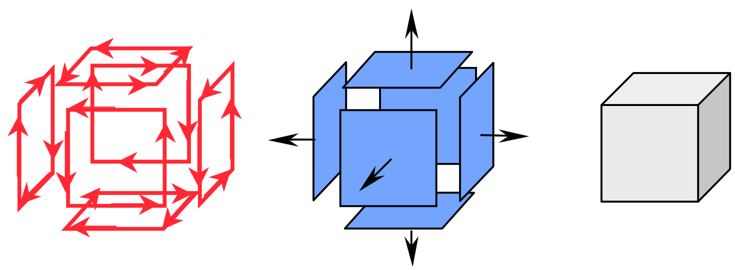
Smooth differential forms	$\Lambda^k(\Omega): \quad x \to \omega(x) \in \Lambda^k(T_x\Omega)$
DeRham complex	$\mathbf{R} \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \to 0$
Metric conjugation	$*: \Lambda^k(T_x\Omega) \to \Lambda^{n-k}(T_x\Omega) \Leftrightarrow \omega \wedge *\xi = (\omega,\xi)_x \omega_n$
L² inner product on Λ*k*(Ω)	$(\omega,\xi)_\Omega = \int_\Omega (\omega,\xi)_x \omega_n \quad \Rightarrow \quad (\omega,\xi)_\Omega = \int_\Omega \omega \wedge *\xi$
Codifferential	$d^*: \Lambda^{k+1}(\Omega) \to \Lambda^{k+1}(\Omega) \Leftrightarrow (d\omega,\xi)_\Omega = (\omega, d^*\xi)_\Omega$
Hodge Laplacian	$\Delta: \Lambda^k(\Omega) \to \Lambda^k(\Omega) \quad \to \quad \Delta = dd^* + d^*d$
Completion of Λ*k*(Ω)	$\Lambda^k(L^2,\Omega)$
Sobolev spaces	$\Lambda^k(d,\Omega) = \{\omega \in \Lambda^k(L^2,\Omega) \mid d\omega \in \Lambda^{k+1}(L^2,\Omega)\}$

Chains and cochains

Computational grid = Chain complex

$$\partial : C_k \to C_{k-1}$$

$$\partial\partial = 0 \qquad C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} C_3$$



$$0 = \partial\partial K^3 \xleftarrow{\partial} \partial K^3 \xleftarrow{\partial} K^3$$

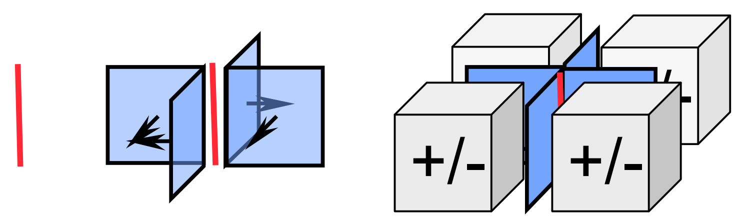
Field representation = Cochain complex

$$C^k = L(C_k, \mathbf{R}) = C_k^{\ *} \qquad \langle \sigma^i, \sigma_j \rangle = \delta_{ij}$$

$$K^1 \xrightarrow{\delta} \delta K^1 \xrightarrow{\delta} \delta\delta K^1 = 0$$

$$\delta : C^k \to C^{k+1} \qquad \langle \omega, \partial\eta \rangle = \langle \delta\omega, \eta \rangle$$

$$\delta\delta = 0 \qquad C^0 \xrightarrow{\delta} C^1 \xrightarrow{\delta} C^2 \xrightarrow{\delta} C^3$$



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Basic mappings

Reduction

$$\mathcal{R} : \Lambda^k\left(L^2,\Omega\right) \to C^k$$

Natural choice

$$\left\langle \mathcal{R}\omega,\sigma \right\rangle = \int_\sigma \omega$$

DeRham map

$$\mathcal{R}d = \delta\mathcal{R}$$

Proof

$$\left\langle \delta\mathcal{R}\omega,c \right\rangle = \left\langle \mathcal{R}\omega,\partial c \right\rangle = \int_{\partial c} \omega = \int_c d\omega = \left\langle \mathcal{R}d\omega,c \right\rangle$$

$$\begin{array}{ccccc} \cdots & \Lambda^k & \xrightarrow{d} & \Lambda^{k+1} & \cdots \\ & \downarrow & & \downarrow & \\ \cdots & C_k & \xleftarrow{\partial} & C_{k+1} & \cdots \\ & \updownarrow & & \updownarrow & \\ \cdots & C^k & \xrightarrow{\delta} & C^{k+1} & \cdots \end{array}$$

$$\begin{array}{ccc} \Lambda^k & \xrightarrow{d} & \Lambda^{k+1} \\ \mathcal{R}\downarrow & \textbf{CDP I} & \downarrow\mathcal{R} \\ C^k & \xrightarrow{\delta} & C^{k+1} \end{array} \qquad \begin{array}{ccc} \Lambda^k & \xrightarrow{d} & \Lambda^{k+1} \\ \mathcal{I}\downarrow & \textbf{CDP2} & \downarrow\mathcal{I} \\ C^k & \xrightarrow{\delta} & C^{k+1} \end{array}$$

natural **required**

$$\text{Range } \mathcal{I}\mathcal{R} = \Lambda^k\left(L^2,K\right) \subset \Lambda^k\left(L^2,\Omega\right)$$

$$\text{Range } \mathcal{I}\mathcal{R} = \Lambda^k\left(d,K\right) \subset \Lambda^k\left(d,\Omega\right)$$

Reconstruction

$$\mathcal{I} : C^k \to \Lambda^k\left(L^2,\Omega\right)$$

No natural choice

$$\mathcal{R}\mathcal{I} = id$$

$$\mathcal{I}\mathcal{R} = id + O(h^s)$$

$$\ker \mathcal{I} = 0$$

Conforming

$$\mathcal{I} : C^k \to \Lambda^k\left(d,\Omega\right)$$

$$\mathcal{I}d = \delta\mathcal{I}$$

Combinatorial operations

Discrete derivative

Forms are dual to **manifolds** $\langle d\omega, \Omega \rangle = \langle \omega, \partial\Omega \rangle$

Cochains are dual to **chains** $\langle \delta a, \sigma \rangle = \langle a, \partial\sigma \rangle$

δ approximates *d* on cochains

Discrete integral

$$\int_\sigma a = \langle a, \sigma \rangle$$

Stokes theorem

$$\langle \delta a, \sigma \rangle = \langle a, \partial\sigma \rangle$$

grad

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

curl

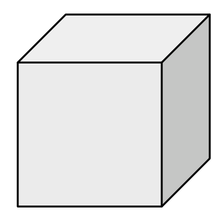
$$\begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

div

$$\begin{pmatrix} -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix}$$

$$\delta\delta = 0$$

$$\mathcal{R}d = \delta\mathcal{R}$$



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Natural and derived operations

Natural Inner product $(a,b)_x = (\mathcal{I}a, \mathcal{I}b)_x$ $\quad (a,b)_\Omega = \int_\Omega (a,b)_x \omega_n = (\mathcal{I}a, \mathcal{I}b)_\Omega$

Wedge product $\wedge : C^k \times C^l \mapsto C^{k+l}$ $\quad a \wedge b = \mathcal{R}(\mathcal{I}a \wedge \mathcal{I}b)$

Derived Adjoint derivative $\delta^* : C^{k+1} \mapsto C^k$ $\quad (\delta^* a, b)_\Omega = (a, \delta b)_\Omega$

Provides a second set of **grad**, **div** and **curl** operators. Scalars encoded as 0 or 3-forms, vectors as 1 or 2-forms, derivative choice depends on encoding.

Discrete Laplacian $D : C^k \mapsto C^k$ $\quad D = \delta^*\delta + \delta\delta^*$

Derived operations are necessary to avoid **internal inconsistencies** between the discrete operations: $\mathcal{I}$ is only **approximate inverse** of $\mathcal{R}$ and natural operations will clash

Example Natural adjoint $d^* = (-1)^k * d *$ $\longrightarrow$ $\delta^* = (-1)^k \mathcal{R} * d * \mathcal{I}$

$\mathcal{I}$ must be regular and $(\delta^* a, b)_\Omega = (a, \delta b)_\Omega + O(h^s)$ $\quad \Rightarrow \delta^*$ not true adjoint

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Mimetic properties (I)

Discrete Poincare lemma (existence of potentials in contractible domains)

$$d\omega_k = 0 \quad \Rightarrow \quad \omega_k = d\omega_{k+1} \qquad\qquad \delta c^k = 0 \quad \Rightarrow \quad c^k = \delta c^{k+1}$$

Discrete Stokes Theorem

$$\langle d\omega_{k-1}, c_k \rangle = \langle \omega_{k-1}, \partial c_k \rangle \qquad\qquad \langle \delta c^{k-1}, c_k \rangle = \langle c^{k-1}, \partial c_k \rangle$$

Discrete "Vector Calculus"

$$dd = 0 \qquad\qquad \delta\delta = \delta^* \delta^* = 0$$

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega \qquad\qquad a \wedge b = (-1)^{kl} b \wedge a$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \qquad\qquad \delta(a \wedge b) = \delta a \wedge b + (-1)^k a \wedge \delta b$$

(Regular $\mathcal{I}$)

Any feature of the continuum system that is implied by differential forms calculus is inherited by the discrete model

Called ***mimetic*** property by Hyman and Scovel (1988)

Mimetic properties (II)

Inner product induces combinatorial Hodge theory on cochains

$$\text{Co-cycles of } (\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3) \xrightarrow{\mathcal{R}} \text{co-cycles of } (C^0, C^1, C^2, C^3)$$

$$d\omega = 0 \quad \Rightarrow \quad \delta\mathcal{R}\omega = 0$$

Discrete Harmonic forms

$$H^k(\Omega) = \{\eta \in \Lambda^k(\Omega) \mid d\eta = d^*\eta = 0\} \qquad H^k(K) = \{c^k \in C^k \mid \delta c^k = \delta^* c^k = 0\}$$

Discrete Hodge decomposition

$$\omega = d\rho + \eta + d^*\sigma \qquad a = \delta b + h + \delta^* c$$

Theorem $\quad \dim\ker(\Delta) = \dim\ker(D)$

Remarkable property of the mimetic *D* - kernel size is a **topological invariant!**

Discrete $*$ operation

Natural definition

$$*_N : C^k \mapsto C^{n-k} \qquad *_N = \mathcal{R} * \mathcal{I}$$

Derived definition

$$*_D : C^k \mapsto C^{n-k} \qquad \int_\Omega a \wedge *_D b = (a,b)_\Omega$$

mimics

$$(\omega,\xi)_\Omega = \int_\Omega \omega \wedge *\xi$$

Theorem

$$*_N \mathcal{R}\omega^h = \mathcal{R} * \omega^h \quad \forall \omega^h \in \mathrm{Range}(\mathcal{I}\mathcal{R})$$

CDP on the range

$$\int_\Omega \mathcal{I}\mathcal{R}(\mathcal{I}a \wedge \mathcal{I} *_D b) = \int_\Omega (\mathcal{I}a \wedge *\mathcal{I}b)$$

Weak CDP

$$\int_\Omega b \wedge *_N b = (a,b)_\Omega + O(h^s)$$

$$*_N = *_D + O(h^s)$$

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Problems with the discrete $*$

Action of $*$ must be coordinated with the other discrete operations

	$(\bullet,\bullet)$	$\wedge$	δ^*	$\mathcal{R}$	$\mathcal{I}$
$*_N$	—	—	—	✓	—
$*_D$	✓	✓	—	—	—

Analytic $*$ is a **local, invertible** operation $\Rightarrow$ **positive diagonal** matrix

$\dim C^k \neq \dim C^{n-k}$ $\Rightarrow$ $*_N : C^k \mapsto C^{n-k}$ **cannot be a square matrix!**

Construction of $*$ is nontrivial task unless primal-dual grid is used!

Implications

A consistent discrete framework requires a choice of a primary operation
either $*$ or $(\cdot,\cdot)$ but not both

A **discrete** $*$ is the primary concept in Hiptmair (2000), Bossavit (1999)

- Inner product derived from discrete $*$
- Used in **explicit discretization of material laws**

The **natural inner product** is the primary operation in our approach

- **Sufficient** to give rise to combinatorial Hodge theory on cochains
- **Easier** to define than a discrete $*$ operation
- **Incorporate** material laws in the natural inner product, or
- **Enforce** material laws **weakly** (justified by their approximate nature)

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Algebraic equivalents

Operation	Matrix form	type
δ	$\mathbf{D}_k$	{-1,0,1}
$(\cdot,\cdot)$	$\mathbf{M}_k$	SPD
$a_1 \wedge b_1$	$\sum \mathbf{W}_{11}$	Skew symm.
$a_1 \wedge b_2$	$\sum \mathbf{W}_{12}$	$\mathbf{W}_{12}^{T}=\mathbf{W}_{21}$
$b_2 \wedge a_1$	$\sum \mathbf{W}_{21}$	
δ^*	$\mathbf{M}_k^{-1}\, \mathbf{D}_k^{T}\mathbf{M}_{k+1}$	rectangular
$\mathcal{D}$	$\mathbf{M}_k^{-1}\, \mathbf{D}_k^{T}\mathbf{M}_{k+1}\mathbf{D}_k + \mathbf{D}_{k-1}\mathbf{M}_{k-1}^{-1}\, \mathbf{D}_{k-1}^{T}\mathbf{M}_k$	square
$*_D$	$\mathbf{W}_{12}(*_D\mathbf{a})=\mathbf{M}_3\mathbf{a}$	pair

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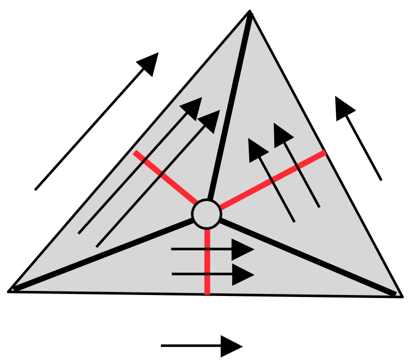
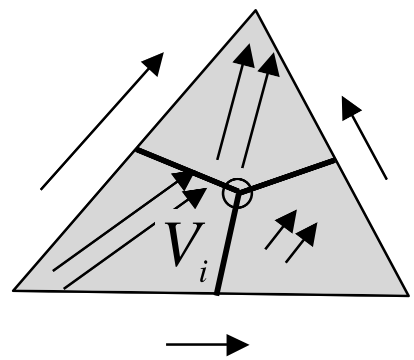
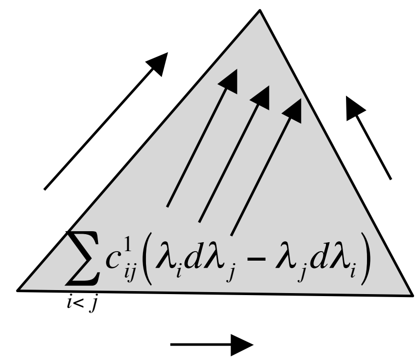
Reconstruction and natural inner products

	Co-volume	Mimetic	Whitney
$\mathcal{I}$		 	
	Nicolaides, Trapp (1992-04)	*Hyman, Shashkov, Steinberg (1985-04)*	*Dodzuik (1976) Hyman, Scovel (1988)*
			$\omega^1_{ij} = \lambda_i d\lambda_j - \lambda_j d\lambda_i$
M	$\begin{pmatrix} h_1 h_1^\perp & & \\ & h_2 h_2^\perp & \\ & & h_3 h_3^\perp \end{pmatrix}$	$\begin{pmatrix} \frac{V_2}{\sin^2\phi_2}+\frac{V_3}{\sin^2\phi_3} & \frac{V_3\cos\phi_3}{\sin^2\phi_3} & \frac{V_2\cos\phi_2}{\sin^2\phi_2} \\ \frac{V_3\cos\phi_3}{\sin^2\phi_3} & \frac{V_1}{\sin^2\phi_1}+\frac{V_3}{\sin^2\phi_3} & \frac{V_1\cos\phi_1}{\sin^2\phi_1} \\ \frac{V_2\cos\phi_2}{\sin^2\phi_2} & \frac{V_1\cos\phi_1}{\sin^2\phi_1} & \frac{V_1}{\sin^2\phi_1}+\frac{V_2}{\sin^2\phi_2} \end{pmatrix}$	$\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \left(w_{ij}, w_{kl}\right) & \\ \cdots & \cdots & \cdots \end{pmatrix}$
δ^*	**local**	**non-local**	**non-local**

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Mimetic discretization: translation to forms

1st order PDE with material laws

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \mathbf{J} = \sigma\mathbf{E}$$
$$\nabla\times\mathbf{H} = \mathbf{J} \qquad \mathbf{B} = \mu\mathbf{H}$$

$$de = -d_t b \qquad *_{\sigma^{-1}} j = e$$
$$dh = j \qquad *_{\mu^{-1}} b = h$$

primal → (e,b)

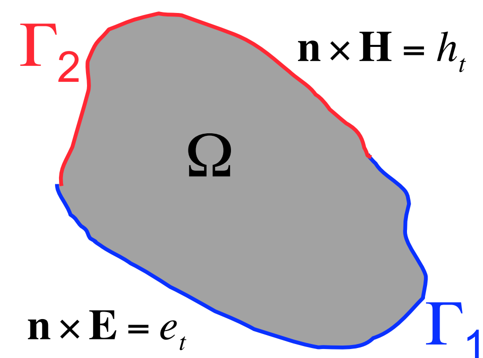
domain **curl** range

dual → (h,j)

1st order PDE with codifferentials

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\frac{1}{\sigma}\nabla\times\frac{1}{\mu}\mathbf{B} = \mathbf{E}$$

$$de = -d_t b$$
$$e = *_{\sigma^{-1}} d *_{\mu^{-1}} b$$



2nd order PDE

$$\nabla\times\frac{1}{\sigma}\nabla\times\frac{1}{\mu}\mathbf{B} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$d *_{\sigma^{-1}} d *_{\mu^{-1}} b = -d_t b$$

NOTE: we could have eliminated the primal pair (E,B) and obtain the last two equations in terms of the dual pair (H,J).

Direct and conforming mimetic models

$$\delta\delta^* b^2 = -\delta_t b^2$$

$$e^1 \in C^2; \quad b^2 \in C^2$$

$$\delta e^1 = -\delta_t b^2$$
$$e^1 = \delta^* b^2$$

$$d *_{\sigma^{-1}} d *_{\mu^{-1}} b = -d_t b$$

Direct

Conforming

$$de = -d_t b$$
$$e = *_{\sigma^{-1}} d *_{\mu^{-1}} b$$

$$\left(db_h^2, d\hat{b}_h^2\right)_\Omega = \left(-d_t b_h^2, \hat{b}_h^2\right)_\Omega$$

$$e_h^1 \in \Lambda^1(d,K); \quad b_h^2 \in \Lambda^2(d,K)$$

$$de_h^1 = -d_t b_h^2$$
$$\left(e_h^1, \hat{e}_h^1\right)_\Omega = \left(b_h^2, d\hat{e}_h^1\right)_\Omega$$

Theorem (Bochev & Hyman)

Assume that $\mathcal{I}$ is **regular** reconstruction operator. Then, the **direct** and the **conforming** mimetic methods are completely equivalent.

Mimetic models with weak material laws

Translate to an equivalent 4-field constrained optimization problem

$$de = -d_t b \quad *_{\sigma^{-1}} j = e$$
$$dh = j \quad *_{\mu^{-1}} b = h$$

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(*_{\sigma^{-1}} j - e \right) \right\|^2 + \left\| \sqrt{\mu}\left(*_{\mu^{-1}} b - h \right) \right\|^2 \right)$$

subject to $de = -d_t b$ and $dh = j$

Discretize in time

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(*_{\sigma^{-1}} j - e \right) \right\|^2 + \left\| \sqrt{\mu\gamma}\left(*_{\mu^{-1}} b - h \right) \right\|^2 \right)$$ subject to $de = -\gamma\left(b - \bar{b}\right)$ and $dh = j$

Discretize in space (fully mimetic)

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^2 - e_h^1 \right) \right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^1 \right) \right\|^2 \right)$$

subject to $de_h^1 = -\gamma\left(b_h^2 - \bar{b}_h^2\right)$ and $dh_h^1 = j_h^2$

Conforming

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j^2 - e^1 \right) \right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b^2 - h^1 \right) \right\|^2 \right)$$

subject to $\delta e^1 = -\gamma\left(b^2 - \bar{b}^2\right)$ and $\delta h^1 = j^2$

Direct

Advantages

– Does not require a primal-dual grid complex
– Explicit discretization of material laws is avoided
– Construction of a discrete $*$ operation not required

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So, where are the least-squares?

We start from the (fully) mimetic discrete 4-field principle

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^2 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^1\right)\right\|^2 \right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

Properties of discrete spaces imply that constraints can be satisfied **exactly**.

$\Rightarrow$ we can **eliminate** the variables in the **ranges** of the differential operators:

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \Rightarrow b_h^2 = \overline{b}_h^2 - \gamma^{-1} de_h^1 \Rightarrow \mu^{-1} b_h^2 - h_h^1 = \mu^{-1}\overline{b}_h^2 - (\mu\gamma)^{-1} de_h^1 - h_h^1$$

$$dh_h^1 = j_h^2 \Rightarrow j_h^2 = dh_h^1 \Rightarrow \sigma^{-1} j_h^2 - e_h^1 = \sigma^{-1} dh_h^1 - e_h^1$$

The **constrained** 4 field principle reduces to the **unconstrained** (least-squares) problem

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} dh_h^1 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left((\mu\gamma)^{-1} de_h^1 + h_h^1 - \mu^{-1}\overline{b}_h^2\right)\right\|^2 \right)$$

$\Rightarrow$ **LSP is equivalent to a mimetic discretization of the 4-field principle**

Properties

Theorem 1 The mimetic LSP has the following properties:

It is **conservative** in the sense that there exist $b_h^2, j_h^2 \in \Lambda^2(d,K)$ such that the pairs (e_h^1, b_h^2), (h_h^1, j_h^2) solve the **discrete eddy current equations**

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

The least-squares solution $(e_h^1, b_h^2), (h_h^1, j_h^2)$ has the discrete **involutions**

$$db_h^2 = 0 \quad \text{and} \quad dj_h^2 = 0$$

The least-squares solution converges optimally.

Duality is fundamental to Maxwell's equations

- $(\boldsymbol{e},\boldsymbol{j})$ provide two **complementary** viewpoints on **electric fields** (line vs. surface)
- $(\boldsymbol{h},\boldsymbol{b})$ provide two **complementary** viewpoints on **magnetic fields** (line vs. surface)
- In the continuum world the two viewpoints **can coexist** at the same point in space
- In the discrete world the two viewpoints **cannot coexist**:
  - **Line fields** are encoded by **circulations** $\Rightarrow$ **live on edges**
  - **Surface fields** are encoded by **fluxes** $\Rightarrow$ **live on faces**

Least-squares "as usual" vs. the 4-field principle

4-field variational principle	(**E**,**B**),(**H**,**J**) mimetic discretization	Elimination of **range** vars (**B**,**J**)	Mimetic least-squares in (**E**,**H**)

Completely equivalent

Eddy current eqs with material laws	Elimination of **some** variables	A least-squares principle	Conforming least-squares

Variables used

$$\nabla\times\mathbf{E}=-\dot{\mathbf{B}} \qquad \mathbf{B}=\mu\mathbf{H}$$
$$\nabla\times\mathbf{H}=\mathbf{J} \qquad \mathbf{J}=\sigma\mathbf{E}$$

Equations	Least-squares functional	Variables used
$\nabla\times\mathbf{E}=-\mu\dot{\mathbf{H}}$, $\nabla\times\mathbf{H}=\sigma\mathbf{E}$	$\left\|\nabla\times\mathbf{E}+\mu\dot{\mathbf{H}}\right\|^2+\left\|\nabla\times\mathbf{H}-\sigma\mathbf{E}\right\|^2$	**In the domain of curl**
$\nabla\times\sigma^{-1}\mathbf{J}=-\mu\dot{\mathbf{H}}$, $\nabla\times\mathbf{H}=\mathbf{J}$	$\left\|\nabla\times\sigma^{-1}\mathbf{J}+\mu\dot{\mathbf{H}}\right\|^2+\left\|\nabla\times\mathbf{H}-\mathbf{J}\right\|^2$	**Dual**
$\nabla\times\mathbf{E}=-\dot{\mathbf{B}}$, $\nabla\times\mu^{-1}\mathbf{B}=\sigma\mathbf{E}$	$\left\|\nabla\times\mathbf{E}+\dot{\mathbf{B}}\right\|^2+\left\|\nabla\times\mu^{-1}\mathbf{B}-\sigma\mathbf{E}\right\|^2$	**Primal**

What can go wrong?

Mimetic least-squares are a result of a **fully mimetic** discretization of the 4-field principle followed by **elimination of range** variables (**B**,**J**)

⇒ **Duality** of viewpoints is **inherited** and represented in mimetic least-squares by (**E**,**H**)

Elimination followed by a **least-squares** principle **may obscure** the origin of the fields: For example, the primal first-order system gives rise to a functional in (**E,B**)

$$\left\|\nabla \times \mathbf{E} + \dot{\mathbf{B}}\right\|^2 + \left\|\nabla \times \mu^{-1}\mathbf{B} - \sigma\mathbf{E}\right\|^2$$

where the primal range variable **B** is also asked to be in the domain of curl (the home of the other primal variable **E**), i.e., **B** has to be simultaneously a **surface** and a **line** field. Recall that in the **discrete world** a vector field can be either surface or line field **but not both**!

Attempts to fix this problem by using C^0 elements and adding a $\operatorname{div} \mathbf{B}=0$ term completely miss the point that $\operatorname{div} \mathbf{B}=0$ is an **involution** rather than a **constraint**!

The **inadequacy** of C^0 spaces is further **obscured** by the fact that **coercivity** of the LSP is **inherited** on any proper subspace of H(curl), and so it seems OK to use such spaces. However this introduces spurious modes because C^0 elements **cannot approximate** well the (large) **nullspace of the curl**!

Where are the mixed methods?

A **fully mimetic** discretization of the semidiscrete 4-field principle

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(*_{\sigma^{-1}} j - e \right) \right\|^2 + \left\| \sqrt{\mu\gamma}\left(*_{\mu^{-1}} b - h \right) \right\|^2 \right) \quad \text{subject to} \quad de = -\gamma\left(b - \bar{b}\right) \quad \text{and} \quad dh = j$$

uses mimetic approximations for both the primal and the dual variables:

$$\Lambda^1(d,K) \times \Lambda^2(d,K) \Leftarrow \left(e_h^1, b_h^2\right) \quad \Longleftarrow \quad (e,b);(h,j) \quad \Longrightarrow \quad \left(h_h^1, j_h^2\right) \Rightarrow \Lambda^1(d,K) \times \Lambda^2(d,K)$$

and **reduces to a mimetic least-squares**. However, we can apply mimetic discretization to just one of the two pairs of variables, either the primal or the dual:

$$\Lambda^1(d,K) \times \Lambda^2(d,K) \Leftarrow \left(e_h^1, b_h^2\right) \quad \Longleftarrow \quad (e,b) \quad \Longleftarrow \quad \left(e_h^2, b_h^1\right) \Rightarrow \Lambda^2(d,K) \times \Lambda^1(d,K)$$

$$\Lambda^2(d,K) \times \Lambda^1(d,K) \Leftarrow \left(h_h^2, j_h^1\right) \quad \Longrightarrow \quad (h,j) \quad \Longrightarrow \quad \left(h_h^1, j_h^2\right) \Rightarrow \Lambda^1(d,K) \times \Lambda^2(d,K)$$

A **primal mimetic** method — A **dual mimetic** method

It turns out that this leads to methods of the mixed type!

The primal mimetic method

We start from the primal mimetic discrete 4-field principle

$$\min \frac{1}{2}\left(\left\| \sqrt{\sigma}\left(\sigma^{-1} j_h^1 - e_h^1\right)\right\|^2 + \left\| \sqrt{\mu\gamma}\left(\mu^{-1} b_h^2 - h_h^2\right)\right\|^2 \right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad d^* h_h^2 = j_h^1$$

Clearly, the minimum is achieved when $j_h^1 = \sigma e_h^1$ and $h_h^2 = \mu^{-1} b_h^2$. After the dual variables
are eliminated from the constraints we are left with the discrete equations

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad d^* \mu^{-1} b_h^2 = \sigma e_h^1$$

Using that $\left(d^* h_h^2, \hat{e}_h^1\right) = \left(h_h^2, d\hat{e}_h^1\right) + \left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_2} \ \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$ gives the **mixed problem**

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad \left(\mu^{-1} b_h^2, d\hat{e}_h^1\right) + \left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_1} = \left(\sigma e_h^1, \hat{e}_h^1\right) \quad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$$

The **range** variable can be **eliminated** to obtain a **Rayleigh-Ritz** type equation

$$\gamma\left(\sigma e_h^1, \hat{e}_h^1\right) + \left(\mu^{-1} de_h^1, d\hat{e}_h^1\right) = \gamma\left\langle h_t, \hat{e}_h^1 \right\rangle_{\Gamma_1} + \gamma\left(\mu^{-1}\overline{b}_h^2, d\hat{e}_h^1\right) \qquad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$$

It is a fully discrete version of the equivalent, **second order** eddy current equation

$$\sigma\dot{\mathbf{E}} + \nabla \times \mu^{-1} \nabla \times \mathbf{E} = 0$$

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The three methods: summary

Fully mimetic

$$\min \frac{1}{2}\left(\left\|\sqrt{\sigma}\left(\sigma^{-1}j_h^2 - e_h^1\right)\right\|^2 + \left\|\sqrt{\mu\gamma}\left(\mu^{-1}b_h^2 - h_h^1\right)\right\|^2\right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

$$\min \frac{1}{2}\left(\left\|\sqrt{\sigma}\left(\sigma^{-1}dh_h^1 - e_h^1\right)\right\|^2 + \left\|\sqrt{\mu\gamma}\left((\mu\gamma)^{-1}de_h^1 + h_h^1 - \mu^{-1}\overline{b}_h^2\right)\right\|^2\right)$$

Primal mimetic

$$\min \frac{1}{2}\left(\left\|\sqrt{\sigma}\left(\sigma^{-1}j_h^1 - e_h^1\right)\right\|^2 + \left\|\sqrt{\mu\gamma}\left(\mu^{-1}b_h^2 - h_h^2\right)\right\|^2\right) \quad \text{subject to} \quad de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad d^* h_h^2 = j_h^1$$

$$de_h^1 = -\gamma\left(b_h^2 - \overline{b}_h^2\right) \quad \text{and} \quad \left(\mu^{-1}b_h^2, d\hat{e}_h^1\right) + \left\langle h_t, \hat{e}_h^1\right\rangle_{\Gamma_1} = \left(\sigma e_h^1, \hat{e}_h^1\right) \quad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K)$$

$$\gamma\left(\sigma e_h^1, \hat{e}_h^1\right) + \left(\mu^{-1}de_h^1, d\hat{e}_h^1\right) = \gamma\left\langle h_t, \hat{e}_h^1\right\rangle_{\Gamma_1} + \gamma\left(\mu^{-1}\overline{b}_h^2, d\hat{e}_h^1\right) \quad \forall\, \hat{e}_h^1 \in \Lambda^1(d,K) \qquad \sigma\dot{\mathbf{E}} + \nabla \times \mu^{-1}\nabla \times \mathbf{E} = 0$$

Dual mimetic

$$\min \frac{1}{2}\left(\left\|\sqrt{\sigma}\left(\sigma^{-1}j_h^2 - e_h^2\right)\right\|^2 + \left\|\sqrt{\mu\gamma}\left(\mu^{-1}b_h^1 - h_h^1\right)\right\|^2\right) \quad \text{subject to} \quad d^* e_h^2 = -\gamma\left(b_h^1 - \overline{b}_h^1\right) \quad \text{and} \quad dh_h^1 = j_h^2$$

$$dh_h^1 = j_h^2 \quad \text{and} \quad \left(\sigma^{-1}j_h^2, d\hat{h}_h^1\right) + \left\langle e_t, \hat{h}_h^1\right\rangle_{\Gamma_2} = -\gamma\left(\mu h_h^1 + \overline{b}_h^1, \hat{h}_h^1\right) \quad \forall\, \hat{h}_h^1 \in \Lambda^1(d,K)$$

$$\gamma\left(\mu h_h^1, \hat{h}_h^1\right) + \left(\sigma^{-1}dh_h^1, d\hat{h}_h^1\right) = -\left\langle e_t, \hat{h}_h^1\right\rangle_{\Gamma_2} + \gamma\left(\overline{b}_h^1, \hat{h}_h^1\right) \quad \forall\, \hat{h}_h^1 \in \Lambda^1(d,K) \qquad \mu\dot{\mathbf{H}} + \nabla \times \sigma^{-1}\nabla \times \mathbf{H} = 0$$

A surprise!

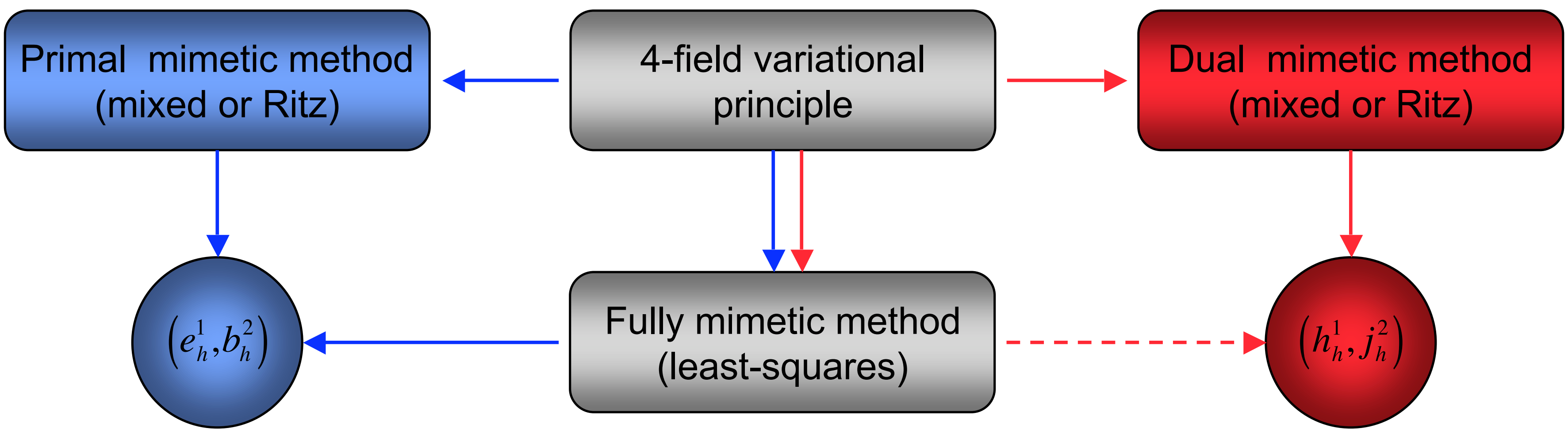
Theorem 2 Let $\left(e_h^1, b_h^2\right), \left(h_h^1, j_h^2\right)$ be the mimetic **least-squares** solution. Then

$\left(e_h^1, b_h^2\right)$ is the solution of the **primal** mimetic method

If $\boldsymbol{b(x,0)=0}$, or we solve in frequency domain, we also have that

$\left(h_h^1, j_h^2\right)$ is the solution of the **dual** mimetic method

This means, mimetic LS is equivalent to simultaneous solution of the primal and dual methods



Proof

The first order necessary condition for the least-squares principle is

$$\left(\sigma^{-1/2}dh_h^1-\sigma^{1/2}e_h^1,\sigma^{-1/2}d\hat{h}_h^1-\sigma^{1/2}\hat{e}_h^1\right)+\left((\mu\gamma)^{-1/2}de_h^1+(\mu\gamma)^{1/2}h_h^1,(\mu\gamma)^{-1/2}d\hat{e}_h^1+(\mu\gamma)^{1/2}\hat{h}_h^1\right)$$
$$=\left(\mu^{-1}(\mu\gamma)^{1/2}\bar{b}_h^2,(\mu\gamma)^{-1/2}d\hat{e}_h^1+(\mu\gamma)^{1/2}\hat{h}_h^1\right)$$

Expand each term

$$\left(\sigma^{-1/2}dh_h^1-\sigma^{1/2}e_h^1,\sigma^{-1/2}d\hat{h}_h^1-\sigma^{1/2}\hat{e}_h^1\right)=\left(\sigma e_h^1,\hat{e}_h^1\right)+\left(\sigma^{-1}dh_h^1,d\hat{h}_h^1\right)-\left(dh_h^1,\hat{e}_h^1\right)-\left(e_h^1,d\hat{h}_h^1\right)$$
$$\left((\mu\gamma)^{-1/2}de_h^1+(\mu\gamma)^{1/2}h_h^1,(\mu\gamma)^{-1/2}d\hat{e}_h^1+(\mu\gamma)^{1/2}\hat{h}_h^1\right)=\gamma\left(\mu h_h^1,\hat{h}_h^1\right)+\gamma^{-1}\left(\mu^{-1}de_h^1,d\hat{e}_h^1\right)+\left(de_h^1,\hat{h}_h^1\right)+\left(h_h^1,d\hat{e}_h^1\right)$$

The least-squares optimality system **uncouples** into two independent equations

$$\gamma\left(\sigma e_h^1,\hat{e}_h^1\right)+\left(\mu^{-1}de_h^1,d\hat{e}_h^1\right)=\gamma\left\langle h_t,\hat{e}_h^1\right\rangle_{\Gamma_2}+\gamma\left(\mu^{-1}\bar{b}_h^2,d\hat{e}_h^1\right)\quad\forall\hat{e}_h^1\in\Lambda^1(d,K)$$

Primal mimetic

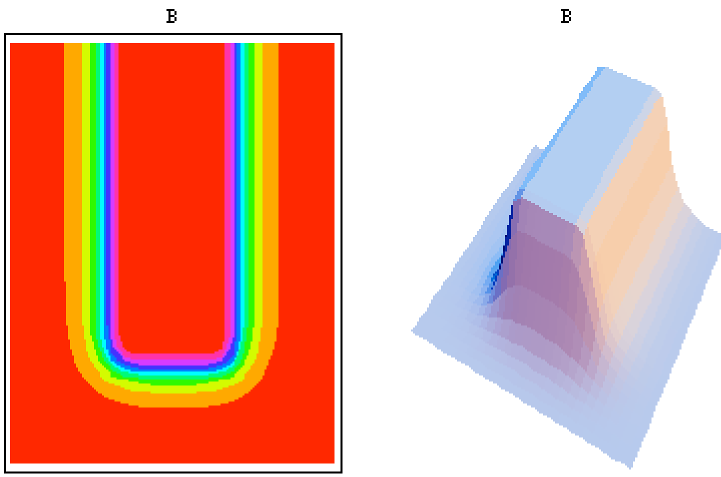
$$\gamma\left(\mu h_h^1,\hat{h}_h^1\right)+\left(\sigma^{-1}dh_h^1,d\hat{h}_h^1\right)=-\left\langle e_t,\hat{h}_h^1\right\rangle_{\Gamma 1}+\gamma\left(\bar{b}_h^2,\hat{h}_h^1\right)\quad\forall\hat{h}_h^1\in\Lambda^1(d,K)$$

If $b(x,0)=0$, or in frequency domain, then the 2nd LS equation is identical to

$$\gamma\left(\mu h_h^1,\hat{h}_h^1\right)+\left(\sigma^{-1}dh_h^1,d\hat{h}_h^1\right)=-\left\langle e_t,\hat{h}_h^1\right\rangle_{\Gamma 1}+\gamma\left(\bar{b}_h^1,\hat{h}_h^1\right)\quad\forall\hat{h}_h^1\in\Lambda^1(d,K)$$

Dual mimetic

Compatibility matters even for least-squares!

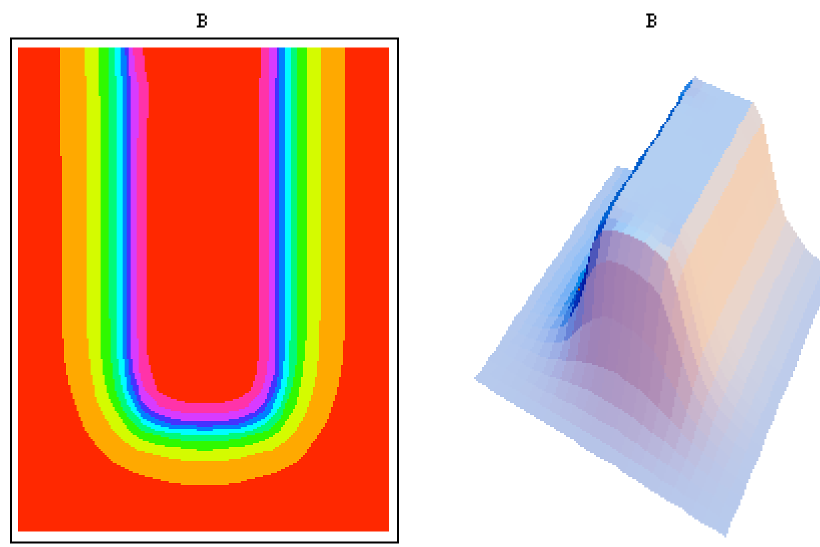


Mimetic LSFEM solution

Ker(curl)={grad p}

Nodal C^0 LSFEM solution

Ker(curl)={0}



Conclusions

Even in least-squares:

Compatibility pays and there's no free lunch

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